Introduction to nonlinear elliptic PDEs

1. | Introduction

Definition 1. Let a_{ij}, b_j, c, f be known scalar functions defined on $\Omega \subseteq \mathbb{R}^d$. Usually we will denote $\mathbf{A} = (a_{ij})$ and $\mathbf{b} = (b_j)$. A *linear second-order PDE* is an equation of the form:

$$-\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\partial_{ij}^2 u(\mathbf{x}) + \sum_{j=1}^{d} b_j(\mathbf{x})\partial_j u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})$$

where $u : \Omega \to \mathbb{R}$ is the unknown function. This form is called *non-divergence form*. If we write the equation in the form:

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} a_{ij}(\mathbf{x})\partial_j u(\mathbf{x})\right) + \sum_{j=1}^{d} b_j(\mathbf{x})\partial_j u(\mathbf{x}) + \\ + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x})$$

then we say that the equation is in *divergence form*. Together with the PDE we usually impose boundary conditions on $\partial\Omega$. The *Dirichlet boundary condition* is:

$$u|_{\partial\Omega} = g$$

and it is called *homogeneous* if $g = 0$. The *Neumann boundary condition* is:

$$\langle \mathbf{n}, \mathbf{A}\boldsymbol{\nabla} u\rangle|_{\partial\Omega} = g$$

where we have assumed that the boundary of Ω is smooth enough to define the normal vector $\mathbf{n}$. The condition is called *homogeneous* if $g = 0$. Note that if $\mathbf{A} = \mathbf{I}_d$, then the Neumann boundary condition is just $\partial_{\mathbf{n}} u = g$.

Remark. If the coefficients $a_{ij} \in \mathcal{C}^1$, then we are able to convert the equation from non-divergence form to divergence form and vice versa.

Definition 2. Let a_{ij}, b_j, c be known functions on $\Omega \subseteq \mathbb{R}^d$. We say that the operator

$$L = -\sum_{i,j=1}^{d} a_{ij}\partial_{ij}^2 + \sum_{j=1}^{d} b_j\partial_j + c \tag{1}$$

is *uniformly elliptic* if there exists $\theta > 0$ such that for all $x \in \Omega$ and all $\mathbf{p} \in \mathbb{R}^d$ we have:

$$Q_x(\mathbf{p}) := \mathbf{p}^{\mathrm{T}}\mathbf{A}(\mathbf{x})\mathbf{p} = \sum_{i,j=1}^{d} a_{ij}(\mathbf{x})p_i p_j \geq \theta \sum_{i=1}^{d} {p_i}^2 = \theta \left\|\mathbf{p}\right\|^2$$

Remark. Geometrically speaking, this implies that the sets

$$\xi_{x,h} = \{\mathbf{p} \in \mathbb{R}^d : Q_x(\mathbf{p}) = h\}$$

are ellipsoids.

Definition 3. Consider the problem

$$\mathcal{D}_f := \begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where L is as in Eq. (1). The *weak formulation* (or *variational formulation*) of the problem is:

$$\langle \boldsymbol{\nabla} u, \boldsymbol{\nabla} v\rangle_2 + \langle \mathbf{b}\cdot\boldsymbol{\nabla} u, v\rangle_2 + \langle cu, v\rangle_2 = \langle f, v\rangle_2 \quad \forall v \in H_0^1(\Omega)$$

A solution of such problem is called a *weak solution* of $\mathcal{D}_f$.

Definition 4. If the weak solution u_f of the problem $\mathcal{D}_f$ is in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for some $p \in [1, \infty)$, then u_f is called a *strong solution* of $\mathcal{D}_f$. If $u_f \in \mathcal{C}^2(\Omega) \cap H_0^1(\Omega)$, then we say that u_f is a *classical solution* of $\mathcal{D}_f$.

Proposition 5. Let H be Hilbert and $K : H \to H$ be a continuous linear operator. Then, the following are equivalent:

1. K is compact.
2. For any bounded sequence $(u_n) \in H$, the sequence (Ku_n) has a convergent subsequence.
3. For any sequence $(u_n) \in H$ such that $u_n \rightharpoonup u$, we have $Ku_n \to Ku$.

2. | Hilbert space methods for divergence form linear PDEs

In this section, we will assume that $\Omega \subset \mathbb{R}^d$ is an open, bounded subset, $a_{ij} = a_{ji}$ and $a_{ij}, b_j, c \in L^\infty(\Omega)$.

Lax-Milgram theorem

Remark. Instead of the usual norm for $H_0^1(\Omega)$, here we will use the following one:

$$\|u\|_{H_0^1(\Omega)}^2 = \|\boldsymbol{\nabla} u\|_{L^2(\Omega)}^2$$

Definition 6. Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bilinear map. We say that a is *continuous* if $\exists C > 0$ such that $\forall u, v \in H$ we have:

$$|a(u, v)| \leq C \|u\| \|v\|$$

Definition 7. Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bilinear map. We say that a is *coercive* if $\exists \alpha > 0$ such that $\forall u \in H$ we have:

$$a(u, u) \geq \alpha \|u\|^2$$

Definition 8. Let H be a Hilbert space and $a : H \times H \to \mathbb{C}$ be a bilinear map. We say that a is *symmetric* if $\forall u, v \in H$ we have:

$$a(u, v) = \overline{a(v, u)}$$

Theorem 9 (Lax-Milgram theorem). Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear map. Then, $\forall f \in H^* \; \exists! u_f \in H$ such that:

$$a(u_f, v) = f(v) \quad \forall v \in H$$

In addition, if H is a real Hilbert space and a is symmetric, then u is the unique minimizer of:

$$\min_{v\in H} \left\{ \frac{1}{2} a(v,v) - f(v) \right\}$$

Proposition 10. Consider the problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $L = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j)$ and $f \in L^2(\Omega)$. Then, the problem has a unique weak solution $u \in H_0^1(\Omega)$ and

$$\|u\|_{H_0^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

Proof. Consider the bilinear form

$$a(u,v) := \int_\Omega \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j v$$

We check the hypotheses of 9 Lax-Milgram theorem:

1. a is continuous:

$$\begin{aligned} |a(u,v)| &\leq \sum_{i,j=1}^d \|a_{ij}\|_\infty \|\nabla u\|_2 \|\nabla v\|_2 \\ &\leq C \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \end{aligned}$$

2. a is coercive:

$$\begin{aligned} a(u,u) &= \int_\Omega \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j u \\ &\geq \theta \int_\Omega \sum_{i=1}^d |\partial_i u|^2 \\ &= \theta \|u\|_{H_0^1(\Omega)}{}^2 \end{aligned}$$

   by the uniform ellipticity of L.

Moreover, since $a(u,u) = \langle f, u\rangle_2$ we have that:

$$\theta \|u\|_{H_0^1(\Omega)}{}^2 \leq \langle f, u\rangle_2 \leq \|f\|_2 \|u\|_2 \leq C \|f\|_2 \|u\|_{H_0^1(\Omega)}$$

by the ?? ??. □

Abstract Fredholm alternative

Remark. One can check that if we try to apply 9 Lax-Milgram theorem to the problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $L = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{j=1}^d b_j \partial_j$, it fails due to the coercivity condition.

Proposition 11. Consider the problem:

$$\mathcal{D}_{\mu,f} := \begin{cases} L_\mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $L_\mu = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{j=1}^d b_j \partial_j + \mu$. Then, if $\mu > 0$ is large enough, the problem has a unique weak solution in $H_0^1(\Omega)$

Sketch of the proof. Taking the natural bilinear map a, the coercivity condition becomes:

$$a_\mu(u,u) \geq \theta \|u\|_{H_0^1(\Omega)}^2 - C \|u\|_{H_0^1(\Omega)} \|u\|_2 + \mu \|u\|_2^2$$

which for μ large enough it is bigger than $\delta \|u\|_{H_0^1(\Omega)}^2$ for some $\delta > 0$. □

Lemma 12. Let H be Hilbert and $K : H \to H$ be a compact linear operator. Then, $\dim\ker(\mathrm{id} - K) < \infty$.

Proof. If $\dim\ker(\mathrm{id} - K) = \infty$, then $\exists (u_n) \in \ker(\mathrm{id} - K)$ orthonormal, and thus bounded. In particular, $u_n = Ku_n$ and since K is compact, we have that (Ku_n) has a convergent subsequence. But:

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left\|Ku_{n_k} - Ku_{n_{k+1}}\right\|^2 \\ &= \lim_{k\to\infty} \left\|u_{n_k} - u_{n_{k+1}}\right\|^2 \\ &= \lim_{k\to\infty} \|u_{n_k}\|^2 + \left\|u_{n_{k+1}}\right\|^2 \\ &= 2 \end{aligned}$$

by ?? ??. □

Lemma 13. Let H be Hilbert and $K : H \to H$ be a compact linear operator. Then, $\exists c > 0$ such that $\forall u \in \ker(\mathrm{id} - K)^\perp$ we have $\|u - Ku\| \geq c\|u\|$.

