Fluid mechanics

1. | Equations of motion

Euler's equations

In this section we will describe the motion of a fluid with a set of equation that result from the conservation of mass, momentum and energy. From what follows, let $D \subseteq \mathbb{R}^3$ be a region filled with a fluid. For each time t and $\mathbf{x} \in D$ we assume that the fluid has a well-defined mass density $\rho(\mathbf{x}, t)$[1]. Finally, we denote by $\mathbf{u}(\mathbf{x}, t)$ the velocity of the fluid at time t and position $\mathbf{x}$. For the moment, we will also assume that ρ and $\mathbf{u}$ are smooth functions.

Proposition 1 (Conservation of mass). Let $W \subseteq D$ be a fixed subregion of D. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho \,\mathrm{d}V = -\int_{\partial W} \rho \mathbf{u} \cdot \mathrm{d}\mathbf{S}$$

Or equivalently:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \mathbf{div}(\rho \mathbf{u}) = 0 \tag{1}$$

This latter equation is called the *continuity equation*.

Proof. The variation of mass in W is given by:

$$\frac{\mathrm{d}m_W}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho \,\mathrm{d}V$$

But on the other hand, the flow of mass through the boundary of W is given by:

$$\frac{\mathrm{d}m_W}{\mathrm{d}t} = -\int_{\partial W} \rho \mathbf{u} \cdot \mathrm{d}\mathbf{S}$$

where the minus sign accounts for the fact the inward flow should be positive (increases the mass) and the outward flow should be negative (decreases the mass). From here the result follows. The differential form is a consequence of **?? ??**. □

Lemma 2. Let $\mathbf{x}(t)$ be the path followed by a fluid particle. Then its acceleration is given by:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} =: \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$$

where $\mathbf{u} \cdot \boldsymbol{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ if $\mathbf{u} = (u, v, w)$. Here the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})$$

is called the *material derivative*.

Sketch of the proof. Compute the time derivative of $\mathbf{u}(t, \mathbf{x}(t))$ using the Chain rule. □

For any continuum, forces acting on a piece of material are of two types. First, there are forces of stress, whereby the piece of material is acted on by forces across its surface by the rest of the continuum. Second, there are external or body, forces such as gravity or a magnetic field, which exert a force per unit volume on the continuum.

Definition 3 (Ideal fluid). An *ideal fluid* has the following property: for any motion of the fluid there is a function $p(\mathbf{x}, t)$ called the *pressure* such that if S is a surface in the fluid with a chosen unit normal $\mathbf{n}$, the force of stress exerted across the surface S per unit of area at $\mathbf{x} \in S$ at time t is $p(\mathbf{x}, t)\mathbf{n}$. Thus, the total force of stress exerted inside a region $W \subseteq D$ is given by:

$$\mathbf{A}_{\partial W} := \text{Force on } W = -\int_{\partial W} p\mathbf{n}\,\mathrm{d}S$$

where the minus sign is because $\mathbf{n}$ points outwards.

Proposition 4 (Conservation of momentum). The balance of momentum for an ideal fluid is given by:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho \mathbf{f}$$

where $\mathbf{f}$ is the external force per unit of mass.

Proof. Let $\mathbf{e}$ be any fixed vector in space. By **?? ??** we have:

$$\mathbf{e}\cdot\mathbf{A}_{\partial W} = -\int_{\partial W} p\mathbf{n}\cdot\mathbf{e}\,\mathrm{d}S = -\int_W \mathbf{div}(p\mathbf{e})\,\mathrm{d}V = -\int_W \boldsymbol{\nabla} p\cdot\mathbf{e}\,\mathrm{d}V$$

Hence:

$$\mathbf{A}_{\partial W} = -\int_W \boldsymbol{\nabla} p \,\mathrm{d}V$$

On the other hand, the total external body acting on W is given by:

$$\mathbf{F} = \int_W \rho \mathbf{f}\,\mathrm{d}V$$

Thus, using the **?? ??** the result follows, as $\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ accounts for the variation of momentum per unit of volume. □

Corollary 5. The integral form of the conservation of momentum is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W \rho\mathbf{u}\,\mathrm{d}V = -\int_{\partial W} (p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\,\mathrm{d}S + \int_W \rho\mathbf{f}\,\mathrm{d}V$$

Proof. From Eq. (1) and the material derivative we have:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) = -\mathbf{div}(\rho\mathbf{u})\mathbf{u} - \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla} p + \rho\mathbf{f}$$

[1] The assumption that ρ exists is a continuum assumption. Clearly, it does not hold if the molecular structure of matter is taken into account. For most macroscopic phenomena occurring in nature, it is believed that this assumption is extremely accurate.

