Stochastic calculus

Along the document we assume that we work in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that all the random variables are defined on this space.

1. | Preliminaries

Stochastic processes

Proposition 1. A stochastic process $X = (X_t)_{t\in\mathbb{T}}$ is Gaussian if and only if $\forall n \in \mathbb{N}$, $\forall t_1, \ldots, t_n \in \mathbb{T}$, $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$Z := \lambda_1 X_{t_1} + \cdots + \lambda_n X_{t_n}$$

is a Gaussian random variable. In particular, we have:

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}Z}) = \mathrm{e}^{\mathrm{i}\mathbb{E}(Z) - \frac{1}{2}\mathrm{Var}(Z)}$$

Remark. A stochastic process $X = (X_t)_{t\in\mathbb{T}}$ can also be viewed as a single random variable taking values in $\mathbb{R}^{\mathbb{T}}$, equipped with the product σ-algebra $\bigotimes_{t\in\mathbb{T}} \mathcal{B}(\mathbb{R})$.

Proposition 2. Let $m : \mathbb{T} \to \mathbb{R}$ be a measurable function and $\gamma : \mathbb{T}^2 \to \mathbb{R}$ be a symmetric positive-definite function. Then, there exists a Gaussian process $(X_t)_{t\in\mathbb{T}}$ such that $\mathbb{E}(X_t) = m(t)$ and $\mathrm{Cov}(X_s, X_t) = \gamma(s,t)$.

Definition 3. Let $(X_t)_{t\in\mathbb{T}}$, $(Y_s)_{s\in\mathbb{S}}$ be two stochastic processes. We say that they are *jointly Gaussian* if the concatenated process $((X_t)_{t\in\mathbb{T}}, (Y_s)_{s\in\mathbb{S}})$ is Gaussian.

Lemma 4. Two jointly Gaussian stochastic processes $(X_t)_{t\in\mathbb{T}}$, $(Y_s)_{s\in\mathbb{S}}$ are independent if and only if $\forall t \in \mathbb{T}$, $\forall s \in \mathbb{S}$, $\mathrm{Cov}(X_t, Y_s) = 0$.

Proposition 5. Two stochastic processes $(X_t)_{t\in\mathbb{T}}$, $(Y_s)_{s\in\mathbb{S}}$ are independent if and only if $\forall n \in \mathbb{N}$, $\forall t_1, \ldots, t_n \in \mathbb{T}$, $\forall s_1, \ldots, s_n \in \mathbb{S}$ and $\forall f, g : \mathbb{R}^n \to \mathbb{R}$ bounded and measurable functions, we have:

$$\mathbb{E}(f(X_{t_1}, \ldots, X_{t_n}) g(Y_{s_1}, \ldots, Y_{s_n})) = \\ = \mathbb{E}(f(X_{t_1}, \ldots, X_{t_n}))\mathbb{E}(g(Y_{s_1}, \ldots, Y_{s_n}))$$

Brownian motion

Definition 6. A *Brownian motion* is a stochastic process $(B_t)_{t\geq 0}$ such that:

1. B is Gaussian with $\mathbb{E}(B_t) = 0$ and $\mathrm{Cov}(B_s, B_t) = s \wedge t$.
2. B has continuous paths.

Proposition 7. Let B be a Brownian motion. Then:

1. $B_0 = 0$ a.s.
2. B has independent increments.
3. B has stationary increments.

Conversely, any stochastic process with these properties has the law of a Brownian motion.

Theorem 8 (Strong law of large numbers for Brownian motion). Let $(B_t)_{t\geq 0}$ be a Brownian motion. Then:

$$\frac{B_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} 0$$

Proof. We already now that the process $s \to sB_{1/s}\mathbf{1}_{s>0}$ is a Brownian motion. In particular, we must have continuity at $0 = B_0$. □

Theorem 9 (Markov property for Brownian motion). Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and $a \geq 0$ fixed. Then, the Brownian motion $(B_{t+a} - B_a)_{t\geq 0}$ is independent of $(B_s)_{s\in[0,a]}$.

Proof. The processes $(B_s)_{s\in[0,a]}$ and $(B_{t+a} - B_a)_{t\geq 0}$ are jointly Gaussian, because their coordinates are linear combinations of coordinates of the same Gaussian process B. Thus, by Theorem 4 it reduces to compute the following correlation:

$$\mathrm{Cov}(B_s, B_{t+a} - B_a) = s \wedge (t+a) - s \wedge a = 0$$

□

Remark. Recall that $s \wedge t := \min(s,t)$ and $s \vee t := \max(s,t)$.

Martingales

Definition 10. Let $(X_t)_{t\geq 0}$ be a stochastic process. We define the *natural filtration* of X as $\mathcal{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$, where $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$.

From now on, we will assume that we work in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$.

Definition 11 (Martingale). A stochastic process $(X_t)_{t\geq 0}$ is a *martingale* if:

1. it is *adapted*, i.e. X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.
2. $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$.
3. $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$ for all $0 \leq s \leq t$.

The process is called a *sub-martingale* if the last condition is replaced by $\mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s$ for all $0 \leq s \leq t$ and a *super-martingale* if $\mathbb{E}(X_t \mid \mathcal{F}_s) \leq X_s$ for all $0 \leq s \leq t$.

Proposition 12. Let $B = (B_t)_{t\geq 0}$ be a Brownian motion. Then, the following processes are martingales $(M_t)_{t\geq 0}$ with respect to the natural filtration induced by B:

- $M_t = B_t$
- $M_t = B_t^2 - t$
- $M_t = \mathrm{e}^{\theta B_t - \frac{1}{2}\theta^2 t}$, for any fixed $\theta \in \mathbb{R}$.

Proposition 13. Let $A \subseteq \mathbb{R}$ be a closed set and $X = (X_t)_{t\geq 0}$ be an adapted continuous process. Then, the *hitting time* of A by X, defined as:

$$T_A := \inf\{t \geq 0 : X_t \in A\}$$

is a stopping time.

Proof. Using the continuity of X and the fact that A is closed, one can easily check that:

$$\{T_A \leq t\} = \bigcap_{k\in\mathbb{N}} \bigcup_{s\in[0,t]\cap\mathbb{Q}} \left\{d(X_s, A) \leq \frac{1}{k}\right\}$$

Now, $\left\{d(X_s, A) \leq \frac{1}{k}\right\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$ because X is adapted and $z \mapsto d(z, A)$ is measurable. Thus, $\{T_A \leq t\} \in \mathcal{F}_t$ because it is a countable union and intersection of events in $\mathcal{F}_t$. □

Theorem 14 (Doob's optional sampling theorem). Let $(M_t)_{t\geq 0}$ be a continuous martingale and T be a stopping time. Then, the *stopped process* $M^T := (M_{t\wedge T})_{t\geq 0}$ is a continuous martingale. In particular, $\forall t \geq 0$, $\mathbb{E}(M_{t\wedge T}) = \mathbb{E}(M_0)$. If M^T is uniformly integrable and $T \overset{\text{a.s.}}{\leq} \infty$, then taking $t \to \infty$ we have $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

Lemma 15 (Orthogonality of martingales). Let $(M_t)_{t\geq 0}$ be a continuous martingale and let $0 \leq s \leq t$. Then:

$$\mathbb{E}((M_t - M_s)^2 \mid \mathcal{F}_s) = \mathbb{E}({M_t}^2 - {M_s}^2 \mid \mathcal{F}_s)$$

Proof. We have that:

$$\mathbb{E}((M_t - M_s)^2 \mid \mathcal{F}_s) = \mathbb{E}({M_t}^2 - 2M_tM_s + {M_s}^2 \mid \mathcal{F}_s) =$$
$$= \mathbb{E}({M_t}^2 + {M_s}^2 \mid \mathcal{F}_s) - 2M_s\mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}({M_t}^2 - {M_s}^2 \mid \mathcal{F}_s)$$

□

Theorem 16 (Doob's maximal inequality). If M is a continuous square-integrable martingale, then $\forall a, t \geq 0$ we have:

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |M_s| \geq a\right) \leq \frac{\mathbb{E}({M_t}^2)}{a^2}$$

Proposition 17. Let (M^n) be a sequence of continuous square-integrable martingales and suppose that for each $t \geq 0$, the limit $M_t \overset{L^2}{:=} \lim_{n\to\infty} M_t^n$ exists. Then, $M = (M_t)_{t\geq 0}$ is a continuous square-integrable martingale.