Proof. We proceed by contradiction. Suppose we have a sequence $(u_n) \in \ker(\mathrm{id} - K)^\perp$ with $\|u_n\| = 1$ such that $\|u_n - Ku_n\| \to 0$. Since (u_n) is bounded, we have that (u_n) has a weakly convergent subsequence (u_{n_k}) to $u \in H$. Since K is compact, we have that $Ku_{n_k} \to Ku$, and thus by continuity of the norm, $u = Ku$. Thus $u \in \ker(\mathrm{id} - K)$ and $u \in \ker(\mathrm{id} - K)^\perp$, which implies $u = 0$, a contraction with $\|u\| = 1$. □

Lemma 14. Let H be Hilbert and $K : H \to H$ be a compact linear operator. Then, $\mathrm{im}(\mathrm{id} - K)$ is closed.

Proof. Let $(v_n) \in \mathrm{im}(\mathrm{id} - K)$ be such that $v_n \to v \in H$. Then, $\exists (u_n) \in H$ such that $v_n = (\mathrm{id} - K)u_n$. By ?? ??, we can write $u_n = u_n^{\ker} + u_n^{\ker^\perp}$, where $u_n^{\ker} \in \ker(\mathrm{id} - K)$ and $u_n^{\ker^\perp} \in \ker(\mathrm{id} - K)^\perp$. Thus, $v_n = (\mathrm{id} - K)u_n^{\ker^\perp}$ and by Theorem 13, we have:

$$\|v_n - v_m\| \geq c \left\| u_n^{\ker^\perp} - u_m^{\ker^\perp} \right\|$$

Since (v_n) is Cauchy, so it is $(u_n^{\ker^\perp})$, and thus $(u_n^{\ker^\perp})$ converges to some $u \in \ker(\mathrm{id} - K)^\perp$. Thus, $v = (\mathrm{id} - K)u \in \mathrm{im}(\mathrm{id} - K)$. □

Theorem 15 (Abstract Fredholm alternative). Let H be Hilbert and $K : H \to H$ be a compact linear operator. Then:

1. $\ker(\mathrm{id} - K)$ and $\ker(\mathrm{id} - K^*)$ are both finite dimensional, and they have the same dimension.

2. $\mathrm{im}(\mathrm{id} - K) = \ker(\mathrm{id} - K^*)^\perp$. In particular, $\mathrm{im}(\mathrm{id} - K)$ is closed.

3. Either $\ker(\mathrm{id}-K) \neq \{0\}$ or $\mathrm{id}-K$ is an isomorphism.

Proof.

2. From **??** we have that $\overline{\mathrm{im}\, A} = (\ker A^*)^\perp$ for any general operator A between Hilbert spaces. Thus, $\mathrm{im}(\mathrm{id} - K) = \ker(\mathrm{id} - K^*)^\perp \iff \mathrm{im}(\mathrm{id} - K)$ is closed, which reduces to Theorem 14.

3. We first show that $\ker(\mathrm{id} - K) = \{0\} \iff \ker(\mathrm{id} - K^*) = \{0\}$. The argument is symmetric because $K^{**} = K$ and the fact that K is compact $\iff K^*$ is compact. So suppose $\ker(\mathrm{id} - K) = \{0\}$. Then, $\mathrm{id} - K$ is injective. Assume $\ker(\mathrm{id} - K^*) \neq \{0\}$. Then, $\mathrm{im}(\mathrm{id} - K) = \ker(\mathrm{id} - K^*)^\perp \neq H$ and so $\mathrm{im}((\mathrm{id} - K)^2) \subsetneq \mathrm{im}(\mathrm{id} - K)$. Indeed, if we had equality, then for any $u \in H$, we would have $(\mathrm{id} - K)u \in \mathrm{im}((\mathrm{id} - K)^2)$, and thus $\exists v \in H$ such that $(\mathrm{id} - K)u = (\mathrm{id} - K)^2 v$, which implies $u = (\mathrm{id} - K)v$ because $\ker(\mathrm{id} - K) = \{0\}$. Now, recursively, we have an infinite sequence $\mathrm{im}((\mathrm{id} - K)^{n+1}) \subsetneq \mathrm{im}((\mathrm{id} - K)^n)$, which implies that $\forall n\ \exists u_n \in \mathrm{im}((\mathrm{id} - K)^n) \cap \mathrm{im}((\mathrm{id} - K)^{n+1})^\perp$ with $\|u_n\| = 1$. Thus, $\langle u_n, u_m\rangle = \delta_{n,m}$. But $u_n - Ku_n \in \mathrm{im}((\mathrm{id} - K)^{n+1})$ so, $u_n - Ku_n \perp u_n$. This implies, by **?? ??**, that $\|Ku_n\|^2 = \|u_n - Ku_n\|^2 + \|u_n\|^2 \geq 1$, which is a contradiction with the compactness of K because any orthonormal sequence always converges weakly to zero (and so $Ku_n \to 0$). So either $\ker(\mathrm{id} - K) \neq \{0\}$ or $\mathrm{id} - K$ is bijective.

   To finish this point, we need to prove that if $\ker(\mathrm{id} - K) = \{0\}$, then $(\mathrm{id} - K)^{-1}$ is a bounded linear operator. But this is a consequence of Theorem 13: if $u \in H$, then $u \in \ker(\mathrm{id} - K)^\perp$ and thus $\|(\mathrm{id} - K)u\| \geq c\,\|u\|$, which implies that $\|v\| \geq c\left\|(\mathrm{id} - K)^{-1}v\right\|$ taking $v = (\mathrm{id} - K)u$.

1. Assume without loss of generality that $\dim\ker(\mathrm{id} - K) < \dim\ker(\mathrm{id} - K^*)$. Then, there exists a linear injective map $A : \ker(\mathrm{id} - K) \to \ker(\mathrm{id} - K^*) = \mathrm{im}(\mathrm{id} - K)^\perp$. Let $\tilde{K}$ be the operator defined by $\tilde{K}u = Ku + Au^{\ker}$, where $u^{\ker}$ is the projection of u onto $\ker(\mathrm{id}-K)$. Then, $\tilde{K}$ is compact (because K is compact and so is A, because it has finite range). Moreover, if $u \in \ker(\mathrm{id} - \tilde{K})$, then $(\mathrm{id} - K)u - Au^{\ker} = 0$, which since $(\mathrm{id} - K)u \in \mathrm{im}(\mathrm{id} - K)$ and $Au^{\ker} \in \mathrm{im}(\mathrm{id} - K)^\perp$ implies that both terms are zero. So $u = u^{\ker} \in \ker(\mathrm{id} - K)$ and since A is injective, $u = u^{\ker} = 0$. Thus, $\ker(\mathrm{id} - \tilde{K}) = \{0\}$ and by the previous point, $\mathrm{id} - \tilde{K}$ is an isomorphism from H to itself. So, for every $w \in \ker(\mathrm{id} - K^*)$, $\exists u \in H$ such that $w = (\mathrm{id} - \tilde{K})u$. Projecting both sides onto $\ker(\mathrm{id} - K^*) = \mathrm{im}(\mathrm{id} - K)^\perp$, we have $w = -Au^{\ker}$, which implies that A is onto, and so $\dim\ker(\mathrm{id} - K) = \dim\ker(\mathrm{id} - K^*)$. Theorem 12 finishes the proof.

□

Definition 16. Consider the operator L as in Eq. (1). We define the *formal adjoint* of L as:

$$
\begin{aligned}
L^*v &:= -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j v) - \sum_{j=1}^{d} \partial_j(b_j v) + cv \\
&= -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j v) - \sum_{j=1}^{d} b_j\partial_j v + \left(c - \sum_{j=1}^{d} \partial_j b_j\right) v
\end{aligned}
$$

It satisfies $\langle Lu, v\rangle = \langle u, L^*v\rangle$ for all $u, v \in H_0^1(\Omega)$.

Proposition 17. The *homogeneous adjoint problem*

$$
\mathcal{D}_0^* := \begin{cases} L^*v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}
$$

whose weak formulation is

$$
\langle \nabla v, \nabla w\rangle_2 + \langle \mathbf{b}\cdot\nabla v, w\rangle_2 = 0 \quad \forall w \in H_0^1(\Omega)
$$

has a finite dimensional solution space W_0, the space V_0 of solutions of $\mathcal{D}_0$ has also finite dimension and $\dim W_0 = \dim V_0$. Moreover, if $f \in L^2(\Omega)$, $\mathcal{D}_f$ is solvable if and only if $\langle f, v\rangle = 0$ for all $v \in W_0$.