Let $\mathbf{e} \in \mathbb{R}^3$ be a fixed vector. Then:

$$\begin{aligned}\mathbf{e}\cdot \frac{\partial}{\partial t}(\rho\mathbf{u}) &= -\mathbf{e}\cdot \mathbf{div}(\rho\mathbf{u})\mathbf{u} - \mathbf{e}\cdot\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \mathbf{e}\cdot\boldsymbol{\nabla}p + \mathbf{e}\cdot\rho\mathbf{f} \\ &= -\,\mathbf{div}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})) + \rho\mathbf{e}\cdot\mathbf{f}\end{aligned}$$

Integrating over W and using the **?? ??** we obtain the result. □

Definition 6. Let $\mathbf{x} \in D$. We denote by $\boldsymbol{\varphi}(\mathbf{x}, t)$ the position of the fluid particle $\mathbf{x}$ at time t and fixed $t \in \mathbb{R}$, $\boldsymbol{\varphi}_t : \mathbf{x} \to \boldsymbol{\varphi}(\mathbf{x}, t)$. If $W \subseteq D$, we denote $W_t := \boldsymbol{\varphi}_t(W)$ the volume W moving with the fluid.

Lemma 7. Let $J(\mathbf{x}, t)$ be the Jacobian determinant of $\boldsymbol{\varphi}_t$. Then:

$$\frac{\partial}{\partial t}J(\mathbf{x}, t) = J(\mathbf{x}, t)(\mathbf{div}\,\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)$$

Proof. We have that $J = \det \mathbf{D}\boldsymbol{\varphi} = \det\left(\frac{\partial\phi_1}{\partial\mathbf{x}}, \frac{\partial\phi_2}{\partial\mathbf{x}}, \frac{\partial\phi_3}{\partial\mathbf{x}}\right)$, where $\boldsymbol{\varphi} = (\phi_1, \phi_2, \phi_3)$ and $\frac{\partial\phi_i}{\partial\mathbf{x}} := \left(\frac{\partial\phi_i}{\partial x}, \frac{\partial\phi_i}{\partial y}, \frac{\partial\phi_i}{\partial z}\right)^{\mathrm{T}}$. Hence, from the multilineary property of the determinant we have:

$$\frac{\partial}{\partial t}J = \det\left(\frac{\partial}{\partial t}\frac{\partial\phi_1}{\partial\mathbf{x}}, \frac{\partial\phi_2}{\partial\mathbf{x}}, \frac{\partial\phi_3}{\partial\mathbf{x}}\right) + \det\left(\frac{\partial\phi_1}{\partial\mathbf{x}}, \frac{\partial}{\partial t}\frac{\partial\phi_2}{\partial\mathbf{x}}, \frac{\partial\phi_3}{\partial\mathbf{x}}\right) + \det\left(\frac{\partial\phi_1}{\partial\mathbf{x}}, \frac{\partial\phi_2}{\partial\mathbf{x}}, \frac{\partial}{\partial t}\frac{\partial\phi_3}{\partial\mathbf{x}}\right) \quad (2)$$

Now if $\mathbf{u} = (u_1, u_2, u_3)$, then:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\phi_i}{\partial\mathbf{x}} &= \frac{\partial}{\partial\mathbf{x}}u_i(\boldsymbol{\varphi}(\mathbf{x}, t), t) \\ &= \frac{\partial u_i}{\partial\phi_1}\frac{\partial\phi_1}{\partial\mathbf{x}} + \frac{\partial u_i}{\partial\phi_2}\frac{\partial\phi_2}{\partial\mathbf{x}} + \frac{\partial u_i}{\partial\phi_3}\frac{\partial\phi_3}{\partial\mathbf{x}}\end{aligned}$$

because $\frac{\partial\phi_i}{\partial t} = u_i(\boldsymbol{\varphi}(\mathbf{x}, t), t)$. Introducing this into Eq. (2) we obtain:

$$\frac{\partial}{\partial t}J = J\left(\frac{\partial u_1}{\partial\phi_1} + \frac{\partial u_2}{\partial\phi_2} + \frac{\partial u_3}{\partial\phi_3}\right) = J(\mathbf{div}\,\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)$$

□

Corollary 8. We have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\,\mathrm{d}V = \int_{W_t}\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V$$

Proof. Using the **?? ??** we have that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho\mathbf{u}\,\mathrm{d}V &= \int_{W}\Big[\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t) + (\rho\mathbf{u})\cdot \\ &\qquad \cdot(\mathbf{div}\,\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}, t), t)\Big]J(\mathbf{x}, t)\,\mathrm{d}V \\ &= \int_{W_t}\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}) + (\rho\,\mathbf{div}\,\mathbf{u})\mathbf{u}\,\mathrm{d}V \\ &= \int_{W_t}\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V\end{aligned}$$

where the last equality follows from the Theorem 1:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathbf{div}\,\mathbf{u} = \frac{\partial\rho}{\partial t} + \mathbf{div}(\rho\mathbf{u}) = 0$$

□

Corollary 9 (Transport theorem). For any smooth enough function $f(\mathbf{x}, t)$ we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\rho f\,\mathrm{d}V &= \int_{W_t}\rho\frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}V \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}f\,\mathrm{d}V &= \int_{W_t}\left[\frac{\mathrm{d}f}{\mathrm{d}t} + \mathbf{div}(f\mathbf{u})\right]\mathrm{d}V\end{aligned}$$

Definition 10. A flow is called *incompressible* if for any fluid subregion $W \subseteq D$ we have:

$$\mathrm{vol}(W_t) = \mathrm{vol}(W) = \mathrm{const.}$$

Otherwise, the flow is called *compressible*.