Proof. By 16 Doob's maximal inequality applied to $M^n - M^m$ we have that for fixed $t \geq 0$ and $k \in \mathbb{N}$:

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |M_s^n - M_s^m| \geq \frac{1}{k^2}\right) \leq k^2\mathbb{E}((M_t^n - M_t^m)^2) \leq \frac{1}{k^2}$$

where in the last inequality we have used that (M^n) converges in L^2 and so we have chosen n, m large enough so that the inequality holds. Thus, there is an increasing sequence (n_k) such that:

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |M_s^{n_{k+1}} - M_s^{n_k}| \geq \frac{1}{k}\right) \leq \frac{1}{k^2}$$

By ?? ??, we deduce that

$$\sum_{k=1}^{\infty} \sup_{0\leq s\leq t} |M_s^{n_{k+1}} - M_s^{n_k}| < \infty$$

which ensures that (M^{n_k}) is continuous in the space of continuous functions equipped with the topology of uniform convergence on every compact set. But the limit is necessarily a version of M, because for each $t \geq 0$ we have $M_t^n \to M_t$ in L^2. □

Quadratic variation

Definition 18. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function. We define the *absoulte variation* of f on the interval $[s, t]$ as:

$$V(f, s, t) := \sup_{(t_k)_{0\leq k\leq n}\in \mathrm{P}([s,t])} \sum_{k=1}^{n} |f(t_{k+1}) - f(t_k)|$$

where $\mathrm{P}([s, t])$ is the set of all partitions of $[s, t]$. A function has *finite variation* if $V(f, s, t) < \infty$ for all $0 \leq s \leq t$.

Lemma 19. Let $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function and $0 \leq s \leq t$. Then:

- $V(f, s, t) = V(f, s, u) + V(f, u, t)$, for all $s \leq u \leq t$.
- If $f \in C^1$, then $V(f, s, t) = \int_s^t |f'(u)| \, du$.
- If f is monotone, then $V(f, s, t) = |f(t) - f(s)|$.
- $V(f + g, s, t) \leq V(f, s, t) + V(g, s, t)$.
- Finite variation functions form a vector space.

Proposition 20. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then, f has finite variation if and only if it can be written as the difference of two non-decreasing functions.

Sketch of the proof. Theorem 19 gives us the implication to the left. For the other one, note that the functions $f_1(t) := V(f, 0, t)$ and $f_2(t) := V(f, 0, t) - f(t)$ are non-decreasing. □

Theorem 21 (Quadratic variation). Let $M = (M_t)_{t\geq 0}$ be a continuous square-integrable martingale. Then, for each $t \geq 0$ the limit

$$\langle M\rangle_t := \lim_{n\to\infty} \sum_{k=1}^{n} \left|M_{t_k^n} - M_{t_{k-1}^n}\right|^2$$

exists in L^1 and does not depend on the partition $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0, t])$ chosen as long as the *mesh* $\Delta_n := \max_{1\leq k\leq n}(t_k^n - t_{k-1}^n)$ goes to 0 as $n \to \infty$. Moreover, $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ has the following properties:

1. $\langle M\rangle_0 = 0$
2. $\langle M\rangle$ is non-decreasing.
3. The function $t \mapsto \langle M\rangle_t$ is continuous.
4. $({M_t}^2 - \langle M\rangle_t)_{t\geq 0}$ is a martingale.

Proof. We omit the proof of the existence and continuity. We will only prove the last property. Let $0 \leq s \leq t$ and $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([s,t])$ be such that $\Delta_n \to 0$. Then:

$$\begin{aligned}\mathbb{E}({M_t}^2 - {M_s}^2 \mid \mathcal{F}_s) &= \sum_{k=1}^{n} \mathbb{E}({M_{t_k^n}}^2 - {M_{t_{k-1}^n}}^2 \mid \mathcal{F}_s) \\ &= \sum_{k=1}^{n} \mathbb{E}\left((M_{t_k}^n - M_{t_{k-1}}^n)^2 \mid \mathcal{F}_s\right)\end{aligned}$$

by the 15 Orthogonality of martingales. Now since we have convergence of $\sum_{k=1}^n (M_{t_k}^n - M_{t_{k-1}}^n)^2$ to $\langle M\rangle_t - \langle M\rangle_s$ in L^1, we get the result:

$$\mathbb{E}({M_t}^2 - {M_s}^2 \mid \mathcal{F}_s) = \mathbb{E}(\langle M\rangle_t - \langle M\rangle_s \mid \mathcal{F}_s)$$

□

Proposition 22. Let B be a Brownian motion. Then:

$$\mathbb{P}(\forall s,t \geq 0, V(B,s,t) = \infty) = 1$$

But, $\langle B\rangle_t = t$ for all $t \geq 0$.

Proof. Let $B = (B_t)_{t\geq 0}$. Then:

$$V(B,s,t) \geq \sum_{k=1}^{n} \left|B_{s+k\frac{t-s}{n}} - B_{s+(k-1)\frac{t-s}{n}}\right| = \sqrt{\frac{t-s}{n}} \sum_{k=1}^{n} |\xi_k|$$

where ξ_k are i.i.d. $N(0,1)$. By the **??** **??** we get the result. The second part is similar, but we get convergence instead. □

Proposition 23. If a function f has finite variation and g is continuous, then:

$$\sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))(g(t_k) - g(t_{k-1})) \xrightarrow{n\to\infty} 0$$

Proof. Note that:

$$\begin{aligned}&\left|\sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))(g(t_k) - g(t_{k-1}))\right| \leq \\ &\qquad \leq V(f,0,t) \max_{\substack{0\leq u\leq v\leq t \\ |u-v|\leq\Delta_n}} |g(u) - g(v)|\end{aligned}$$

which goes to zero by uniform continuity of g at $[0,t]$. □

Corollary 24. Let $M = (M_t)_{t\geq 0}$ be a continuous square-integrable martingale with finite variation a.s. Fix $t \geq 0$. By Theorem 23 we have that $\langle M\rangle_t = 0$ Then:

$$\mathbb{P}(\forall t \geq 0,\ M_t = M_0) = 1$$

Proof. By 15 Orthogonality of martingales, we have:

$$\mathbb{E}((M_t - M_0)^2) = \mathbb{E}({M_t}^2) - \mathbb{E}({M_0}^2) = \mathbb{E}(\langle M\rangle_t) = 0$$

where the penultimate equality follows from the fact that ${M_t}^2 - \langle M\rangle_t$ is a martingale and so it has constant expectation. This shows that $\mathbb{P}(\forall t \geq 0,\ M_t = M_0) = 1$. Now we can use the fact that M is continuous to conclude using $t \in \mathbb{Q}$. □

Proposition 25. The quadratic variation is the unique process that satisfies Items 21-1 to 21-4.

Proof. Let A be another process satisfying such properties. Then, $M^2 - \langle M\rangle$ and $M^2 - A$ are both martingales. Thus, $A - \langle M\rangle$ is also a martingale. But it is also continuous and has finite variation (by Theorem 20). So by Theorem 24, $A = \langle M\rangle$. □

Local martingales

Definition 26. A stochastic process $(M_t)_{t\geq 0}$ is a *continuous local martingale* if there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ (called *localizing sequence*) such that:

1. $T_n \nearrow \infty$ a.s.
2. $M^{T_n} := (M_{t\wedge T_n})_{t\geq 0}$ is a martingale for all $n \in \mathbb{N}$.

Remark. If M is a martingale, then M is a local martingale by taking $T_n = +\infty$ for all $n \in \mathbb{N}$.

Remark. Any local martingale is adapted because it is the pointwise limit of M^{T_n}, which are adapted by definition.