Proof. We saw in Theorem 11 that for $\mu \geq \mu_0 > 0$, L_μ is an isomorphism. Now we want to solve $L_0 u = f$. Consider the change of variables $u = {L_{\mu_0}}^{-1}w$, with $w = L_{\mu_0}u \in H^{-1}(\Omega)$. Thus, the equation becomes:

$$
f = (L_{\mu_0} - \mu_0){L_{\mu_0}}^{-1}w = w - \mu_0{L_{\mu_0}}^{-1}w = (\mathrm{id} - K)w
$$

with $K = \mu_0{L_{\mu_0}}^{-1}$. We claim that $K : L^2(\Omega) \to L^2(\Omega)$ is compact. Note that $K = \mu_0\iota_{H_0^1\hookrightarrow L^2} \circ {L_{\mu_0}}^{-1} \circ \iota_{L^2\hookrightarrow H^{-1}}$, so since ${L_{\mu_0}}^{-1}$ and $\iota_{L^2\hookrightarrow H^{-1}}$ are bounded, and we have a compact embedding $H_0^1 \hookrightarrow L^2$, we have that K is compact. Finally, one can check that:

$$
\mathrm{id} - K^* = (\mathrm{id} - K)^* = \left(L_0{L_{\mu_0}}^{-1}\right)^* = ({L_{\mu_0}}^*)^{-1}{L_0}^*
$$

By 15 Abstract Fredholm alternative, we have that $(\mathrm{id} - K)w = f$ has a solution if and only if $(\mathrm{id} - K^*)h = 0 \implies \langle f, h\rangle_{L^2} = 0$ for all $h \in L^2$. But ${L_{\mu_0}}^*$ is an isomorphism, so $(\mathrm{id} - K^*)h = 0 \iff {L_0}^*h = 0$. □

Definition 18. We define the following problem:

$$
\mathcal{N}_f := \begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}
$$

and $\mathcal{N}_f^* = \mathcal{N}_f$. The weak formulation of the problem is:

$$
\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \quad \forall v \in H^1(\Omega)
$$

Proposition 19. $\mathcal{N}_f$ has at least one solution if and only if for any weak solution v of $\mathcal{N}_0$ we have $\langle f, v\rangle = 0$.

Spectrum of compact operators

In this section $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 20. Let H be a $\mathbb{K}$-Hilbert space and $K : H \to H$ be a bounded operator. We define the *resolvent set* of K as:

$$\rho(K) = \{\lambda \in \mathbb{K} : \lambda \mathrm{id} - K \text{ is invertible}\}$$

and the *spectrum* of K as:

$$\sigma(K) = \mathbb{K} \setminus \rho(K)$$

Proposition 21. Let H be a $\mathbb{K}$-Hilbert space and $T : H \to H$ be a bounded operator. Then, $\sigma(T)$ is closed.

Proof. Note that $\rho(K)$ is open because if $\lambda \in \rho(T)$, then $\exists \varepsilon \in \mathbb{R}$ such that $|\varepsilon| < \left\| (\lambda \mathrm{id} - K)^{-1} \right\|$. And so $(\lambda + \varepsilon)\mathrm{id} - K$ is invertible. Thus, $\sigma(K)$ is closed. □

Theorem 22. Let H be an infinite-dimensional separable Hilbert space and $K : H \to H$ be a compact operator. Then:

1. $0 \in \sigma(K)$.
2. If $\lambda \in \sigma(K) \setminus \{0\}$, then λ is an eigenvalue of K.
3. $\sigma(K)$ is closed and at most countable.
4. If $\sigma(K) \cap \mathbb{R}$ is infinite, then $\sigma(K) \setminus \{0\}$ is of the form $\{\lambda_n\}_{n \in \mathbb{N}}$ with $\lambda_n \to 0$.
5. If $\lambda \in \sigma(K) \setminus \{0\}$, then:

$$\dim \left(\bigcup_{p \geq 1} \ker (\lambda \mathrm{id} - K)^p \right) < \infty$$

Proof.

1. Assume $0 \notin \sigma(K)$. Then, K is bijective and so $\mathrm{id} = K \circ K^{-1}$ is compact, as it is the composition of a compact operator and a bounded operator. But this is a contradiction with **?? ??** because the image of any bounded set under a compact operator is relatively compact (or *precompact*).
2. If $\ker(\lambda \mathrm{id} - K) = \{0\}$, then by 15 Abstract Fredholm alternative, $\lambda \mathrm{id} - K$ is an isomorphism, and thus $\lambda \in \rho(K)$.

□

Lemma 23. Let H be a Hilbert space and $T : H \to H$ be a continuous self-adjoint operator. Then:

$$\|T\| = \sup_{\|x\|=1} |\langle x, Tx \rangle|$$

Proof. Clearly $\alpha := \sup_{\|x\|=1} |\langle x, Tx \rangle| \leq \|T\|$. For the converse, it suffices to show that $|\langle Tx, y \rangle| \leq \alpha$ for all $\|x\| = \|y\| = 1$. We have:

$$\langle Tx, y \rangle = \frac{1}{4} \left(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle \right)$$

And then, by **?? ??**:

$$|\langle Tx, y \rangle| \leq \frac{\alpha}{4} \left(\|x+y\|^2 + \|x-y\|^2 \right) = \alpha$$

□

Lemma 24. Let $H \neq \{0\}$ be Hilbert and $K : H \to H$ be a compact and self-adjoint operator. Then:

$$\sup_{\|x\|=1} \langle x, Kx \rangle = \lambda$$

where λ is the largest eigenvalue of K.

Proof. Let (x_n) be a maximizing sequence with $\|x_n\| = 1$. After extraction, we can assume that $x_n \rightharpoonup x_*$ and so $Kx_n \to Kx_*$. Thus, $\langle x_n, Kx_n \rangle \to \langle x_*, Kx_* \rangle$. So x_* is a maximizer. Now, take $h \perp x_*$ and $\|h\| = 1$. Then, $x_t := \frac{x_* + th}{\sqrt{1+t^2}}$ satisfies $\|x_t\| = 1$ and:

$$\langle x_t, Kx_t \rangle = \langle x_*, Kx_* \rangle + 2t \langle h, Kx_* \rangle + \mathrm{o}(t) \leq \langle x_*, Kx_* \rangle$$

because of the maximality. So we must have $\langle h, Kx_* \rangle = 0$, which implies $Kx_* \in \left(\langle x_* \rangle^\perp \right)^\perp = \langle x_* \rangle$. Thus, $Kx_* = \lambda x_*$. □

Regularity theorems for weak solutions of divergence-form elliptic PDEs

Theorem 25 (Inner regularity). Assume, in addition to the usual assumptions, that $a_{ij} \in \mathcal{C}^1(\Omega)$. Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ be a weak solution of $Lu = f$. Then, $u \in H^2_{\mathrm{loc}}(\Omega)$ and for any compact embedding $\omega \subset\subset \Omega$, meaning that $\overline{\omega} \subset \Omega$ compact, we have $u \in H^2(\omega)$ and:

$$\|u\|_{H^2(\omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Corollary 26. Assume that $a_{ij} \in \mathcal{C}^{m+1}(\Omega)$ for some $m \in \mathbb{N}$, and $b_j, c \in \mathcal{C}^m(\Omega)$. Let $f \in H^m(\Omega)$ and $u \in H^1$ be a weak solution of $Lu = f$. Then, $u \in H^{m+2}_{\mathrm{loc}}(\Omega)$ and for any $\omega \subset\subset \Omega$ we have $u \in H^{m+2}(\omega)$ and:

$$\|u\|_{H^{m+2}(\omega)} \leq C \left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Corollary 27. Assume $a_{ij}, b_j, c, f \in \mathcal{C}^\infty(\Omega)$. Let $u \in H^1(\Omega)$ be a weak solution of $Lu = f$. Then, $u \in \mathcal{C}^\infty(\Omega)$.