Proposition 11. Consider the flow $\boldsymbol{\varphi}$ and its Jacobian J. Then, the following are equivalent:

1. The flow is incompressible.
2. $\mathbf{div}\,\mathbf{u} = 0$.
3. $J = 1$.

Proof. Note that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{W}J\,\mathrm{d}V = \int_{W_t}J\,\mathbf{div}\,\mathbf{u}\,\mathrm{d}V$$

Hence, if $\mathbf{div}\,\mathbf{u} = 0$ then the flow is incompressible. Now, if the flow is incompressible we have that:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t}\mathrm{d}V = \int_{W}\frac{\mathrm{d}J}{\mathrm{d}t}\,\mathrm{d}V$$

which is implies that $J = \mathrm{const.}$ by **?? ??**. Since $J(\mathbf{x}, 0) = 1$ we have that $J = 1$. Finally, from Theorem 7 we have that if $J = 1$ then $\mathbf{div}\,\mathbf{u} = 0$. □

Definition 12. A fluid is called *homogeneous* if $\rho = \rho(t)$, that is, if ρ is constant in space.

Proposition 13. A fluid is incompressible if and only if $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$. In particular, if the fluid is homogeneous, then it is incompressible if and only if $\rho = \mathrm{const.}$ (i.e. it is also constant in time).

Proof. We can write Eq. (1) as:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathbf{div}\,\mathbf{u} = 0$$

And the result follows from Theorem 11. □

Proposition 14. Let J be the Jacobian of the flow $\boldsymbol{\varphi}$. Then:

$$\rho(\boldsymbol{\varphi}(\mathbf{x}, t), t)J(\mathbf{x}, t) = \rho(\mathbf{x}, 0)$$

Sketch of the proof. From 9 Transport theorem with $f = 1$ we have:

$$\int_{W_0} \rho(\mathbf{x}, 0)\,\mathrm{d}V = \int_{W_t} \rho\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_0} \rho J\,\mathrm{d}V$$

Since, W_0 is arbitrary, the result follows from **??** **??**. □

Remark. As a corollary, a fluid that is homogeneous at $t = 0$ but is compressible, will generally not remain homogeneous. However, the fluid will remain homogeneous if it is incompressible.

Definition 15. The *kinetic energy* of a moving portion W_t of a fluid is defined as:

$$E_{\text{kinetic}} = \frac{1}{2}\int_{W_t} \rho\,\|\mathbf{u}\|^2\,\mathrm{d}V$$

where the norm is the Euclidean norm.

Lemma 16. The rate of change of kinetic energy is given by:

$$\frac{\mathrm{d}E_{\text{kinetic}}}{\mathrm{d}t} = \int_{W_t} \rho\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V$$

Proof. From 9 Transport theorem we have that:

$$\frac{\mathrm{d}E_{\text{kinetic}}}{\mathrm{d}t} = \frac{1}{2}\int_{W_t} \rho\frac{\mathrm{D}\,\|\mathbf{u}\|^2}{\mathrm{D}t}\,\mathrm{d}V$$

Now use the linearity of the material derivative and the dot product. □

Theorem 17. Consider an incompressible fluid such that the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work:

$$\frac{\mathrm{d}E_{\text{kinetic}}}{\mathrm{d}t} = -\int_{\partial W_t} p\mathbf{u}\cdot\mathrm{d}\mathbf{S} + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V$$

Then, the Euler equations that completely describe the motion of the fluid are:

$$\begin{cases} \rho\dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{f} \\ \dfrac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \\ \mathbf{div}\,\mathbf{u} = 0 \end{cases}$$

with the boundary conditions $\mathbf{u}\cdot\mathbf{n} = 0$ on ∂D.

Proof. From 16 and using the **??** **??** we have:

$$\int_{W_t} \rho\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V = -\int_{W_t} [\mathbf{div}(p\mathbf{u}) - \rho\mathbf{u}\cdot\mathbf{f}]\,\mathrm{d}V$$
$$= -\int_{W_t} [\mathbf{u}\cdot\boldsymbol{\nabla}p - \rho\mathbf{u}\cdot\mathbf{f}]$$

because $\mathbf{div}\,\mathbf{u} = 0$. This equation is a consequence of balance of momentum. □

Remark. This argument, in addition, shows that if we assume $E = E_{\text{kinetic}}$, then the fluid must be incompressible.

Definition 18. A compressible flow is called *isentropic* if there exists a function w, called the *enthalpy*, such that:

$$\boldsymbol{\nabla}w = \frac{1}{\rho}\boldsymbol{\nabla}p$$

Remark. From this part we will need some basic concepts of thermodynamics, that we review here. Recall that:

$$p = \text{pressure} \quad \rho = \text{density} \quad T = \text{temperature} \quad s = \text{entropy}$$
$$w = \text{enthalpy} \quad \epsilon = \text{internal energy per unit mass}$$

These quantities are related by the First Law of Thermodynamics:

$$\mathrm{d}w = T\,\mathrm{d}s + \frac{1}{\rho}\,\mathrm{d}p \tag{3}$$

which using that $\epsilon = w - p/\rho$ can be written as:

$$\mathrm{d}\epsilon = T\,\mathrm{d}s + \frac{p}{\rho^2}\,\mathrm{d}\rho$$

Remark. Note that if the pressure is a function of ρ only, then the flow is isentropic by defining $w = \int \frac{p'(\rho)}{\rho}\,\mathrm{d}\rho$ which is the integrated version of Eq. (3).