Proposition 27. Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale. Then, if $\forall t \geq 0$ we have

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |M_s|\right) < \infty$$

then M is a martingale.

Proof. We've argued that local martingales are automatically adapted. Moreover:

$$\mathbb{E}(|M_t|) \leq \mathbb{E}\left(\sup_{0\leq s\leq t} |M_s|\right) < \infty$$

Finally, fix $0 \leq s \leq t$. For all $n \in \mathbb{N}$ we have:

$$\mathbb{E}(M_{t\wedge T_n} \mid \mathcal{F}_s) = M_{s\wedge T_n}$$

And using the **??** **??** with $M_{t\wedge T_n} \leq \sup_{0\leq s\leq t} |M_s|$ we conclude the result. □

Remark. Note that if M is a continuous local martingale with $M_0 = 0$, then we can always take $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$ as a localizing sequence.

Theorem 28 (Doob's optional sampling theorem for local martingales). Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale and T be a stopping time. Then, the stopped process $M^T := (M_{t\wedge T})_{t\geq 0}$ is a continuous local martingale.

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence for M. Since M^{T_n} is a continuous martingale, by 14 Doob's optional sampling theorem we have that $M^{T_n\wedge T}$ is a continuous martingale. Thus, M^T is a local martingale with localizing sequence $(T_n)_{n\in\mathbb{N}}$. □

Proposition 29. Continuous local martingales form a vector space.

Proof. Let M, $\tilde{M}$ be continuous local martingales with localizing sequences $(T_n)_{n\in\mathbb{N}}$ and $(\tilde{T}_n)_{n\in\mathbb{N}}$ respectively and $\lambda, \tilde{\lambda} \in \mathbb{R}$. Then, $(T_n \wedge \tilde{T}_n)_{n\in\mathbb{N}}$ is a localizing sequence for both M and $\tilde{M}$ and so $\lambda M^{T_n\wedge\tilde{T}_n} + \tilde{\lambda}\tilde{M}^{T_n\wedge\tilde{T}_n}$ is a martingale. □

Proposition 30. If M is a continuous local martingale which has finite variation a.s., then:

$$\mathbb{P}(\forall t \geq 0,\ M_t = M_0) = 1$$

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence for M. Then, M^{T_n} is a martingale and $V(M^{T_n}, 0, t) = V(M, 0, t\wedge T_n) < \infty$. Thus, by Theorem 24 we have that $M_t^{T_n} = M_0^{T_n}$ $\forall t \geq 0$ and $n \in \mathbb{N}$. Taking $n \to \infty$ we get the result. □

Proposition 31. Let M be a continuous local martingale. Then, the limit

$$\langle M \rangle_t := \lim_{n\to\infty} \sum_{k=1}^{n} \left| M_{t_k^n} - M_{t_{k-1}^n} \right|^2$$

exists in probability for any $t \geq 0$ and does not depend on the partition $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0,t])$ chosen as long as $\Delta_n \to 0$. Moreover, $\langle M \rangle = (\langle M \rangle_t)_{t\geq 0}$ is the unique process (up to modification) such that:

1. $\langle M \rangle_0 = 0$
2. $t \mapsto \langle M \rangle_t$ is a.s. continuous.
3. $\langle M \rangle$ is a.s. non-decreasing.
4. $({M_t}^2 - \langle M \rangle_t)_{t\geq 0}$ is a continuous local martingale.

Theorem 32 (Levy's characterization of Brownian motion). Let $M = (M_t)_{t\geq 0}$ be a stochastic process. Then, the following are equivalent:

1. M is a continuous local square-integrable martingale with $M_0 = 0$ and $\langle M \rangle_t = t$.
2. M is a $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

2. | Stochastic integration

Wiener isometry

Definition 33. Let H, H' be Hilbert. A map $I : H \to H'$ is called *isometry* if it is linear and $\forall x \in H$ we have:

$$\|I(x)\|_{H'} = \|x\|_H$$

We speak of *partial isometry* when I is only defined on a subspace of H.

Theorem 34. Let H, H' be Hilbert, $V \subseteq H$ be a dense subspace and $I : V \to H'$ be a partial isometry. Then, there exists a unique continuous isometry extension of I to H.

Proof. Let $x \in H \setminus V$. Then, $\exists (x_n) \in V$ such that $x_n \to x$. Clearly, any continuous extension must satisfy $I(x) := \lim_{n\to\infty} I(x_n)$, so we take it as a definition. Note that, first, the limit exists because $(I(x_n))$ is Cauchy and moreover this definition does not depend on the sequence (x_n). From this definition, the extension is automatically linear and norm-preserving (because of the continuity). □

Definition 35. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $f \in \mathcal{S}(\mathbb{R}_{\geq 0})$ be a simple function such that $f = \sum_{k=1}^{n} a_k \mathbf{1}_{(t_{k-1},t_k]}$ with $0 = t_0 \leq t_1 \leq \cdots \leq t_n$. We define the *Wiener integral* of f as:

$$I(f) := \sum_{k=1}^{n} a_k (B_{t_k} - B_{t_{k-1}})$$

Remark. Recall that simple functions are dense in L^p (**??**).

Theorem 36. Let $(B_t)_{t\geq 0}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exists a unique linear and continuous map $I : L^2(\mathbb{R}_{\geq 0}) \to L^2((\Omega, \mathcal{F}, \mathbb{P}))$ such that for all $0 \leq s \leq t$:

$$I(\mathbf{1}_{(s,t]}) = B_t - B_s$$

Moreover, I is an isometry. The map I is called *Wiener isometry* (or *Wiener integral*) and denoted by $I(f) = \int_0^\infty f(u)\,\mathrm{d}B_u$.

Remark. Recall that the limit of Gaussian variables is Gaussian.

Proposition 37. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Then, the following are satisfied:

- For any $f \in L^2(\mathbb{R}_{\geq 0})$ we have:

$$\int_0^\infty f(u)\,\mathrm{d}B_u \overset{L^2}{=} \lim_{n\to\infty} \sum_{k=1}^{n^2} a_{n,k}(f)(B_{\frac{k+1}{n}} - B_{\frac{k}{n}})$$

  where $a_{n,k}(f) := n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\,\mathrm{d}u$ is an approximation of f in the interval $[\frac{k}{n}, \frac{k+1}{n}]$.

- The Wiener integral is a Gaussian variable with zero mean and variance $\int_0^\infty f(u)^2\,\mathrm{d}u$.

- For any $f, g \in L^2(\mathbb{R}_{\geq 0})$ we have:

$$\mathrm{Cov}\left(\int_0^\infty f(u)\,\mathrm{d}B_u\,, \int_0^\infty g(u)\,\mathrm{d}B_u\right) = \int_0^\infty f(u)g(u)\,\mathrm{d}u$$

The Wiener integral as a process

Definition 38. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ and $0 \leq s \leq t$. We define the *Wiener integral* of f as:

$$\int_s^t f(u)\,\mathrm{d}B_u := \int_0^\infty f(u)\mathbf{1}_{(s,t]}(u)\,\mathrm{d}B_u$$

Lemma 39 (Chasles relation). Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ and $0 \leq r \leq s \leq t$. Then:

$$\int_r^t f(u)\,\mathrm{d}B_u = \int_r^s f(u)\,\mathrm{d}B_u + \int_s^t f(u)\,\mathrm{d}B_u$$

Proposition 40. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $f \in L^2_{\text{loc}}(\mathbb{R}_{\geq 0})$. Then, the associate process $M^f = (M^f_t)_{t\geq 0}$ defined as:

$$M^f_t := \int_0^t f(u)\,\mathrm{d}B_u$$

is a centered Gaussian process with covariance function:

$$\operatorname{Cov}(M^f_s, M^f_t) = \int_0^{s\wedge t} f(u)^2\,\mathrm{d}u$$

Proof. We'll only proof that M^f is Gaussian (the computation of the mean and covariance functions is easy). Let $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Then:

$$\sum_{k=1}^n \lambda_k M^f_{t_k} = \int_0^\infty g(u)\,\mathrm{d}B_u$$

with $g(u) = \sum_{k=1}^n \lambda_k f(u)\mathbf{1}_{(0,t_k]}(u) \in L^2(\mathbb{R}_{\geq 0})$, and the right-hand side is Gaussian because it is a Wiener integral. □