Theorem 28 (Regularity up to the boundary). Assume that $\partial\Omega$ is $\mathcal{C}^2$ and that $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$, $b_j, c \in L^\infty(\Omega)$. Let $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$ be a weak solution of $\mathcal{D}_f$. Then, $u \in H^2(\Omega)$ and:

$$\|u\|_{H^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Corollary 29. Assume that $\partial\Omega$ is $\mathcal{C}^m$, $m \in \mathbb{N}$, and that $a_{ij} \in \mathcal{C}^{m+1}(\overline{\Omega})$, $b_j, c \in \mathcal{C}^m(\overline{\Omega})$. Let $f \in H^m(\Omega)$ and $u \in H^1_0(\Omega)$ be a weak solution of $\mathcal{D}_f$. Then, $u \in H^{m+2}(\Omega)$ and:

$$\|u\|_{H^{m+2}(\Omega)} \leq C \left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Corollary 30. Assume that $\partial\Omega$ is $\mathcal{C}^\infty$ and that $a_{ij}, b_j, c, f \in \mathcal{C}^\infty(\overline{\Omega})$. Let $u \in H^1_0(\Omega)$ be a weak solution of $\mathcal{D}_f$. Then, $u \in \mathcal{C}^\infty(\Omega)$ and $\forall m \in \mathbb{N}$:

$$\|u\|_{H^m(\Omega)} \leq C \left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

Weak maximum principle for weak solutions of divergence-form elliptic PDEs

Lemma 31. Let $\Omega \subseteq \mathbb{R}^d$ open and $u \in H^1(\Omega)$. Then:

$$u^+ := \begin{cases} u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases} \qquad u^- := \begin{cases} -u & \text{if } u < 0 \\ 0 & \text{if } u \leq 0 \end{cases}$$

are also in $H^1(\Omega)$ and:

$$\boldsymbol{\nabla}(u^+) \overset{\text{a.e.}}{=} \begin{cases} \boldsymbol{\nabla} u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases} \quad \boldsymbol{\nabla}(u^-) \overset{\text{a.e.}}{=} \begin{cases} -\boldsymbol{\nabla} u & \text{if } u < 0 \\ 0 & \text{if } u \geq 0 \end{cases}$$

Corollary 32. Let $\Omega \subseteq \mathbb{R}^d$ open and $u \in H^1(\Omega)$. Then, $|u| \in H^1(\Omega)$ and $\boldsymbol{\nabla}|u| = \operatorname{sgn} \boldsymbol{\nabla} u$.

Lemma 33. Let $(u_n) \in H^1(\Omega)$ be such that $u_n \overset{H^1(\Omega)}{\longrightarrow} u$. Then, ${u_n}^\pm \overset{H^1(\Omega)}{\longrightarrow} u^\pm$.

Corollary 34. Let $u \in H^1(\Omega)$. Then, $\operatorname{Tr}_{\partial\Omega}(u^\pm) = (\operatorname{Tr}_{\partial\Omega} u)^\pm$.

Lemma 35. Let $\Omega \subseteq \mathbb{R}^d$ open with $\mathcal{C}^1$ boundary, $u \in H^1(\Omega)$ and $\operatorname{Tr}_{\partial\Omega} u \overset{\text{a.e.}}{\leq} 0$. Then, $u^+ \in H_0^1(\Omega)$.

Theorem 36 (Weak maximum principle). Let $\Omega \subseteq \mathbb{R}^d$ open and bounded with $\mathcal{C}^1$ boundary, $a_{ij} = a_{ji}, c \in L^\infty(\Omega)$, $c \overset{\text{a.e.}}{\geq} 0$, $L = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + c$ be elliptic and $f \in L^2(\Omega)$ with $f \overset{\text{a.e.}}{\leq} 0$. Let $u \in H^1(\Omega)$ be such that:

- $\displaystyle\int_\Omega \left[\sum_{i,j=1}^d a_{ij}\partial_i u \partial_j v + cuv\right] = \int_\Omega fv \ \forall v \in H_0^1(\Omega)$
- $\operatorname{Tr}_{\partial\Omega} u \overset{\text{a.e.}}{\leq} 0$

Then, $u \overset{\text{a.e.}}{\leq} 0$.

Proof. Take $v = u^+ \in H_0^1(\Omega)$ by Theorem 35. Then, we have:

$$0 \leq \theta \|\boldsymbol{\nabla} u^+\|_{L^2}{}^2 \leq \int_{\{u>0\}} \sum_{i,j=1}^d a_{ij}\partial_i u \partial_j u + cu^2 = \int_{\{u>0\}} fu \leq 0$$

where in the second inequality we used the ellipticity of L. Thus, we must have $\boldsymbol{\nabla} u^+ = 0$ a.e. in Ω, which implies $u^+ = 0$ a.e. in Ω, because $u^+|_{\partial\Omega} = 0$. □

Theorem 37 (Weak maximum principle). Let $\Omega \subseteq \mathbb{R}^d$ open and bounded with $\mathcal{C}^1$ boundary, $a_{ij} = a_{ji}, b_j, c \in L^\infty(\Omega)$, $c \overset{\text{a.e.}}{\geq} 0$, $L = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{j=1}^d b_j\partial_j + c$ be elliptic and $f \in L^2(\Omega)$ with $f \overset{\text{a.e.}}{\leq} 0$. Let $u \in H^1(\Omega)$ be such that:

- $\displaystyle\int_\Omega \left[\sum_{i,j=1}^d a_{ij}\partial_i u \partial_j v + \sum_{j=1}^d b_j v \partial_j u + cuv\right] = \int_\Omega fv$ $\forall v \in H_0^1(\Omega)$
- $\operatorname{Tr}_{\partial\Omega} u \overset{\text{a.e.}}{\leq} 0$

Then, $u \overset{\text{a.e.}}{\leq} 0$.

Proof. Let $m \geq 0$ and $v_m = (u-m)^+$. Proceeding as in the previous proof, we have:

$$\begin{aligned} 0 &\leq \theta \|\boldsymbol{\nabla} v_m\|_{L^2}^2 - d\|\mathbf{b}\|_\infty \|\boldsymbol{\nabla} v_m\|_{L^2} \|v_m\|_{L^2} \\ &\leq \int_{\{u>m\}} \sum_{i,j=1}^d a_{ij}\partial_i u \partial_j v_m + \sum_{j=1}^d b_j \partial_j u v_m + cuv_m \\ &= \int_{\{u>m\}} fv_m \leq 0 \end{aligned}$$

Thus, $\|\boldsymbol{\nabla} v_m\|_{L^2} \leq C\|v_m\|_{L^2}$, with C independent of m. Note that since Ω is bounded, $\lim_{m\to\infty} |\{u > m\}| = 0$ by ?? ??, and so $\lim_{m\to\infty} \operatorname{supp} v_m = 0$ as well since $\operatorname{supp} v_m \subseteq \{u > m\}$. We now continue the proof for $d \geq 3$. By ?? ?? we have a continuous embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}$ with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$. So, $\|v_m\|_{L^{2^*}} \leq \delta \|\boldsymbol{\nabla} v_m\|_{L^2}$. Thus:

$$\begin{aligned} \|\boldsymbol{\nabla} v_m\|_{L^2} &\leq C\|v_m\|_{L^2} \leq C|\operatorname{supp} v_m|^{1-\frac{2}{2^*}} \|v_m\|_{L^{2^*}} \leq \\ &\leq C\delta|\operatorname{supp} v_m|^{1-\frac{2}{2^*}} \|\boldsymbol{\nabla} v_m\|_{L^2} \leq \frac{1}{2}\|\boldsymbol{\nabla} v_m\|_{L^2} \end{aligned}$$

where in the second inequality we used ?? ?? and the last one is valid for $m \geq m_0$ large enough. Thus, $\|\boldsymbol{\nabla} v_m\|_{L^2} = 0$ for $m \geq m_0$, which implies $u \overset{\text{a.e.}}{\leq} m_0$. This means that $|\{u > m_0\}| = 0$ and that $\forall \varepsilon > 0$, $|\{u \geq m_0 - \varepsilon\}| > 0$. Suppose now that $m_0 > 0$ and let $S_\varepsilon = |\{u > m_0 - \varepsilon\}|$. Again by ?? ??, $\lim_{\varepsilon\to 0} S_\varepsilon = 0$. But then, proceeding as in the previous step:

$$\|\boldsymbol{\nabla} v_{m_0-\varepsilon}\|_{L^2} \leq C\delta {S_\varepsilon}^{1-\frac{2}{2^*}} \|\boldsymbol{\nabla} v_{m_0-\varepsilon}\|_{L^2} \leq \frac{1}{2}\|\boldsymbol{\nabla} v_{m_0-\varepsilon}\|_{L^2}$$

by choosing ε small enough. Thus, $\|\boldsymbol{\nabla} v_{m_0-\varepsilon}\|_{L^2} = 0$ and so $u \overset{\text{a.e.}}{\leq} m_0 - \varepsilon$, which is a contradiction. Thus, $m_0 = 0$ (because $m_0 \geq 0$ from the beginning) and so $u \overset{\text{a.e.}}{\leq} 0$. □

Theorem 38 (Weak minimum principle). Let $\Omega \subseteq \mathbb{R}^d$ open and bounded with $\mathcal{C}^1$ boundary, $a_{ij} = a_{ji}, b_j, c \in L^\infty(\Omega)$, $c \overset{\text{a.e.}}{\geq} 0$, $L = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{j=1}^d b_j\partial_j + c$ be elliptic and $f \in L^2(\Omega)$ with $f \overset{\text{a.e.}}{\geq} 0$. Let $u \in H^1(\Omega)$ be such that:

- $\displaystyle\int_\Omega \left[\sum_{i,j=1}^d a_{ij}\partial_i u \partial_j v + \sum_{j=1}^d b_j v \partial_j u + cuv\right] = \int_\Omega fv$ $\forall v \in H_0^1(\Omega)$
- $\operatorname{Tr}_{\partial\Omega} u \overset{\text{a.e.}}{\geq} 0$

Then, $u \overset{\text{a.e.}}{\geq} 0$.