Theorem 19. For isentropic flows, the integral form of the energy balace reads as follows: The rate of change of energy in a portion of fluid equals the rate at which work is done on it.

$$\frac{\mathrm{d}E_{\text{total}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{W_t} \left[\frac{1}{2}\rho\,\|\mathbf{u}\|^2 + \rho\epsilon\right]\mathrm{d}V$$
$$= \int_{W_t} \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}V - \int_{\partial W_t} p\mathbf{u}\cdot\mathrm{d}\mathbf{S}$$

And the Euler equations are:

$$\begin{cases} \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}w + \mathbf{f} \\ \dfrac{\partial\rho}{\partial t} + \mathbf{div}(\rho\mathbf{u}) = 0 \end{cases}$$

and the boundary conditions are $\mathbf{u}\cdot\mathbf{n} = 0$ on ∂D.

Remark. Gases can often be treated as isentropic fluid with $p = A\rho^\gamma$ where A and $\gamma \geq 1$ are constants. Here:

$$w = \frac{\gamma A\rho^{\gamma-1}}{\gamma - 1} \qquad \epsilon = \frac{A\rho^{\gamma-1}}{\gamma - 1}$$

Definition 20. Given a fluid with velocity field $\mathbf{u}(\mathbf{x}, t)$, a *streamline* is a curve $\mathbf{x}(s)$ such that $\mathbf{u}(\mathbf{x}(s), t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$ with t fixed.

Definition 21. We define the trajectory as the curve $\mathbf{x}(t)$ such that $\mathbf{u}(\mathbf{x}(t), t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

Remark. If $\mathbf{u}$ is independent of t, then the streamlines and trajectories coincide. In this case, the fluid is said to be *stationary* or *steady*.

Theorem 22 (Bernoulli's theorem). In a stationary isentropic flow with a present conservative force $\mathbf{f} = -\boldsymbol{\nabla}\psi$, the quantity

$$\frac{1}{2}\|\mathbf{u}\|^2 + w + \psi$$

is constant along streamlines. The same holds for homogeneous incompressible flow with w replaced by p/ρ.

Proof. An easy check shows that:

$$\frac{1}{2}\boldsymbol{\nabla}\left(\|\mathbf{u}\|^2\right) = (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

Because the flow is steady, the equations of motion give $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}w + \mathbf{f}$. Thus:

$$\boldsymbol{\nabla}\left(\frac{1}{2}\|\mathbf{u}\|^2 + w + \psi\right) = \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

Let $\mathbf{x}(s)$ be a streamline. Then:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\frac{1}{2}\|\mathbf{u}\|^2 + w + \psi\right)(\mathbf{x}(s),t)\right] = [\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})]\cdot\mathbf{x}'(s) = 0$$

because $\mathbf{x}'(s) = \mathbf{u}$ is orthogonal to $\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$. □

Rotation and vorticity

Definition 23. Let $\mathbf{u} = (u,v,w)$ be the velocity field of a fluid. The *vorticity* is the vector field $\boldsymbol{\xi} := \boldsymbol{\nabla}\times\mathbf{u}$.

Proposition 24. Let $\mathbf{x}\in\mathbb{R}^3$ and let $\mathbf{u}(\mathbf{x})$ be a smooth vector field. Then, in a small neighbourhood of $\mathbf{x}$, $\mathbf{u}$ is the sum of a translation, a deformation and a rotation (with rotation vector $\boldsymbol{\xi}/2$):

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\cdot\mathbf{h} + \frac{1}{2}\boldsymbol{\xi}(\mathbf{x})\times\mathbf{h} + \mathrm{O}\left(\|\mathbf{h}\|^2\right) \tag{4}$$

where $\mathbf{y} = \mathbf{x}+\mathbf{h}$. The matrix $\mathbf{D}$ is symmetric and is called the *deformation tensor*.

Proof. From **??** **??** we have:

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \boldsymbol{\nabla}\mathbf{u}(\mathbf{x})\cdot\mathbf{h} + \mathrm{O}\left(\|\mathbf{h}\|^2\right)$$

Now let:

$$\mathbf{D} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}) + \boldsymbol{\nabla}\mathbf{u}(\mathbf{x})^{\mathrm{T}}\right] \quad \mathbf{S} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}) - \boldsymbol{\nabla}\mathbf{u}(\mathbf{x})^{\mathrm{T}}\right]$$

Thus, $\boldsymbol{\nabla}\mathbf{u} = \mathbf{D}+\mathbf{S}$ and $\mathbf{S}\cdot\mathbf{h} = \frac{1}{2}\boldsymbol{\xi}(\mathbf{x})\times\mathbf{h}$. □

Remark. The physical intuition behind $\mathbf{D}$ is the following. Because $\mathbf{D}$ is symmetric, for each $\mathbf{x}$ fixed, there is an orthonormal basis $\mathcal{B}$ such that:

$$\mathbf{D}_{\mathcal{B}} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

Now if we ignore all the terms in (4) except $\mathbf{D}(\mathbf{x})\cdot\mathbf{h}$, we see that:

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathbf{D}(\mathbf{x})\cdot\mathbf{h}$$

which is equivalent to three linear differential equations of the form:

$$\frac{\mathrm{d}\tilde{h}_i}{\mathrm{d}t} = d_i\tilde{h}_i$$

Hence, the vector field $\mathbf{D}(\mathbf{x})\cdot\mathbf{h}$ is thus merely expanding or contracting along each of the axis of $\mathcal{B}$. Finally, the rotation term $\frac{1}{2}\boldsymbol{\xi}(\mathbf{x})\times\mathbf{h}$ induces a flow

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \frac{1}{2}\boldsymbol{\xi}(\mathbf{x})\times\mathbf{h}$$

whose solution is a well-known rotation $\mathbf{h}(t) = \mathbf{R}(t,\boldsymbol{\xi}(\mathbf{x}))\mathbf{h}(0)$, where $\mathbf{R}(t,\boldsymbol{\xi}(\mathbf{x}))$ is the matrix that represents a rotation through an angle t about the axis $\boldsymbol{\xi}(\mathbf{x})$.

Lemma 25. Let $\mathbf{u}$ be the velocity field of a flow and C be a closed curve with $C_t := \boldsymbol{\varphi}_t(C)$ the curve transported by the flow. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{s} = \int_{C_t}\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot\mathrm{d}\mathbf{s}$$

Proof. Assume $\mathbf{x}(s)$, $0\le s\le 1$ is a parametrization of C. Then, $\boldsymbol{\varphi}(\mathbf{x}(s),t)$, $0\le s\le 1$ is a parametrization of C_t. Thus, using the 9 Transport theorem we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{s} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}(s),t))\cdot\frac{\partial}{\partial s}\boldsymbol{\varphi}(\mathbf{x}(s),t)\mathrm{d}s \\ &= \int_0^1\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\boldsymbol{\varphi}(\mathbf{x}(s),t))\cdot\frac{\partial}{\partial s}\boldsymbol{\varphi}(\mathbf{x}(s),t)\mathrm{d}s + \\ &\quad + \int_0^1\mathbf{u}(\boldsymbol{\varphi}(\mathbf{x}(s),t))\cdot\frac{\partial}{\partial t}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\varphi}(\mathbf{x}(s),t)\mathrm{d}s\end{aligned}$$

The first term is the desired result. For the second term I_2, note that $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varphi} = \mathbf{u}$ and therefore:

$$I_2 = \frac{1}{2}\int_0^1\frac{\partial}{\partial s}(\mathbf{u}\cdot\mathbf{u})(\boldsymbol{\varphi}(\mathbf{x}(s),t),t)\mathrm{d}s = 0$$

because C_t is closed. □

Theorem 26 (Kelvin's circulation theorem). Consider a isentropic fluid without external forces and C be a closed curve with $C_t := \boldsymbol{\varphi}_t(C)$ the curve transported by the flow. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{s} = 0$$

Proof. Using the Euler equations we know that $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}w$. Thus, from Theorem 25 we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{s} = -\int_{C_t}\boldsymbol{\nabla}w\cdot\mathrm{d}\mathbf{s} = 0$$

because C_t is closed. □

Definition 27. A *vortex line* is a curve ℓ such that it is tangent to the vorticity vector field $\boldsymbol{\xi}$ at each point. A *vortex sheet* is a surface Σ such that it is tangent to the vorticity vector field $\boldsymbol{\xi}$ at each point.

Proposition 28. If a surface (or a curve) move with the flow of an isentropic fluid, and it is a vortex sheet (or a vortex line) at time $t = 0$, then it is a vortex sheet (or a vortex line) at any time t.

Sketch of the proof. Use **??** **??**. □

Proposition 29. For an isentropic fluid (in the absence of external forces) with vorticity $\boldsymbol{\xi}$ and *vorticity per unit of mass* $\boldsymbol{\omega} := \frac{\boldsymbol{\xi}}{\rho}$, the following holds:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} = \mathbf{0}$$
$$\boldsymbol{\omega}(\boldsymbol{\varphi}(\mathbf{x}, t), t) = \mathbf{D}\boldsymbol{\varphi}_t(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}, 0)$$

Definition 30. A *vortex tube* is a two-dimensional surface S formed by all vortex lines passing through a given closed curve C.

Theorem 31 (Helmholtz's theorem). Consider an isentropic fluid. Then:

1. If C_1, C_2 are two closed curves encircling the same vortex tube, then:

$$\int_{C_1} \mathbf{u} \cdot \mathrm{ds} = \int_{C_2} \mathbf{u} \cdot \mathrm{ds}$$

   This value is called *strength of the vortex tube.*

2. The strength of a vortex tube is constant in time as the tube moves with the fluid.

Proof. POSAR FIGURA. Let C_1, C_2 be the oriented curves of **??** and V be the volume enclosed within the tube between the two sections delimited by them. Using that S is a vortex sheet and **??** **??** we have:

$$0 = \int_V \mathbf{div}\,\boldsymbol{\xi}\,\mathrm{d}V = \int_{S_1 \sqcup S_2 \sqcup S} \boldsymbol{\xi} \cdot \mathrm{d}\mathbf{A} = \int_{C_1} \boldsymbol{\xi} \cdot \mathrm{ds} - \int_{C_2} \boldsymbol{\xi} \cdot \mathrm{ds}$$