Theorem 41. Let $f \in L^2_{\text{loc}}(\mathbb{R}_{\geq 0})$. Then, M^f is a continuous square-integrable martingale with:

$$\langle M^f\rangle_t = \int_0^t f(u)^2\,\mathrm{d}u$$

Proof. The integrability and square-integrability is clear because M^f is Gaussian. Note that $t \mapsto M^f_t$ is continuous when $f = \mathbf{1}_{(0,a]}$, because the Brownian motion is continuous. Now using Theorem 17 we get the result true for any $f \in L^2_{\text{loc}}(\mathbb{R}_{\geq 0})$. Now let's prove that M^f is a martingale. We have:

$$\begin{aligned} M^f_t &= \lim_{n\to\infty} \sum_{k=1}^{n^2} a_{n,k}(f\mathbf{1}_{(0,t]})(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n^2} a_{n,k}(f\mathbf{1}_{(0,t]})(B_{\frac{k+1}{n}\wedge t} - B_{\frac{k}{n}} \wedge t) \end{aligned}$$

and the last expression is $\mathcal{F}_t$-measurable. Finally, if $0 \leq s \leq t$ we have that since $M^f_t - M^f_s$ is independent of $\mathcal{F}_s$:

$$\mathbb{E}(M^f_t - M^f_s \mid \mathcal{F}_s) = \mathbb{E}(M^f_t - M^f_s) = 0$$

Moreover, $((M^f_t)^2)_{t\geq 0}$ is clearly adapted and:

$$\begin{aligned} \mathbb{E}\left((M^f_t)^2 - (M^f_s)^2 \mid \mathcal{F}_s\right) &= \mathbb{E}\left((M^f_t - M^f_s)^2 \mid \mathcal{F}_f\right) = \\ &= \mathbb{E}\left((M^f_t - M^f_s)^2\right) = \left\|I(f\mathbf{1}_{(s,t]})\right\|^2_{L^2(\Omega)} = \\ &= \left\|f\mathbf{1}_{(s,t]}\right\|^2_{L^2(\mathbb{R}_{\geq 0})} = \int_s^t f(u)^2\,\mathrm{d}u \end{aligned}$$

where the first equality is due to 15 Orthogonality of martingales and the we used the isometry property of I. This implies that $(M^f_t)^2{}_{t\geq 0} - \int_0^t f(u)^2\,\mathrm{d}u$ is a martingale and by the uniqueness of the quadratic variation we get the result. □

Proposition 42. Let $(B_t)_{t\geq 0}$ be a Brownian motion. For any $f \in L^2_{\text{loc}}(\mathbb{R}_{\geq 0})$, the process $Z^f = (Z^f_t)_{t\geq 0}$ defined as:

$$Z^f_t := \mathrm{e}^{\int_0^t f(u)\mathrm{d}B_u - \frac{1}{2}\int_0^t f(u)^2\mathrm{d}u}$$

is a continuous square-integrable martingale.

Proof. The integrability and adaptedness poses no problem. Now fix $0 \leq s \leq t$. We previously saw that $\int_s^t f(u)\,\mathrm{d}B_u$ is independent of $\mathcal{F}_s$ and so:

$$\mathbb{E}\left(Z^f_t \mid \mathcal{F}_s\right) = Z^f_s\mathbb{E}\left(\mathrm{e}^{\int_s^t f(u)\mathrm{d}B_u - \frac{1}{2}\int_s^t f(u)^2\mathrm{d}u}\right) = Z^f_s$$

because $\int_s^t f(u)\,\mathrm{d}B_u \sim N(0, \int_s^t f(u)^2\,\mathrm{d}u)$. □

Progressive processes

Definition 43. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and $\phi = (\phi_t)_{t\geq 0}$ a stochastic process. We say that ϕ is *progressive* if for fixed $t \geq 0$ the function

$$\begin{aligned} ([0,t]\times\Omega, \mathcal{B}([0,t])\otimes\mathcal{F}_t) &\longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\ (u,\omega) &\longmapsto \phi_u(\omega) \end{aligned}$$

is measurable.

Lemma 44. Let $\phi = (\phi_t)_{t\geq 0}$ be a stochastic process and

$$\mathcal{P} := \cap_{t\geq 0}\{A \subset \mathbb{R}_{\geq 0}\times\Omega : A \cap ([0,t]\times\Omega) \in \mathcal{B}([0,t])\otimes\mathcal{F}_t\}$$

Then, ϕ is progressive if and only if the map $(t,\omega) \mapsto \phi_t(\omega)$ is $\mathcal{P}$-measurable.

Proposition 45. The following stochastic processes $(\phi_t)_{t\geq 0}$ are progressive:

- A deterministic process $\phi_t(\omega) = f(t)$, $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$.
- $\phi_t(\omega) = X(\omega)\mathbf{1}_{(a,b]}(t)$ where $0 \leq a < b$ and X be $\mathcal{F}_a$-measurable.
- $\phi_t(\omega) = X(\omega)\mathbf{1}_{[0,T(\omega)]}(t)$ where T is a stopping time.
- $\phi_t(\omega) = F(\phi^1_t(\omega), \ldots, \phi^n_t(\omega))$ where $F : \mathbb{R}^n \to \mathbb{R}$ is measurable and $(\phi^i_t)_{1\leq i\leq n}$ are progressive.
- A pointwise limit of progressive processes.
- A continuous adapted process.

Itô isometry

Definition 46. We define the set $\mathbb{M}^2(\mathbb{R}_{\geq 0})$ as the set of all progressive processes $\phi = (\phi_t)_{t\geq 0}$ such that:

$$\mathbb{E}\left(\int_0^\infty {\phi_u}^2\,\mathrm{d}u\right) < \infty$$

Remark. Note that $\mathbb{M}^2(\mathbb{R}_{\geq 0}) = L^2(\mathbb{R}_{\geq 0}\times\Omega, \mathcal{P}, \mathrm{d}t\otimes\mathbb{P})$ is Hilbert with the scalar product:

$$\langle\phi,\psi\rangle_{\mathbb{M}^2} := \mathbb{E}\left(\int_0^\infty \phi_u\psi_u\,\mathrm{d}u\right)$$

Theorem 47 (Itô integral). Let $(B_t)_{t\geq 0}$ be a Brownian motion. Then, there exists a unique linear and continuous map $I : \mathbb{M}^2(\mathbb{R}_{\geq 0}) \to L^2((\Omega, \mathcal{F}, \mathbb{P}))$ such that $I(\phi) = X(B_t - B_s)$ whenever $\phi_u(\omega) = X(\omega)\mathbf{1}_{(s,t]}(u)$ for some $0 \leq s \leq t$ and $X \in L^2(\Omega, \mathcal{F}_s, \mathbb{P})$. Moreover, I is an isometry, i.e.:

$$\mathbb{E}\left(\int_0^\infty \phi_u \psi_u \, du\right) = \mathbb{E}\left(I(\phi)I(\psi)\right)$$

We call I the *Itô isometry* (or *Itô integral*) and we denote it by $I(\phi) = \int_0^\infty \phi_u \, dB_u$.