Sketch of the proof. Apply 37 Weak maximum principle to $u \mapsto -u$ with $f \mapsto -f$. □

Corollary 39. If u is a weak solution of $\mathcal{D}_0$ with $c \geq 0$, then $u \overset{\text{a.e.}}{=} 0$.

Proof. If u is a weak solution of $\mathcal{D}_0$, then u is a super- (that is, $Lu \geq 0$) and sub-solution (that is $Lu \leq 0$) of $\mathcal{D}_0$. Thus, using 37 Weak maximum principle and 38 Weak minimum principle we conclude that $u \overset{\text{a.e.}}{=} 0$. □

Corollary 40. For each $f \in L^2(\Omega)$, the problem $\mathcal{D}_f$ has a unique weak solution u_f. Moreover, if $\partial\Omega \in \mathcal{C}^1$, then $u_f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \mapsto u_f$ is a bounded linear operator from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$. If $\partial\Omega \in \mathcal{C}^{m+1}$, $b_j \in \mathcal{C}^{m-1}$ and $f \in H^{m-1}(\Omega)$, then $u_f \in H^{m+1}(\Omega) \cap H_0^1(\Omega)$ and $f \mapsto u_f$ is a bounded linear operator from $H^{m-1}(\Omega)$ to $H^{m+1}(\Omega) \cap H_0^1(\Omega)$.

Sketch of the proof. 15 Abstract Fredholm alternative applied to this problem (check Theorem 17) tells us that either there is a nonzero weak solution to $\mathcal{D}_0$ or $\mathcal{D}_f$ is solvable for all $f \in L^2(\Omega)$. But the first case is impossible by Theorem 39. □

Theorem 41. Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be open and bounded with $\mathcal{C}^{m+1}$ boundary, $m \geq 1$. Let $a_{ij} \in \mathcal{C}^m(\overline{\Omega})$, $b_j, c \in \mathcal{C}^{m-1}(\overline{\Omega})$ and $Lu = -\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j u) + \sum_{j=1}^d b_j \partial_j u + cu$ be an elliptic operator. Then, for any $f \in W^{m-1,p}(\Omega)$, if $u \in H_0^1(\Omega)$ is a weak solution of $\mathcal{D}_f$, then $u \in W^{m+1,p}(\Omega)$ and:

$$\|u\|_{W^{m+1,p}(\Omega)} \leq C\left(\|f\|_{W^{m-1,p}(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

If in addition the weak solution of $\mathcal{D}_0$ is $u = 0$, then $L : W^{m+1,p}(\Omega) \cap W_0^{1,p}(\Omega) \to W^{m-1,p}(\Omega)$ is an isomorphism, where $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

3. | Regularity in $\mathcal{C}^{k,\alpha}$ for non-divergence form elliptic PDEs

In this section we will still always work in $\Omega \subset \mathbb{R}^d$ open and bounded and the elliptic operator L (with ellipticity constant θ) will be in its non-divergence form:

$$L = -\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 + \sum_{j=1}^d b_j\partial_j + c$$

with $a_{ij} = a_{ji}$. Moreover we will not use the usual Hölder norm

$$\|u\|_{\mathcal{C}^{k,\alpha}(\Omega)} = \sup_{\substack{x\neq y\\|\beta|=k}} \frac{\left|\partial^\beta u(x) - \partial^\beta u(y)\right|}{|x-y|^\alpha}$$

but the following one:

$$\|u\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} = \sup_{\substack{x\in\overline{\Omega}\\|\beta|\leq k}} \left|\partial^\beta u(x)\right| + \sup_{\substack{x\neq y\\|\beta|=k}} \frac{\left|\partial^\beta u(x) - \partial^\beta u(y)\right|}{|x-y|^\alpha}$$

Remark. Recall that $(\mathcal{C}^{k,\alpha}(\overline{\Omega}), \|\cdot\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})})$ is a Banach space and that if $0 < \alpha_1 \leq \alpha_2 < 1$, then $\mathcal{C}^{k,\alpha_2}(\overline{\Omega}) \subseteq \mathcal{C}^{k,\alpha_1}(\overline{\Omega})$

Schauder estimates

Theorem 42 (Schauder estimates). Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $\partial\Omega \in \mathcal{C}^{2,\alpha}$ for some $0 < \alpha < 1$. In the elliptic operator L assume that $a_{ij}, b_j, c \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Then, $\exists C > 0$ such that if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ solves $Lu = f$, with $f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, then $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and:

$$\|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq C\left(\|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} + \|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}\right)$$

Moreover we have:

$$\|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq \tilde{C}\left(\|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} + \|u\|_{\mathcal{C}^{0}(\overline{\Omega})}\right)$$

Corollary 43. Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $\partial\Omega \in \mathcal{C}^{k+2,\alpha}$ for some $0 < \alpha < 1$ and $k \geq 0$. In the elliptic operator L assume that $a_{ij}, b_j, c \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$. Then, $\exists c > 0$ such that if $u \in \mathcal{C}^{k+2}(\Omega) \cap \mathcal{C}^k(\overline{\Omega})$ solves $Lu = f$, with $f \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$, then $u \in \mathcal{C}^{k+2,\alpha}(\overline{\Omega})$ and:

$$\|u\|_{\mathcal{C}^{k+2,\alpha}(\overline{\Omega})} \leq C\left(\|f\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} + \|u\|_{\mathcal{C}^{k+1,\alpha}(\overline{\Omega})}\right)$$

Maximum and comparison principles

Lemma 44. If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_d(\mathbb{R})$ are symmetric and $\mathbf{A}, \mathbf{B} \geq 0$, then $\operatorname{tr}(\mathbf{AB}) \geq 0$.

Theorem 45 (Weak maximum principle). Let $u \in \mathcal{C}^2(\Omega)$ be such that $Lu \leq 0$. Then:

- If $c = 0$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
- If $c \geq 0$, then $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$.

Proof. Assume first that $Lu < 0$ and $c = 0$. Suppose $\exists x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$. Then, $\boldsymbol{\nabla} u(x_0) = 0$ and $\mathbf{H}u(x_0) \leq 0$, that is, $\sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)p_i p_j \leq 0$ $\forall \mathbf{p} \in \mathbb{R}^d$. On the other hand:

$$\begin{aligned}\operatorname{tr}(\mathbf{A}(x_0)\mathbf{H}u(x_0)) &= \sum_{i,j=1}^d a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\\ &= -Lu(x_0) > 0\end{aligned}$$

The ellipticity of L implies that $\mathbf{A} > 0$, but this is a contradiction with Theorem 44 because $\mathbf{H}u(x_0) \leq 0$. If we now have $c \geq 0$, assume that $\max_{\overline{\Omega}} u^+ > \max_{\partial\Omega} u^+$. Then, $\exists x_0 \in \Omega$ such that $u(x_0) > 0$ and $u(x_0) = \max_{\overline{\Omega}} u^+$. Similarly, we have:

$$\begin{aligned}\operatorname{tr}(\mathbf{A}(x_0)\mathbf{H}u(x_0)) &= \sum_{i,j=1}^d a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\\ &= -Lu(x_0) + c(x_0)u(x_0) \geq 0\end{aligned}$$

which again leads to a contraction. Now assume $Lu \leq 0$. Take $u_\varepsilon = u + \varepsilon \mathrm{e}^{\lambda x_1}$, with $\varepsilon > 0$ and $\lambda > 0$ yet to be chosen. An easy computation shows that:

$$\begin{aligned}Lu_\varepsilon &\leq \mathrm{e}^{\lambda x_1}[-\lambda^2 a_{11} + b_1\lambda + c]\\ &\leq \mathrm{e}^{\lambda x_1}[-\lambda^2 a_{11} + \|\mathbf{b}\|_\infty \lambda + c] < 0\end{aligned}$$

for λ large enough. We do here the case $c = 0$ (the other is analogous). From what we have previously seen, $\exists y_\varepsilon \in \partial\Omega$ such that $u(x) \leq u_\varepsilon(x) \leq u_\varepsilon(y_\varepsilon)$. And so we can find a sequence y_{ε_n} that converges to some $y_0 \in \partial\Omega$ (because $\partial\Omega$ is compact) as $\varepsilon_n \to 0$, which implies $u(x) \leq u(y_0)$. □

Theorem 46 (Weak minimum principle). Let $u \in \mathcal{C}^2(\Omega)$ be such that $Lu \geq 0$. Then:

- If $c = 0$, then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

- If $c \geq 0$, then $\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-$.