The second part follows from 26 Kelvin's circulation theorem. □

Proposition 32. Consider a two-dimensional homogeneous incompressible fluid in a simply connected domain $D \subseteq \mathbb{R}^2$. Then, the vorticity vector field $\boldsymbol{\xi} = (0, 0, \xi)$ satisfies the following boundary problem:

$$\begin{cases} \dfrac{\mathrm{D}\xi}{\mathrm{D}t} = 0 \\ \Delta\psi = -\xi \\ \mathbf{u} = (\partial_y \psi, -\partial_x \psi) \end{cases}$$

with $\psi = 0$ on ∂D. The function ψ is called *stream function.* These equations completely determine the flow. The first equation is called *vorticity equation.*

Proof. From incompressibility, if $\mathbf{u} = (u, v)$ we have that $\partial_x u = -\partial_y v$. Thus, from vector calculus, we know that there exists a unique function (except for an additive constant) ψ such that $\mathbf{u} = (\partial_y \psi, -\partial_x \psi)$. Moreover, an easy check shows that ψ is contant along streamlines. Thus, we can adjust the constant so that $\psi = 0$ on ∂D. Now, since the scalar vortex function is $\xi = \partial_x v - \partial_y u$, we have that $\Delta\psi = -\xi$. To prove the first equation, adapt Theorem 29. □

Remark. Note that in the previous proposition we also have the following:

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\xi = u\partial_x \xi + v\partial_y \xi = \partial_y \psi \partial_x \xi - \partial_x \psi \partial_y \xi = \det \mathbf{D}(\xi, \psi)$$

Thus, the flow is stationary if and only if ξ and ψ are functionally dependent.

Theorem 33. Consider a three-dimensional homogeneous incompressible fluid in a convex domain $D \subseteq \mathbb{R}^3$. Then, the vorticity vector field $\boldsymbol{\xi}$ satisfies the following boundary problem:

$$\begin{cases} \dfrac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\mathbf{u} = \mathbf{0} \\ \Delta\mathbf{A} = -\boldsymbol{\xi} \\ \mathbf{div}\,\mathbf{A} = 0 \\ \mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A} \end{cases}$$

Remark. The convexity is used to ensure that given $\mathbf{div}\,\mathbf{u} = 0$, there exists a vector field $\mathbf{A}$ such that $\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{div}\,\mathbf{A} = 0$.

Remark. One of the troubles with the 3-dimensional case is that given $\boldsymbol{\xi}$, the vector field A is not uniquely determined (we cannot impose boundary condition such as $\mathbf{A} = 0$ on ∂D because $\mathbf{A}$ need not be constant on ∂D as was the case with ψ).

Navier-Stokes equations

Theorem 34 (Cauchy's stress theorem). The force acting on a surface S of a fluid is a linear function of the normal vector $\mathbf{n}$ to S.

In **??** we defined an ideal fluid as one in which forces across a surface were normal to that surface. We now consider more general fluids. We will assume now that the force exerted across a surface S per unit of area is given by:

$$\text{Force on } S = -p(\mathbf{x}, t)\mathbf{n} + \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}$$

where the matrix $\boldsymbol{\sigma}$ is called *stress tensor*[2].

Remark. We will make the following assumptions to the stress tensor:

- $\boldsymbol{\sigma}$ is a linear function $(ax + b)$ of the velocity gradients $\boldsymbol{\nabla}\mathbf{u}$.
- $\boldsymbol{\sigma}$ is invariant under rigid body rotations. That is, if $\mathbf{U}$ is an orthogonal matrix:

$$\boldsymbol{\sigma}(\mathbf{U} \cdot \boldsymbol{\nabla}\mathbf{u} \cdot \mathbf{U}^{-1}) = \mathbf{U} \cdot \boldsymbol{\sigma}(\boldsymbol{\nabla}\mathbf{u}) \cdot \mathbf{U}^{-1}$$

- $\boldsymbol{\sigma}$ is symmetric.

[2] The new feature is that $\boldsymbol{\sigma} \cdot \mathbf{n}$ need not be parallel to $\mathbf{n}$. The separation of the forces into pressure and other forces in is somewhat ambiguous because $\boldsymbol{\sigma} \cdot \mathbf{n}$ may contain a component parallel to $\mathbf{n}$. This issue will be resolved later when we give a more definite functional form to $\boldsymbol{\sigma}$.

Proposition 35. The stress tensor $\boldsymbol{\sigma}$ can be written as:

$$\boldsymbol{\sigma} = 2\mu\left[\mathbf{D} - \frac{1}{3}(\mathbf{div}\,\mathbf{u})\mathbf{I}\right] + \zeta(\mathbf{div}\,\mathbf{u})\mathbf{I}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{D}$ is the deformation tensor, μ is the *first viscosity coefficient* and $\zeta = \lambda + \frac{2}{3}\mu$ is the *second viscosity coefficient*.