Proposition 48. Let $(\phi_u), (\psi_u) \in \mathbb{M}^2(\mathbb{R}_{\geq 0})$. Then, the following are satisfied:

- $$\int_0^\infty \phi_u \, dB_u \overset{L^2}{=} \lim_{n\to\infty} \sum_{k=1}^{n^2} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} \phi_u \, du \right) (B_{\frac{k+1}{n}} - B_{\frac{k}{n}})$$
- If $\phi_u(\omega) = f(t)$, $f \in L^2(\mathbb{R}_{\geq 0})$, then we recover the Wiener integral.
- $$\mathbb{E}\left(\int_0^\infty \phi_u \, dB_u\right) = 0$$
- $$\operatorname{Cov}\left(\int_0^\infty \phi_u \, dB_u , \int_0^\infty \psi_u \, dB_u\right) = \mathbb{E}\left(\int_0^\infty \phi_u \psi_u \, du\right)$$

The Itô integral as a process

Definition 49. Let (ϕ_u) be a progressive process and $0 \leq s \leq t$. We define:

$$\int_s^t \phi_u \, dB_u := \int_0^\infty \phi_u \mathbf{1}_{(s,t]}(u) \, dB_u$$

The set of such processes such that $\forall t \geq 0$, $\mathbb{E}\left(\int_0^t {\phi_u}^2 \, du\right) < \infty$ is denoted by $\mathbb{M}^2$. The set of such processes such that $\forall t \geq 0$, $\int_0^t {\phi_u}^2 \, du < \infty$ is denoted by $\mathbb{M}^2_{\text{loc}}$.

Remark. Note that $\mathbb{M}^2(\mathbb{R}_{\geq 0}) \subsetneq \mathbb{M}^2 \subsetneq \mathbb{M}^2_{\text{loc}}$.

Theorem 50. Let $(\phi_u) \in \mathbb{M}^2$. Then, the associate process $M^\phi = (M_t^\phi)_{t\geq 0}$ defined as:

$$M_t^\phi := \int_0^t \phi_u \, dB_u$$

is a continuous square-integrable martingale with:

$$\langle M^\phi \rangle_t = \int_0^t {\phi_u}^2 \, du$$

Remark. Note that the by **?? ??** we have that:

$$\langle M^\phi, M^\psi \rangle_t = \int_0^t \phi_u \psi_u \, du$$

Generalized Itô integral

Proposition 51. Let $(\phi_u) \in \mathbb{M}^2_{\text{loc}}$. Consider the stopping time

$$T_n := \inf\{t \geq 0 : \int_0^t {\phi_u}^2 \, du \geq n\}$$

and the truncated progressive process $\phi_t^n(\omega) := \phi_t(\omega)\mathbf{1}_{[0,T_n(\omega)]}(t)$. Then, $\phi^n \in \mathbb{M}^2(\mathbb{R}_{\geq 0})$.

Definition 52. Let $(\phi_u) \in \mathbb{M}^2_{\text{loc}}$. We define the *generalized Itô integral* of ϕ as:

$$\int_0^\infty \phi_u \, dB_u := \lim_{n\to\infty} \int_0^\infty \phi_u \mathbf{1}_{[0,T_n]}(u) \, dB_u$$

which is well-defined.

Theorem 53. Let $(\phi_u) \in \mathbb{M}^2_{\text{loc}}$. Then, the associate process $M^\phi = (M_t^\phi)_{t\geq 0}$ defined as:

$$M_t^\phi := \int_0^t \phi_u \, dB_u$$

is a continuous local martingale with:

$$\langle M^\phi \rangle_t = \int_0^t {\phi_u}^2 \, du$$

Theorem 54 (Stochastic dominated convergence theorem). Let $t \geq 0$ and $(\phi_u^n) \in \mathbb{M}^2_{\text{loc}}$ be a sequence of progressive processes such that $\phi_u^n \xrightarrow[n\to\infty]{\mathbb{P}} \phi_u$ for all a.e. $u \in [0, t]$. Suppose that $\forall u \in [0, t]$ and $\forall n \in \mathbb{N}$ we have $|\phi_u^n| \overset{\text{a.e.}}{\leq} \Psi_u$, with $\Psi \in \mathbb{M}^2_{\text{loc}}$. Then:

$$\int_0^t \phi_u^n \, dB_u \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t \phi_u \, dB_u$$

Corollary 55. If (ϕ_u) is a continuous and adapted process, then $\forall t \geq 0$ and any subdivision $(t_k^n) \in \mathrm{P}([0, t])$ such that $\Delta_n \to 0$ we have:

$$\sum_{k=0}^{n-1} \phi_{t_{k+1}^n}(B_{t_{k+1}^n} - B_{t_k^n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t \phi_u \, dB_u$$

3. | Stochastic differentiation

Itô processes

Proposition 56. Let $\psi = (\psi_t)_{t\geq 0}$ be a stochastic process such that $\forall t \geq 0$ we have

$$\int_0^t |\psi_u| \, du < \infty$$

In this case we say that $\psi \in \mathbb{M}^1_{\text{loc}}$. Then, the process

$$t \mapsto \int_0^t \psi_u \, dB_u$$

is an adapted continuous process.

Definition 57. An *Itô process* is a stochastic process $(X_t)_{t\geq 0}$ of the form:

$$X_t = X_0 + \int_0^t \phi_u \, \mathrm{d}B_u + \int_0^t \psi_u \, \mathrm{d}u \tag{1}$$

with $\phi \in \mathbb{M}^2_{\mathrm{loc}}$ and $\psi \in \mathbb{M}^1_{\mathrm{loc}}$. The two integrals are called *martingale term* and *drift term* respectively. Instead of Eq. (1) we usually write:

$$\mathrm{d}X_t = \phi_t \, \mathrm{d}B_t + \psi_t \, \mathrm{d}t$$

This expression is called *stochastic differential*.

Remark. Itô processes form a vector space. That is, if X and Y are Itô processes and $\lambda, \mu \in \mathbb{R}$, then $Z = \lambda X + \mu Y$ is an Itô process and:

$$\mathrm{d}Z_t = \lambda \, \mathrm{d}X_t + \mu \, \mathrm{d}Y_t$$

Moreover they are always continuous adapted processes.

Proposition 58. Let $X = (X_t)_{t\geq 0}$ be an Itô process such that $\forall t \geq 0$ we have:

$$\mathrm{d}X_t = \phi_t \, \mathrm{d}B_t + \psi_t \, \mathrm{d}t = \tilde{\phi}_t \, \mathrm{d}B_t + \tilde{\psi}_t \, \mathrm{d}t$$

for some $\phi, \tilde{\phi} \in \mathbb{M}^2_{\mathrm{loc}}$ and $\psi, \tilde{\psi} \in \mathbb{M}^1_{\mathrm{loc}}$. Then, ϕ, $\tilde{\phi}$ are indistinguishable and so are ψ, $\tilde{\psi}$.

Proof. By assumption, we have that a.e. $\forall t \geq 0$:

$$\int_0^t (\phi_u - \tilde{\phi}_u) \, \mathrm{d}B_u = \int_0^t (\psi_u - \tilde{\psi}_u) \, \mathrm{d}u$$

But since the right-hand side of the equation is a local martingale and the left-hand side has finite variation, we have that both sides must be 0 a.e. in t. Moreover, by the uniqueness of the quadratic variation we have that:

$$\int_0^t (\phi_u - \tilde{\phi}_u)^2 \, \mathrm{d}u = 0$$

Letting $t \to \infty$ we get that ϕ, $\tilde{\phi}$ are indistinguishable. Finally, from the Lebesgue integral, we have that ψ, $\tilde{\psi}$ are indistinguishable. □

Proposition 59. Let $X = (X_t)_{t\geq 0}$ be an Itô process such that $\mathrm{d}X_t = \phi_t \, \mathrm{d}B_t + \psi_t \, \mathrm{d}t$. Then:

- X is a local martingale if and only if $X_0 \in L^1$ and $\psi = 0$.
- X is a square-integrable martingale if and only if $X_0 \in L^2$, $\phi \in \mathbb{M}^2$ and $\psi = 0$.

Definition 60. Let $X = (X_t)_{t\geq 0}$ be an Itô process such that $\mathrm{d}X_t = \phi_t \, \mathrm{d}B_t + \psi_t \, \mathrm{d}t$, and $Y = (Y_t)_{t\geq 0}$ be a continuous adapted process. Then, $Y\phi \in \mathbb{M}^2_{\mathrm{loc}}$ and $Y\psi \in \mathbb{M}^1_{\mathrm{loc}}$ and we define:

$$\int_0^t Y_u \, \mathrm{d}X_u := \int_0^t Y_u \phi_u \, \mathrm{d}B_u + \int_0^t Y_u \psi_u \, \mathrm{d}u$$

Remark. Note that using ?? ?? we also have:

$$\int_0^t Y_u \, \mathrm{d}X_u \overset{\mathbb{P}}{=} \lim_{n\to\infty} \sum_{k=0}^{n-1} Y_{t_k^n} (X_{t_{k+1}^n} - X_{t_k^n})$$

along any subdivision $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0,t])$ such that $\Delta_n \to 0$.