Sketch of the proof. Apply 45 Weak maximum principle to $-u$ using that $(-u)^+ = u^-$. $\square$

Remark. Nothing can be said if $c < 0$. For example, consider $-u'' - u = 0$, which has $u(x) = \sin(x)$ as a solution, and take $\Omega = (0, \pi)$.

Lemma 47 (Hopf's lemma). Let $u \in \mathcal{C}^2(\Omega)$ be such that $Lu \leq 0$ and suppose that the region Ω is connected and that satisfies the *interior ball condition*: for any $x \in \partial\Omega$ there exists $r > 0$ and $y \in \Omega$ such that $B(y,r) \subset \Omega$ and $\overline{B(y,r)} \cap \partial\Omega = \{x\}$. Suppose in addition that $c = 0$ and $x_0 \in \partial\Omega$ is such that $u(x_0) = \max_{\overline{\Omega}} u$. Then, either u is constant in Ω or

$$\liminf_{t\to 0^+} \frac{u(x_0) - u(x_0 + t\mathbf{n})}{t} > 0$$

for any vector $\mathbf{n}$ of the form $\mathbf{n} = \frac{x_0 - y_0}{\|x_0 - y_0\|}$ with $B(y_0, r) \subset \Omega$ and $\overline{B(y_0, r)} \cap \partial\Omega = \{x_0\}$.

Remark. In particular, if $\partial\Omega \in \mathcal{C}^1$ and $u \in \mathcal{C}^1(\overline{\Omega})$, then 47 Hopf's lemma implies that either u is constant in Ω or $\partial_{\mathbf{n}} u(x_0) = \boldsymbol{\nabla} u(x_0) \cdot \mathbf{n} > 0$.

Theorem 48 (Strong maximum principle). Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected, and $u \in \mathcal{C}^2(\Omega)$ be such that $Lu \leq 0$. Then:

1. If $c = 0$ and $\exists x_0 \in \Omega$ such that $u(x_0) \geq u(x)$ $\forall x \in \Omega$, then $u = \text{const.}$ in Ω.
2. If $c \geq 0$ and $\exists x_0 \in \Omega$ such that $u(x_0) \geq 0$ and $u(x_0) \geq u(x)$ $\forall x \in \Omega$, then $u = \text{const.}$ in Ω.

Proof. Assume $c = 0$, the other case is similar. Let $M = \max_{\Omega} u$, $C := \{u = M\}$ and $V := \{u < M\}$. Take $y \in V$ satisfying $d(y, C) < d(y, \partial\Omega)$ and let B be the largest ball with center at y whose interior lies in V. Then, there exists $x_0 \in C$ with $x_0 \in \partial B$. Clearly V satisfies the interior ball condition at x_0, whence 47 Hopf's lemma implies that $\partial_{\mathbf{n}} u(x_0) > 0$. But $\partial_{\mathbf{n}} u(x_0) = 0$ because $x_0 \in C$, which is a contradiction. Thus, $V = \varnothing$ and so $u = \text{const.}$ in Ω. $\square$

Theorem 49 (Strong minimum principle). Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected, and $u \in \mathcal{C}^2(\Omega)$ be such that $Lu \geq 0$. Then:

1. If $c = 0$ and $\exists x_0 \in \Omega$ such that $u(x_0) \leq u(x)$ $\forall x \in \Omega$, then $u = \text{const.}$ in Ω.
2. If $c \geq 0$ and $\exists x_0 \in \Omega$ such that $u(x_0) \leq 0$ and $u(x_0) \leq u(x)$ $\forall x \in \Omega$, then $u = \text{const.}$ in Ω.

Sketch of the proof. Apply 48 Strong maximum principle to $-u$. $\square$

Theorem 50 (A priori estimate). Suppose that $c \geq 0$ and $u \in \mathcal{C}^2(\Omega)$ is a solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u|_{\partial\Omega} = h & \text{on } \partial\Omega \end{cases}$$

with $f \in \mathcal{C}^0(\overline{\Omega})$ and $h \in \mathcal{C}^0(\partial\Omega)$. Then, $\forall x \in \overline{\Omega}$:

$$u(x) \leq \max_{\partial\Omega} h^+ + C \max_{\overline{\Omega}} f^+$$

with C independent of u, f and h. Moreover, we have:

$$|u| \leq \max_{\partial\Omega} |h| + C \max_{\overline{\Omega}} |f|$$

Proof. Let

$$w(x) = \max_{\partial\Omega} h^+ + \max_{\overline{\Omega}} f^+ (\cosh(\lambda r) - \cosh(\lambda x_1))$$

with $r = \max\{|x_1| : x \in \overline{\Omega}\}$. An easy check shows that for $\lambda = \lambda_0 > 0$ large enough we have:

$$\begin{cases} Lw \geq \max_{\overline{\Omega}} f^+ \\ \max_{\partial\Omega} h^+ \leq w \leq \max_{\partial\Omega} h^+ + \max_{\overline{\Omega}} f^+ \cosh(\lambda_0 r) \end{cases}$$

Let $v = u - w$. Then, $Lv \leq 0$ and $v|_{\partial\Omega} \leq 0$. Thus, 45 Weak maximum principle implies that $v \leq 0$ in Ω, that is, $u \leq w$ in Ω. $\square$

Continuation method

Theorem 51 (Continuation method). Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $\partial\Omega \in \mathcal{C}^{2,\alpha}$ for some $0 < \alpha < 1$. Consider the problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u|_{\partial\Omega} = h & \text{on } \partial\Omega \end{cases}$$

with $f, a_{ij}, b_j, c \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $h \in \mathcal{C}^{0,\alpha}(\partial\Omega)$. Then, there exists a solution to this problem in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$.

Proof. We will do it for $h = 0$. Let $t \in [0,1]$ and consider the problem:

$$\mathcal{D}_t := \begin{cases} L_t u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

with $L_t = tL - (1-t)\Delta$. We know that $\mathcal{D}_0$ has a unique weak solution $u_0 \in H_0^1(\Omega)$. The idea of the continuation method is that if $\mathcal{D}_t$ is solvable for all f, then for $k > 0$ small enough, $\mathcal{D}_{t+k}$ is solvable for all f too. Rewrite $\mathcal{D}_{t+k}$ as:

$$\begin{cases} L_t u = f - k(L + \Delta)u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

We need to solve the fixed point problem $u = \phi(u)$, with

$$\begin{aligned} \phi : \mathcal{C}^{2,\alpha} &\longrightarrow \mathcal{C}^{2,\alpha} \\ u &\longmapsto \phi(u) = {L_t}^{-1} f - k{L_t}^{-1}(L + \Delta)u \end{aligned}$$

From 42 Schauder estimates and 50 A priori estimate we deduce that $\|u\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}$. So $\forall \varphi \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ we have $\|{L_t}^{-1}\varphi\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \leq C\|\varphi\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}$ (it can be seen that the constant does not depend on t). We will show that ϕ is a contraction for k small enough. Let $u, v \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. Then:

$$\begin{aligned} \|\phi(u) - \phi(v)\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} &\leq Ck\|(L + \Delta)(u - v)\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} \\ &\leq \tilde{C}k\|u - v\|_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \end{aligned}$$

So take $k \leq \frac{1}{2\tilde{C}}$. Repeating this argument a finite number of times ($\tilde{C}$ does not depend on t) we conclude that $\mathcal{D}_1$ is solvable. $\square$

4. | Existence theorems for nonlinear elliptic PDEs by fixed point methods

In this section we will mostly consider almost linear elliptic PDEs of the form:

$$\begin{cases} Lu = f(x,u) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

with L either $-\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) + \sum_{j=1}^d b_j\partial_j$ or $-\sum_{i,j=1}^d a_{ij}\partial_{ij}^2 + \sum_{j=1}^d b_j\partial_j$, and $f : \Omega\times\mathbb{R}\to\mathbb{R}$.