Proof. Since $\boldsymbol{\sigma}$ is symmetric, it follows that only depends on the symmetric part of $\boldsymbol{\nabla}\mathbf{u}$, that is, on the deformation tensor $\mathbf{D}$. Thus, the eigenvalues of $\boldsymbol{\sigma}$ are linear functions of those of $\mathbf{D}$. By property 2, they must also be symmetric because we can choose $\mathbf{U}$ to permute two eigenvalues of $\mathbf{D}$ and this must permute the corresponding eigenvalues of $\boldsymbol{\sigma}$. Thus, the eigenvalues σ_i of $\boldsymbol{\sigma}$ can be written as $\sigma_i = ad_i + b$, or equivalently:

$$\sigma_i = \lambda\,\mathbf{div}\,\mathbf{u} + 2\mu d_i$$

for some constants λ and μ and $i = 1, 2, 3$. Using again, property 2, we can reconstruct $\boldsymbol{\sigma}$ as:

$$\boldsymbol{\sigma} = \lambda(\mathbf{div}\,\mathbf{u})\mathbf{I} + 2\mu\mathbf{D}$$

Or equivalently:

$$\boldsymbol{\sigma} = 2\mu\left[\mathbf{D} - \frac{1}{3}(\mathbf{div}\,\mathbf{u})\mathbf{I}\right] + \zeta(\mathbf{div}\,\mathbf{u})\mathbf{I}$$

□

Corollary 36. Introducing the stress tensor, the balance of momentum equation yields the *Navier-Stokes equations*:

$$\rho\frac{\mathrm{Du}}{\mathrm{Dt}} = -\boldsymbol{\nabla}p + (\lambda + \mu)\boldsymbol{\nabla}(\mathbf{div}\,\mathbf{u}) + \mu\Delta\mathbf{u}$$

Sketch of the proof. Follow the proof of 4 Conservation of momentum adapted to the stress tensor. □

Remark. This equation together with the continuity equation and energy equation completely describe the flow in a compressible viscous fluid. In the case of an incompressible homogeneous fluid with $\rho = \rho_0 = \text{const.}$, the complete set of equations becomes the *Navier-Stokes equations for incompressible flow*:

$$\begin{cases} \dfrac{\mathrm{Du}}{\mathrm{Dt}} = -\boldsymbol{\nabla}p' + \nu\Delta\mathbf{u} \\ \mathbf{div}\,\mathbf{u} = 0 \end{cases}$$

where $p' = p/\rho_0$ and $\nu = \mu/\rho_0$ is the *kinematic viscosity*. To this we should add boundary condition, which for an ideal fluid we use $\mathbf{u}\cdot\mathbf{n} = 0$ and if the solid wall that bounds the fluid is stationary, we use $\mathbf{u} = \mathbf{0}$ on the walls.

Reynolds number

Proposition 37. For a given problem, let L be the unit of length (*characteristic length*) and U be the unit of velocity (*characteristic velocity*). This choice determines then the unit of time $T = L/U$. If we convert $\mathbf{u}$, $\mathbf{x}$ and t to dimensionless quantities by

$$\mathbf{u}' := \frac{\mathbf{u}}{U} \quad \mathbf{x}' := \frac{\mathbf{x}}{L} \quad t' := \frac{t}{T}$$

then the Navier-Stokes equations for incompressible flow become:

$$\begin{cases} \dfrac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\boldsymbol{\nabla}')\mathbf{u} = -\boldsymbol{\nabla}p' + \dfrac{\nu}{LU}\Delta'\mathbf{u}' \\ \mathbf{div}\,\mathbf{u}' = 0 \end{cases}$$

where $p' = p/(\rho_0 U^2)$. In this equation, the term $(\mathbf{u}'\cdot\boldsymbol{\nabla}')\mathbf{u}'$ is called the *convective term* (or *inertia term*) and the term $\nu\Delta'\mathbf{u}'$ is called the *diffusive term* (or *dissipation term*).

Sketch of the proof. Use the **?? ??**. □

Definition 38. For a given problem, let L be the unit of length (*characteristic length*) and U be the unit of velocity (*characteristic velocity*). We define the *Reynolds number* as:

$$\mathrm{Re} = \frac{LU}{\nu}$$

Remark. Note that the dimensionless Navier-Stokes equations for incompressible flow only depend on the Reynolds number.

Definition 39. Two flows with the same geometry and the same Reynolds number are said to be *similar*. More precisely, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two flows on regions D_1 and D_2 that are related by a scale factor λ so that $L_1 = \lambda L_2$. Let choices of U_1 and U_2 be made for each flow, and let the viscosities be ν_1 and ν_2, respectively. If $\mathrm{Re}_1 = \mathrm{Re}_2$, then the dimensionless velocity fields $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy exactly the same equation on the same region.

Remark. This idea of the similarity of flows is used in the design of experimental models. For example, suppose we are contemplating a new design for an aircraft wing, and we wish to know the behavior of a fluid flow around it. Rather than build the wing itself, it may be faster and more economical to perform the initial tests on a scaled-down version. We design our model so that it has the same geometry as the full-scale wing, and we choose values for the undisturbed velocity, coefficient of viscosity, and so on, such that the Reynolds number for the flow in our experiment matches that of the actual flow. We can then expect the results of our experiment to be relevant to the actual flow over the full-scale wing.