Quadratic variation of Itô processes

Lemma 61. Let $X = (X_t)_{t\geq 0}$, $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ be two Itô processes with differentials:

$$\mathrm{d}X_t = \phi_t \, \mathrm{d}B_t + \psi_t \, \mathrm{d}t \qquad \mathrm{d}\tilde{X}_t = \tilde{\phi}_t \, \mathrm{d}B_t + \tilde{\psi}_t \, \mathrm{d}t$$

Then, for any $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0,t])$ such that $\Delta_n \to 0$ we have:

$$\sum_{k=0}^{n-1} (X_{t_{k+1}^n} - X_{t_k^n})(\tilde{X}_{t_{k+1}^n} - \tilde{X}_{t_k^n}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t \phi_u \tilde{\phi}_u \, \mathrm{d}u =: \\ =: \langle X, \tilde{X} \rangle_t$$

In particular:

$$\langle X \rangle_t := \langle X, X \rangle_t = \int_0^t {\phi_u}^2 \, \mathrm{d}u$$

and we call it the *quadratic variation* of X.

Proof. We saw it for $X = \tilde{X}$, and the general formula follows from ?? ??. Now, if $\psi = 0$, X is a continuous local martingale with quadratic variation $t \mapsto \int_0^t {\phi_u}^2 \, \mathrm{d}u$. Now if $\phi = 0$, we know it because $t \mapsto \int_0^t \psi_u \, \mathrm{d}u$ has finite variation, and therefore, null quadratic variation. Finally in the general case we have:

$$\sum_{k=0}^{n-1} (X_{t_{k+1}^n} - X_{t_k^n})^2 = \sum_{k=0}^{n-1} \left(\int_{t_k^n}^{t_{k+1}^n} \phi_u \, \mathrm{d}B_u \right)^2 + \\ + \sum_{k=0}^{n-1} \left(\int_{t_k^n}^{t_{k+1}^n} \psi_u \, \mathrm{d}B_u \right)^2 + 2 \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \phi_u \, \mathrm{d}B_u \int_{t_k^n}^{t_{k+1}^n} \psi_u \, \mathrm{d}u$$

The first part tends to $\int_0^t {\phi_u}^2 \, \mathrm{d}u$, the second part tends to 0 and for the last part use Theorem 23. □

Theorem 62 (Stochastic integration by parts). Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two Itô processes. Then, $(X_t Y_t)_{t\geq 0}$ is an Itô process and:

$$\mathrm{d}(X_t Y_t) = X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + \mathrm{d}\langle X, Y \rangle_t$$

The last term $\mathrm{d}\langle X, Y \rangle_t$ is called *Itô term*.

Proof. Let $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0,t])$ such that $\Delta_n \to 0$. Then:

$$X_t Y_t - X_0 Y_0 = \sum_{k=0}^{n-1} (X_{t_{k+1}^n} Y_{t_{k+1}^n} - X_{t_k^n} Y_{t_k^n}) =$$

$$= \sum_{k=0}^{n-1} (X_{t_{k+1}^n} - X_{t_k^n}) Y_{t_{k+1}^n} + \sum_{k=0}^{n-1} X_{t_k^n}(Y_{t_{k+1}^n} - Y_{t_k^n}) + \\ + \sum_{k=0}^{n-1} (X_{t_{k+1}^n} - X_{t_k^n})(Y_{t_{k+1}^n} - Y_{t_k^n})$$

Letting $n \to \infty$ and using Theorem 61 and a previous remark we get the result. □

Corollary 63. Let $X = (X_t)_{t\geq 0}$ be an Itô process. Then, $(X_t^2)_{t\geq 0}$ is an Itô process and:

$$\mathrm{d}X_t^2 = 2X_t \,\mathrm{d}X_t + \mathrm{d}\langle X\rangle_t$$

Itô's formula

Theorem 64 (Itô's formula). Let $X = (X_t)_{t\geq 0}$ be an Itô process and $f \in C^2(\mathbb{R})$. Then, $(f(X_t))_{t\geq 0}$ is an Itô process and:

$$\mathrm{d}f(X_t) = f'(X_t)\,\mathrm{d}X_t + \frac{1}{2} f''(X_t)\,\mathrm{d}\langle X\rangle_t$$

Proof. Let $t \geq 0$ and $(t_k^n)_{0\leq k\leq n} \in \mathrm{P}([0,t])$ such that $\Delta_n \to 0$. Then, using the Taylor expansion of f we have:

$$f(X_t) - f(X_0) = \sum_{k=0}^{n-1} f(X_{t_{k+1}^n}) - f(X_{t_k^n}) = \\ = \sum_{k=0}^{n-1} f'(X_{t_k^n})(X_{t_{k+1}^n} - X_{t_k^n}) + \\ + \frac{1}{2}\sum_{k=0}^{n-1} f''(X_{u_k^n})(X_{t_{k+1}^n} - X_{t_k^n})^2$$

with $u_k^n \in [t_k^n, t_{k+1}^n]$. By a previous remark we have that:

$$\sum_{k=0}^{n-1} f'(X_{t_k^n})(X_{t_{k+1}^n} - X_{t_k^n}) \xrightarrow{\mathbb{P}} \int_0^t f'(X_u)\,\mathrm{d}X_u$$

Now, by Theorem 61 we have that:

$$\sum_{k=0}^{n-1} Y_{t_k^n}(X_{t_{k+1}^n} - X_{t_k^n})^2 \xrightarrow{\mathbb{P}} \int_0^t Y_u\,\mathrm{d}\langle X\rangle_u$$

in the elementary case where $Y_u = \mathbf{1}_{(0,s]}(u)$, $s \geq 0$. By linearity, this immediately extends to the case where Y is a random step function. By density, it further extends to the case where Y is any continuous and adapted process. In particular, we may take $Y_u = f''(X_u)$, and the formula holds by uniform continuity:

$$\max_{0\leq k\leq n} \left| f''(X_{t_k^n}) - f''(X_{u_k^n}) \right| \xrightarrow{\text{a.e.}} 0$$

□

Theorem 65. Let $X^1, \ldots, X^d$ be Itô processes and $f \in C^2(\mathbb{R}^d)$. Then, $(f(X_t^1,\ldots,X_t^d))_{t\geq 0}$ is an Itô process and:

$$\mathrm{d}f(\mathbf{X}) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\mathbf{X})\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{X})\,\mathrm{d}\langle X^i, X^j\rangle_t$$

where $\mathbf{X} := (X^1, \ldots, X^d)$.

Exponential martingales

Lemma 66 (Doléans-Dade exponential). For any $\phi \in \mathbb{M}^2_{\text{loc}}$, the process $Z^\phi = (Z_t^\phi)_{t\geq 0}$ defined as

$$Z_t^\phi := \mathrm{e}^{\int_0^t \phi_u \mathrm{d}B_u - \frac{1}{2}\int_0^t {\phi_u}^2 \mathrm{d}u}$$

is a continuous local martingale.