Method of subsoltions and supersolutions

Theorem 52. Suppose that an operator L is uniformly elliptic on an open bounded set $\Omega\subset\mathbb{R}^d$ with $\partial\Omega\in\mathcal{C}^2$, with $c=0$ and either in divergence form (with $a_{ij}\in\mathcal{C}^1$) or non-divergence form (with $a_{ij}, b_j\in\mathcal{C}^{0,\alpha}$). Suppose that $f\in\mathcal{C}^1(\overline{\Omega}\times\mathbb{R})$ and assume that the problem of Eq. (2) has a bounded subsolution $\underline{u}$ and a bounded supersolution $\overline{u}$ such that $\underline{u}\le\overline{u}$. Then, there exists a solution u to Eq. (2) such that $\underline{u}\le u\le\overline{u}$, which is in $H_0^1(\Omega)\cap H_0^2(\Omega)$ if L is in divergence form and in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ if L is in non-divergence form.

Proof. Let $M := \max\{\|\underline{u}\|_\infty, \|\overline{u}\|_\infty\}$ and modify f outside the set $\overline{\Omega}\times[-M,M]$ so that the modified function $\tilde{f}$ is globally Lipschitz in u and $\sup_{\overline{\Omega}\times\mathbb{R}}\left|\frac{\tilde{f}}{u}\right| \le \sup_{\overline{\Omega}\times[-M-2,M+2]}\left|\frac{f}{u}\right| + 1 =: k$. Then, the function $g(x,t) = \tilde{f}(x,t)+kt$ is non-decreasing in t, and we can rewrite the problem as:

$$\begin{cases} (L+k)u = g(x,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

Now we construct a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ as follows. Let $u_0 = \underline{u}$ and $\forall n\in\mathbb{N}\cup\{0\}$, u_{n+1} be the solution of:

$$\begin{cases} (L+k)u = g(x,u_n) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

Take $w = u_n - u_{n+1}$. By induction, using the monotonicity of g we have that w solves:

$$\begin{cases} (L+k)w \le 0 & \text{in } \Omega \\ w|_{\partial\Omega} \le 0 & \text{on } \partial\Omega \end{cases}$$

So by the 45 Weak maximum principle we have that $w\le 0$ in Ω. Similarly, taking $v = u_{n+1}-\overline{u}$ we have that v solves the same problem, so $v\le 0$ in Ω. Summarizing, one can check we have $\underline{u}\le u_n\le u_{n+1}\le\overline{u}$ for all $n\in\mathbb{N}$. So $\exists u(x) := \lim_{n\to\infty} u_n(x)$, which is a solution to the problem. It suffices to see that $u_n \xrightarrow{\mathcal{C}^{0,\alpha}} u$ because then we'd have $g(x,u_n)\xrightarrow{\mathcal{C}^{0,\alpha}} g(x,u)$ and so $u_{n+1} = (L+k)^{-1}g(x,u_n)\xrightarrow{\mathcal{C}^{2,\alpha}} (L+k)^{-1}g(x,u) = u$. But this is clear because $u_n\xrightarrow{W^{1,p}} u$ (because of the compact embedding $W^{2,p}\subset W^{1,p}$) for all $p<\infty$, and we have an embedding $W^{1,p}(\Omega)\subset\mathcal{C}^{0,\theta}(\overline{\Omega})$ for $p>d$ and for some particular $\theta = 1-\frac{d}{p}$ (see **?? ??**). Now given $\theta=\alpha$ choose p according that relation. □

Topological fixed point theorems

Theorem 53 (Brower fixed point). Let $C\subset\mathbb{R}^n$ be a closed convex bounded set and $f : C\to C$ be a continuous function. Then, f has at least a fixed point.

Theorem 54 (Schauder fixed point). Let C be a convex set in a Banach space $(E,\|\cdot\|)$ and $f : C\to C$ be a continuous function. Assume one of the following two assumptions:

- C is compact for $\|\cdot\|$.
- C is closed and bounded and f is compact.

Then, f has at least a fixed point.

Proof. We will prove it in a Hilbert space $(E,\|\cdot\|)$. Assume the first assumption. Let $\varepsilon>0$. Then, by compactness $\exists N_\varepsilon\in\mathbb{N}$ and $x_1^\varepsilon,\ldots,x_{N_\varepsilon}^\varepsilon\in C$ such that $C\subset\bigcup_{i=1}^{N_\varepsilon} B(x_i^\varepsilon,\varepsilon)$. Let $V_\varepsilon = \langle x_1^\varepsilon,\ldots,x_{N_\varepsilon}^\varepsilon\rangle$ be the linear span of these vectors and $C_\varepsilon := V_\varepsilon\cap C$. Then, $\forall x\in C$ $d(x,C_\varepsilon)<\varepsilon$ because $d(x,x_j^\varepsilon)<\varepsilon$ for some j and $x_j^\varepsilon\in C_\varepsilon$. Let $p_\varepsilon : E\to C_\varepsilon$ be the nonlinear projection on the closed convex bounded set C_ε. For all $x\in C$ we have $\|x-p_\varepsilon(x)\|\le d(x,C_\varepsilon)<\varepsilon$. Now define $f_\varepsilon : C_\varepsilon\to C_\varepsilon$ by $f_\varepsilon(x) = p_\varepsilon(f(x))$. Then, f_ε is continuous and by 53 Brower fixed point we have that f_ε has a fixed point $x_\varepsilon\in C_\varepsilon$. Thus:

$$\|f(x_\varepsilon)-x_\varepsilon\| = \|f(x_\varepsilon)-p_\varepsilon(f(x_\varepsilon))\| < \varepsilon$$

By compactness, there is a sequence $\varepsilon_n\to 0$ and $x\in C$ such that $\|x_{\varepsilon_n}-x\|\to 0$. By the continuity of f, x is a fixed point of f.
Now assume the second hypothesis. Let $K = \overline{\mathrm{Conv}(f(C))}$ be the closure of the *convex hull* of $f(C)$, that is the smallest convex set containing $f(C)$. Then, K is compact and convex. Moreover, $K\subseteq C$ since $f(C)\subseteq C$, C is convex and closed. Furthermore, $f(K)\subseteq f(C)\subseteq K$. So f restricts to a continuous function $f|_K : K\to K$. By the first assumption, $f|_K$ has a fixed point $x\in K\subseteq C$. □

Theorem 55 (Schaefer fixed point). Let $(E,\|\cdot\|)$ be Banach and $f : E\to E$ be continuous and compact. Suppose that $\exists M>0$ such that $\forall(\lambda,u)\in[0,1]\times E$ with $u=\lambda f(u)$ we have $\|u\|<M$. Then, f has at least a fixed point, that lies in $\overline{B(0,M)}$.

Proof. Take $C = \overline{B(0,M)}$. For $x\in C$, let:

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x\in C \\ M\frac{f(x)}{\|f(x)\|} & \text{if } \|f(x)\|>M \end{cases}$$

An easy check shows that $f : C\to C$ is continuous and compact. So, by 54 Schauder fixed point $\exists x_*\in C$ such that $x_* = \tilde{f}(x_*)$. If $\|x_*\| = \|f(x_*)\|>M$, then:

$$\|x_*\| = M\frac{\|f(x_*)\|}{\|f(x_*)\|} = M$$

which is absurd. So $f(x_*) = x_*$. □

5. | Variational methods for nonlinear elliptic PDEs

In this section we will solve a PDE $Lu = f(x, u, \nabla u)$ by minimizing a certain functional under some constraints.

Linear case

Proposition 56 (Without constraints). Consider the problem:

$$\begin{cases} Lu := -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) + cu = f \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

with L elliptic, $a_{ij}, c \in L^\infty(\Omega)$ with $a_{ij} = a_{ji}$, $c \geq 0$, and $f \in L^2(\Omega)$. Then, the problem has a unique weak solution $u \in H_0^1(\Omega)$, and it minimizes the functional:

$$I(u) = \frac{1}{2}\beta(u, u) - \int_\Omega fu$$

where $\beta(u, v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\partial_i u \partial_j v + \int_\Omega cuv$.