Theorem 40 (Helmholtz-Hodge decomposition theorem). Let $D \subset \mathbb{R}^3$ be a region with smooth boundary and $\mathbf{w}$ be a vector field on D. Then, there exists a unique decomposition:

$$\mathbf{w} = \mathbf{u} + \boldsymbol{\nabla}p$$

where $\mathbf{u}$ has zero divergences and it's parallel to ∂D, i.e. $\mathbf{u}\cdot\mathbf{n} = 0$ on ∂D.

Proof. We use the fact that the following problem has existence and uniqueness of solutions up to the addition of a constant to p:

$$\begin{cases} \Delta p = \mathbf{div}\,\mathbf{w} & \text{in } D \\ \dfrac{\partial p}{\partial \mathbf{n}} = \mathbf{w}\cdot\mathbf{n} & \text{on } \partial D \end{cases}$$

Now let $\mathbf{u} := \mathbf{w} - \boldsymbol{\nabla} p$, where p is the solution of the above problem. Then, $\mathbf{div}\, \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$. To prove uniqueness, first note that if $\mathbf{w} = \mathbf{u} + \boldsymbol{\nabla} p$, then:

$$\int_D \mathbf{u} \cdot \boldsymbol{\nabla} p = 0$$

which follows automatically from the **?? ??**, the hypothesis on $\mathbf{u}$ and the relation:

$$\mathbf{div}(\mathbf{u} p) = \mathbf{u} \cdot \boldsymbol{\nabla} p + p\, \mathbf{div}\, \mathbf{u}$$

Now, if moreover we have $\mathbf{w} = \mathbf{u}' + \boldsymbol{\nabla} p'$, then $\mathbf{0} = \mathbf{u} - \mathbf{u}' + \boldsymbol{\nabla}(p - p')$ and therefore taking the inner product with $\mathbf{u} - \mathbf{u}'$ and integrating we get:

$$0 = \int_D \|\mathbf{u} - \mathbf{u}'\|^2 + \int_D (\mathbf{u} - \mathbf{u}') \cdot \boldsymbol{\nabla}(p - p') = \int_D \|\mathbf{u} - \mathbf{u}'\|^2$$

Hence, $\mathbf{u} = \mathbf{u}'$ and therefore $\boldsymbol{\nabla} p = \boldsymbol{\nabla} p'$, which implies that $p - p' = \text{const}$. □

Definition 41. Let $D \subset \mathbb{R}^3$ be a region with smooth boundary. Given a vector field $\mathbf{w} = \mathbf{u} + \boldsymbol{\nabla} p$ on D, we can define the orthogonal projector operator $\mathbf{P}$ as:

$$\mathbf{P}(\mathbf{w}) := \mathbf{u}$$

Remark. The Navier-Stokes equations become thus:

$$\partial_t u = \mathbf{P}\left(-(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \frac{1}{\text{Re}} \Delta \mathbf{u}\right)$$

Remark. If $\text{Re} \gg 1$, then the Navier-Stokes equations for incompressible flow can be approximated by the *Stokes' equations*:

$$\begin{cases} \partial_t \mathbf{u} = -\boldsymbol{\nabla} p + \dfrac{1}{\text{Re}} \Delta \mathbf{u} \\ \mathbf{div}\, \mathbf{u} = 0 \end{cases}$$

Proposition 42. Consider an incompressible viscous flow confined in a region $D \subset \mathbb{R}^3$ with smooth boundary. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} = -\mu \int_D \|\boldsymbol{\nabla} \mathbf{u}\|^2 \, \mathrm{d}V$$

Proof. We have that:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_{\text{kinetic}} &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_D \rho \|\mathbf{u}\|^2 \, \mathrm{d}V = \int_D \rho \mathbf{u} \cdot \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \, \mathrm{d}V \\ &= \int_D \rho \mathbf{u} \cdot (-\boldsymbol{\nabla} p + \mu \Delta \mathbf{u}) \, \mathrm{d}V \\ &= \mu \int_D \rho \mathbf{u} \cdot \Delta \mathbf{u} \, \mathrm{d}V \end{aligned}$$

because $\mathbf{div}\, \mathbf{u} = 0$ and the orthogonality of $\mathbf{u}$ and $\boldsymbol{\nabla} p$. Now using that $\mathbf{div}(\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = \mathbf{u} \cdot \Delta \mathbf{u} + \|\boldsymbol{\nabla} \mathbf{u}\|^2$, the **?? ??** and the boundary condition $\mathbf{u} = 0$, we get the result. □

Proposition 43. Consider a three-dimensional viscous incompressible flow. Then, the vorticity equation becomes:

$$\frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\mathbf{u} = \frac{1}{\text{Re}} \Delta \boldsymbol{\xi}$$

Definition 44. Assume we have an *equation of state* $p = p(\rho)$, with $p'(\rho) > 0$. Define $c = \sqrt{p'(\rho)}$, and called it the *speed of sound*. Then, the *Mach number* is defined as:

$$\text{Ma} = \frac{u}{c}$$

where $u = \|\mathbf{u}\|$.