Proof. Applying Itô's formula to $f(x) = \mathrm{e}^x$ and $X_t = \int_0^t \phi_u \,\mathrm{d}B_u - \frac{1}{2}\int_0^t {\phi_u}^2 \,\mathrm{d}u$ we get:

$$\mathrm{d}Z_t^\phi = \mathrm{e}^{X_t}\left(\phi_t\,\mathrm{d}B_t - \frac{1}{2}{\phi_t}^2\,\mathrm{d}t\right) + \frac{1}{2}\mathrm{e}^{X_t}{\phi_t}^2\,\mathrm{d}t = \mathrm{e}^{X_t}\phi_t\,\mathrm{d}B_t$$

Since, $Z_0^\phi = 1$, we obtain that $\forall t \geq 0$:

$$Z_t^\phi = 1 + \int_0^t Z_u^\phi \phi_u\,\mathrm{d}B_u$$

and the result follows from Theorem 59. □

Lemma 67. If M is a non-negative local martingale, then M is a super-martingale. Moreover, for $T \in \mathbb{R}_{\geq 0}$, $(M_t)_{t\in[0,T]}$ is a martingale if and only if $\mathbb{E}(M_T) \geq \mathbb{E}(M_0)$.

Proof. Since M is a local martingale with localizing sequence (T_n), then $\forall t \geq 0$ we have that $\mathbb{E}(M_{t\wedge T_n} \mid \mathcal{F}_s) = M_{s\wedge T_n}$ and so:

$$\begin{cases} M_{s\wedge T_n} \xrightarrow[n\to\infty]{\text{a.s.}} M_s \\ M_{t\wedge T_n} \xrightarrow[n\to\infty]{\text{a.s.}} M_t \end{cases}$$

Now, by **?? ??** we have:

$$\mathbb{E}(M_t \mid \mathcal{F}_s) \leq \liminf_{n\to\infty} \mathbb{E}(M_{t\wedge T_n} \mid \mathcal{F}_s) = \liminf_{n\to\infty} M_{s\wedge T_n} = M_s$$

which shows that M is a super-martingale. Now, fix $T \geq 0$, and suppose that $\mathbb{E}(M_T) \geq \mathbb{E}(M_0)$. This forces the non-increasing map $t \mapsto \mathbb{E}(M_t)$ to be constant on $[0,T]$. In particular, for any $0 \leq s \leq t \leq T$, the non-negative variable $M_s - \mathbb{E}(M_t \mid \mathcal{F}_s)$ has zero mean, hence is null a.s. □

Theorem 68 (Novikov's condition). Let $t \geq 0$ be fixed and assume that:

$$\mathbb{E}\left(\mathrm{e}^{\frac{1}{2}\int_0^t {\phi_u}^2 \mathrm{d}u}\right) < \infty$$

Then, $(Z_s^\phi)_{s\in[0,t]}$ is a martingale.

Proof. Fix $0 < \varepsilon < 1$. We have that for all $s \in [0,t]$:

$$\left(Z_s^{(1-\varepsilon)\phi}\right)^{\frac{1}{1-\varepsilon^2}} = \left(Z_s^\phi\right)^{\frac{1}{1+\varepsilon}} \left(\mathrm{e}^{\frac{1}{2}\int_0^s \phi_u^2 \mathrm{d}u}\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

Now, choosing $s = t\wedge T_n$, where T_n is a localizing sequence for $Z^{(1-\varepsilon)\phi}$, taking expectation and using **?? ??** we get:

$$\mathbb{E}\left[\left(Z_{t\wedge T_n}^{(1-\varepsilon)\phi}\right)^{\frac{1}{1-\varepsilon^2}}\right] \leq \mathbb{E}\left[Z_{t\wedge T_n}^\phi\right]^{\frac{1}{1+\varepsilon}} \mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^{t\wedge T_n} \phi_u^2 \mathrm{d}u}\right]^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\leq \mathbb{E}\left[e^{\frac{1}{2}\int_0^t \phi_u^2 \, \mathrm{d}u}\right]^{\frac{\varepsilon}{1+\varepsilon}}$$

because $\mathbb{E}\left[Z_{t\wedge T_n}^{\phi}\right] = 1$. This implies that $(Z_{t\wedge T_n}^{(1-\varepsilon)\phi})$ is bounded in L^p for $p = \frac{1}{\varepsilon^2} > 1$. Thus:

$$\mathbb{E}(Z_t^{(1-\varepsilon)\phi}) = \lim_{n\to\infty} \mathbb{E}(Z_{t\wedge T_n}^{(1-\varepsilon)\phi}) = 1$$

In particular, $\mathbb{E}\left[\left(Z_t^{(1-\varepsilon)\phi}\right)^p\right] \geq 1$ and so:

$$1 \leq \mathbb{E}\left[Z_t^{\phi}\right]^{\frac{1}{1+\varepsilon}} \mathbb{E}\left[e^{\frac{1}{2}\int_0^t \phi_u^2 \, \mathrm{d}u}\right]^{\frac{\varepsilon}{1+\varepsilon}}$$

Taking $\varepsilon \to 0$, yields $\mathbb{E}(Z_t^{\phi}) \geq 1$, which suffices to conclude by Theorem 67. □

Girsanov's theorem

Theorem 69 (Giranov's theorem). Let $\phi \in \mathbb{M}^2_{\mathrm{loc}}$ and suppose its associated exponential local martingale $(Z_t^{\phi})_{t\geq 0}$ is a martingale. Then, the formula

$$\mathbb{Q}(A) := \mathbb{E}(Z_t^{\phi} \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$

defines a probability measure on $(\Omega, \mathcal{F}_t)$, under which the process $X = (X_s)_{s\in[0,t]}$ defined as

$$X_s := B_s - \int_0^s \phi_u \, \mathrm{d}u$$

is a $(\mathcal{F}_s)_{s\in[0,t]}$-Brownian motion.

Remark. Note that, by linearity we have that for ant $\mathcal{F}_t$-measurable non-negative random variable Y:

$$\mathbb{E}_{\mathbb{Q}}(Y) = \mathbb{E}(Y Z_t^{\phi}) \qquad \mathbb{E}(Y) = \mathbb{E}_{\mathbb{Q}}\left(\frac{Y}{Z_t^{\phi}}\right)$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation with respect to $\mathbb{Q}$. This is useful for transferring computations between $\mathbb{P}$ and $\mathbb{Q}$.

4. | Stochastic differential equations

Introduction

Definition 70. Let $X = (X_t)_{t\geq 0}$ be a stochastic process and $b : \mathbb{R}_{\geq 0}\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}_{\geq 0}\times\mathbb{R}$ be deterministic functions called *drift* and *diffusion*, respectively. A *stochastic differential equation* (*SDE*) is an equation of the form:

$$\mathrm{d}X_t = b(t, X_t)\, \mathrm{d}t + \sigma(t, X_t)\, \mathrm{d}B_t \tag{2}$$

Definition 71. Consider the SDE of Eq. (2). We say that a progressive process $X = (X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a *solution of the SDE* if $(b(t, X_t))_{t\geq 0} \in \mathbb{M}^1_{\mathrm{loc}}$ and $(\sigma(t, X_t))_{t\geq 0} \in \mathbb{M}^2_{\mathrm{loc}}$ and $\forall t \geq 0$:

$$X_t = X_0 + \int_0^t b(s, X_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s)\, \mathrm{d}B_s$$

Existence and uniqueness of solutions

Lemma 72 (Grönwall's lemma). Let $(x_t)_{t\in[0,T]}$ be a non-negative function in $L^1([0,T])$ satisfying that $\forall t \in [0,T]$:

$$x_t \leq \alpha + \beta \int_0^t x_s \, \mathrm{d}s$$

for some constants $\alpha, \beta \geq 0$. Then, $x_t \leq \alpha e^{\beta t}$ for all $t \in [0,T]$.

Theorem 73 (Existence and uniqueness of solutions of SDEs). Let $b, \sigma : \mathbb{R}_{\geq 0}\times\mathbb{R} \to \mathbb{R}$ be a measurable function satisfying:

- Uniform spatial Lipschitz continuity: $\exists C > 0$ such that $\forall t \geq 0$ and $\forall x, y \in \mathbb{R}$ we have:
$$|b(t,x) - b(t,y)| \leq C|x-y|$$
$$|\sigma(t,x) - \sigma(t,y)| \leq C|x-y|$$
- Local square-integrability in time: $\forall t \geq 0$ we have:
$$\int_0^t |b(s,0)|^2 \, \mathrm{d}s < \infty \qquad \int_0^t |\sigma(s,0)|^2 \, \mathrm{d}s < \infty$$

Then, for each initial condition $\zeta \in L^{(}\Omega, \mathcal{F}_0, \mathbb{P})$, there exists a unique (up to indistinguishability) solution $X = (X_t)_{t\geq 0}$ to the SDE of Eq. (2) with $X_0 = \zeta$. Moreover, $X \in \mathbb{M}^2$.

Remark. In the proof of the above theorem, which we omit here, it can be shown that

$$X_t = \Psi_t\left(\zeta, (B_s)_{s\in[0,t]}\right) \tag{3}$$

for some measurable function $\Psi_t : \mathbb{R} \times C([0,t], \mathbb{R}) \to \mathbb{R}$.

Practical examples

Proposition 74 (Langevin equation). Consider the following SDE:

$$\mathrm{d}X_t = -bX_t \, \mathrm{d}t + \sigma \, \mathrm{d}B_t$$

with $b, \sigma > 0$ and $X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$. Then, the solution is given by:

$$X_t = \zeta e^{-bt} + \sigma \int_0^t e^{-b(t-s)} \, \mathrm{d}B_s$$

Remark. This SDE was proposed by Paul Langevin in 1908 to describe the random motion of a small particle in a fluid, due to collisions with the surrounding molecules. Note that the long-term behavior of X_t has law of $N(0, \frac{\sigma^2}{2b})$ (because the second term has law $N(0, \frac{\sigma^2}{2b}(1 - e^{-2bt}))$), independently of the initial condition ζ.

Proposition 75 (Geometric Brownian motion). Consider the following SDE:

$$\mathrm{d}X_t = X_t(b \, \mathrm{d}t + \sigma \, \mathrm{d}B_t)$$

with $X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$. Then, the solution is given by:

$$X_t = \zeta e^{\left(b - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

Proof. This equation has a unique solution and it's natural to expect it is of the form $X_t = \zeta e^{Y_t}$, where Y_t is an Itô process. Identifying $dY_t = \psi_t\,dt + \phi_t\,dB_t$ and using the 64 Itô's formula we get:

$$\begin{aligned} d(\zeta e^{Y_t}) &= \zeta e^{Y_t}\left(dY_t + \frac{1}{2}\,d\langle Y\rangle_t\right) \\ &= \zeta e^{Y_t}\left(\psi_t\,dt + \phi_t\,dB_t + \frac{1}{2}\phi_t^2\,dt\right) \end{aligned}$$

and so $\phi_t = \sigma$ and $\psi_t = b - \frac{\sigma^2}{2}$. □

Proposition 76 (Black-Scholes process). Consider the following SDE:

$$dX_t = X_t(b_t\,dt + \sigma_t\,dB_t)$$

with $X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ and $(b_t)_{t\geq 0}$, $(b_t)_{t\geq 0}$ deterministic measurable bounded functions. Then, the solution is given by:

$$X_t = \zeta e^{\int_0^t \left(b_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s\,dB_s}$$

Proof. This equation has a unique solution and as in the previous example, we expect $X_t = \zeta e^{Y_t}$, where Y_t is an Itô process. Identifying $dY_t = \psi_t\,dt + \phi_t\,dB_t$ and using the 64 Itô's formula we get:

$$d(\zeta e^{Y_t}) = \zeta e^{Y_t}\left(\psi_t\,dt + \phi_t\,dB_t + \frac{1}{2}\phi_t^2\,dt\right)$$

and so $\phi_t = \sigma_t$ and $\psi_t = b_t - \frac{\sigma_t^2}{2}$. □

Markov property for diffusions

Definition 77. Let $b.\sigma : \mathbb{R} \to \mathbb{R}$ be two Lipschitz functions and consider the following *homogeneous SDE*:

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \\ X_0 = \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}) \end{cases} \tag{4}$$

These kinds of problems are called *diffusions*.

Theorem 78 (Invariance under time shift). Let $X = (X_t)_{t\geq 0}$ be a solution to the SDE of Eq. (4) and assume we write X_t as in Eq. (3). Then, for any $s, t \geq 0$ we have:

$$X_{t+s} = \Psi_t(X_s, (B_{u+s} - B_s)_{u\in[0,t]})$$

Definition 79. Let $f \in L^\infty(\mathbb{R})$ and $t \geq 0$. We define the function $P_t f$ as:

$$\begin{aligned} P_t f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \mathbb{E}(f(X_t^x)) \end{aligned}$$

where X_t^x is the solution to the SDE of Eq. (4) with $X_0 = x$.

Corollary 80. For any $s, t \geq 0$ and any $f \in L^\infty(\mathbb{R})$ we have:

$$\mathbb{E}(f(X_{t+s})) = (P_t f)(X_s)$$

Proposition 81. The family $(P_t)_{t\geq 0}$ has the following properties:

1. P_t is a bounded linear operator from $L^\infty(\mathbb{R})$ to itself for each $t \geq 0$.
2. $P_0 = \mathrm{id}$ and $P_{t+s} = P_t \circ P_s$ for all $s, t \geq 0$.
3. If f is continuous, then so is $t \mapsto P_t f(x)$ for each fixed $x \in \mathbb{R}$.
4. If f is monotone, then so is $P_t f$ for each $t \geq 0$.
5. If f is Lipschitz, then so is $P_t f$ for each $t \geq 0$.
6. If $\sigma, b, f \in \mathcal{C}_b^k(\mathbb{R})$ for some $k \geq 1$, then so is $P_t f$ for each $t \geq 0$.

Generator of a diffusion

Definition 82 (Generator). The *generator* of the semigroup $(P_t)_{t\geq 0}$ is the linear operator L defined by:

$$(Lf)(x) := \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}$$

for all $f \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$ such that the limit exists. Those functions form a vector space denoted by $\mathrm{Dom}(L)$.

Theorem 83. Let $f \in \mathcal{C}_b^2(\mathbb{R})$. Then:

1. Lf is well-defined and it is given $\forall x \in \mathbb{R}$ by:

$$Lf(x) = \frac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x)$$

2. For all $t \geq 0$, we have $P_t f \in \mathrm{Dom}(L)$ and it satisfies the *Kolmogorov's equation*:

$$\frac{d}{dt} P_t f = P_t(Lf) = L(P_t f)$$

3. The process $(M_t)_{t\geq 0}$ defined as

$$M_t := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$$

is a continuous square-integrable martingale.

Connection with PDEs

For this section recall the diffusion equation:

$$\begin{cases} dX_t^x = b(X_t^x)\,dt + \sigma(X_t^x)\,dB_t \\ X_0^x = x \end{cases} \tag{5}$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions. Now, fix $f \in L^\infty(\mathbb{R})$ and consider the PDE:

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = b(x)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2}(t,x) \\ v(0,x) = f(x) \end{cases} \tag{6}$$

where $v \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R})$.

Theorem 84.

1. If v is a bounded solution to the PDE of Eq. (6), then we must have $\forall (t,x) \in [0,\infty) \times \mathbb{R}$:

$$v(t,x) = \mathbb{E}(f(X_t^x)) \tag{7}$$

2. If $b, \sigma, f \in \mathcal{C}_{\mathrm{b}}^2(\mathbb{R})$, then conversely, the function v defined in Eq. (7) is a bounded solution of Eq. (6).

Remark. The interest of this connection between SDEs and PDEs is two-fold: on the one hand, one can use tools from PDE theory to understand the distribution of X_t^x. Conversely, the probabilistic representation of Eq. (7) offers a practical way to numerically solve the PDE of Eq. (6), by simulation.

Theorem 85 (Feynman-Kac's formula). Let $v \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R})$ be a bounded solution to the PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -h(x)v(t,x) + b(x)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2}(t,x) \\ v(0,x) = f(x) \end{cases}$$

where $f, h : \mathbb{R} \to \mathbb{R}$ are measurable, with h non-negative. Then, we have the representation

$$v(t,x) = \mathbb{E}\left(f(X_t^x) \mathrm{e}^{-\int_0^t h(X_s^x)\mathrm{d}s} \right)$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$.