Proof. By **?? ??** (using the scalar product β, which is positive definite because $c \geq 0$) we have that this problem has a unique weak solution $u_f \in H_0^1(\Omega)$. Moreover, it minimizes the functional I. Indeed, we have:

$$I(u) - I(u_f) = \beta(u_f, u - u_f) - \int f(u - u_f) + \\ + \frac{1}{2}\beta(u - u_f, u - u_f) = \frac{1}{2}\beta(u - u_f, u - u_f) > 0$$

if $u \neq u_f$. □

Lemma 57. Let X be a Banach space and $\Phi : X \to \mathbb{R}$ be continuous and convex, then it is weakly sequentially lower semicontinuous, that is, if $u_n \rightharpoonup u$ in X, then $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$.

Theorem 58. Let $(X, \|\cdot\|)$ be a reflexive Banach space and $\Phi : X \to \mathbb{R}$ be continuous, convex and such that $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$. Then, Φ has a minimizer. This minimizer is unique if Φ is strictly convex.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a minimizing sequence. Then, $\sup_{n\in\mathbb{N}} \Phi(u_n) < \infty$, so by the coercivity property of Φ we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded, and so $\{u_n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ with limit $u \in X$. By Theorem 57 we have:

$$\Phi(u) \leq \lim_{k\to\infty} \Phi(u_{n_k}) = \inf_{u\in X} \Phi(u)$$

But $\Phi(u) \geq \inf_{u\in X} \Phi(u)$, so u is a minimizer. □

Theorem 59 (With constraints). Consider the problem of Eq. (3). We know that L is invertible with inverse $L^{-1} : L^2(\Omega) \to H_0^1(\Omega)$. But $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ (see **?? ??**), so:

$$\begin{aligned} K : L^2(\Omega) &\longrightarrow L^2(\Omega) \\ f &\longmapsto L^{-1}f \end{aligned}$$

is compact. Thus, a Hilbert basis (u_n) of K with $Ku_n = \mu_n u_n$, $\mu_n > 0$ with $\mu_n \to 0$ as $n \to \infty$ exists (we may assume $\mu_n \searrow 0$). Thus, $Lu_n = \lambda_n u_n$ with $\lambda_n = \frac{1}{\mu_n} \nearrow +\infty$. Then:

$$\lambda_1 = \min_{\substack{u\in H_0^1(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \langle Lu, u\rangle_{H^{-1}\times H_0^1} = \min_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\langle Lu, u\rangle_{H^{-1}\times H_0^1}}{\|u\|_{L^2(\Omega)}^2}$$

And:

$$\begin{aligned} \lambda_k &= \min_{\substack{u\in H_0^1(\Omega) \\ u\in\langle u_1,\ldots,u_{k-1}\rangle^{\perp_{L^2}} \\ \|u\|_{L^2(\Omega)}=1}} \langle Lu, u\rangle_{H^{-1}\times H_0^1} \\ &= \min_{\substack{u\in H_0^1(\Omega)\setminus\{0\} \\ u\in\langle u_1,\ldots,u_{k-1}\rangle^{\perp_{L^2}}}} \frac{\langle Lu, u\rangle_{H^{-1}\times H_0^1}}{\|u\|_{L^2(\Omega)}^2} \\ &= \min_{\substack{V \text{ subspace of } H_0^1(\Omega) \\ \dim(V)=k}} \max_{\substack{u\in V\setminus\{0\} \\ \|u\|_{L^2(\Omega)}=1}} \langle Lu, u\rangle_{H^{-1}\times H_0^1} \\ &= \max_{\substack{W \text{ subspace of } H_0^1(\Omega) \\ \operatorname{codim}(W)=k-1}} \min_{\substack{u\in W\setminus\{0\} \\ \|u\|_{L^2(\Omega)}=1}} \langle Lu, u\rangle_{H^{-1}\times H_0^1} \end{aligned}$$

Proof. We only prove some of them. Recall that $H_0^1 = \overline{\bigoplus_{n\in\mathbb{N}}\langle u_n\rangle}^{H_0^1}$ and $L^2 = \overline{\bigoplus_{n\in\mathbb{N}}\langle u_n\rangle}^{L^2}$. Take $u \in H_0^1(\Omega) \setminus \{0\}$ and write $u = \sum_{n\in\mathbb{N}} \alpha_n u_n$, which converges in both L^2 and H_0^1. We have:

$$\langle Lu, u\rangle_{H^{-1}\times H_0^1} = \sum_{n\in\mathbb{N}} \lambda_n {\alpha_n}^2 \geq \lambda_1 \sum_{n\in\mathbb{N}} {\alpha_n}^2 = \lambda_1 \|u\|_{L^2(\Omega)}^2$$

So the first equality holds since the lower bound is attained by $u = u_1$. Now take $u \perp_{L^2} \langle u_1, \ldots, u_{k-1}\rangle$. Then, $\alpha_1 = \cdots = \alpha_{k-1} = 0$ and so:

$$\langle Lu, u\rangle_{H^{-1}\times H_0^1} = \sum_{n\geq n} \lambda_n {\alpha_n}^2 \geq \lambda_n \sum_{n\geq n} {\alpha_n}^2 = \lambda_n \|u\|_{L^2(\Omega)}^2$$

So the third equality holds since the lower bound is attained by $u = u_k$. □

Nonlinear case without constraints

Definition 60. We say that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is *Carathéodory* if f is measurable in x and continuous in t.

Theorem 61 (Superposition operator). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying the growth condition $|f(x,t)| \leq C(1 + |t|^\theta)$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $\theta \geq 1$. Then, for any $\theta \leq p < \infty$, the *superposition operator*

$$\begin{aligned} \Phi_f : L^p(\Omega) &\longrightarrow L^{p/\theta}(\Omega) \\ u &\longmapsto f(\cdot, u(\cdot)) \end{aligned}$$

is continuous.

Proof. Let $(u_n), u \in L^p(\Omega)$ be such $u_n \xrightarrow{L^p} u$. We will prove that $v_n := f(\cdot, u_n(\cdot))$ is precompact in $L^{p/\theta}(\Omega)$ (that is, any subsequence v_{n_k} has a convergent subsequence) and has only one limit point, which is $v := f(\cdot, u(\cdot))$. Take a subsequence (v_{n_k}) of (v_n). We know that $u_{n_k} \xrightarrow{L^p} u$. We know that in this case there exists a subsequence

$u_{n_{k_j}} \overset{\text{a.e.}}{\to} u$ and $\left|u_{n_{k_j}}\right| \leq h$ with $h \in L^p$. Then, by the continuity of f, $v_{n_{k_j}} \overset{\text{a.e.}}{\to} v$ and by the growth condition, $v_{n_{k_j}} \leq C(1+|h(x)|^\theta) \in L^{p/\theta}$. So, by ?? ??, $v_{n_{k_j}} \overset{L^{p/\theta}}{\longrightarrow} v$. □

Proposition 62. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying the growth condition $|f(x,t)| \leq C(1+|t|^\theta)$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $1 \leq \theta \leq 2^*$, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$, (if $d \geq 3$) and $1 \leq \theta < \infty$ (if $d = 2$). Then, the superposition operator

$$\begin{aligned} \Phi_f : H^1(\Omega) &\longrightarrow L^{p/\theta}(\Omega) \\ u &\longmapsto f(\cdot, u(\cdot)) \end{aligned}$$

is continuous for all $\theta \leq p \leq 2^*$ (if $d \geq 3$) and $\theta \leq p < \infty$ (if $d = 2$). Moreover, Φ_f is compact if $\theta < p < 2^*$ (if $d \geq 3$) or $\theta < p < \infty$ (if $d = 2$).

Definition 63. Let X, Y be normed spaces and $T : X \to Y$. We say that T is *Fréchet differentiable* at $u \in X$ if $\exists L \in \mathcal{L}(X,Y)$ such that:

$$\lim_{\substack{h \to 0 \\ h \in X\setminus\{0\}}} \frac{\|T(u+h) - T(u) - Lh\|}{\|h\|} = 0$$

In this case, we say that L is the *Fréchet derivative* of T at u. We denote it by $\mathrm{d}T(u)$.

Definition 64. Let X, Y be normed spaces and $T : X \to Y$. We say that T is *Gâteaux differentiable* at $u \in X$ if $\exists L \in \mathcal{L}(X,Y)$ such that $\forall h \in X$:

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \frac{T(u+th) - T(u)}{t} = Lh$$

In this case, we say that L is the *Gâteaux derivative* of T at u. We denote it by $DT(u)$.

Lemma 65. Let X, Y be normed spaces and $T : X \to Y$. Then, if the Fréchet and Gâteaux derivatives of T at u exist, they are unique. Moreover we have:

- If T is Fréchet differentiable at u, then it is Gâteaux differentiable at u and both differentials coincide.
- If T is Fréchet differentiable at u, T is continuous at u.
- If T is Gâteaux differentiable at $u \in U$ and the map

  $$\begin{aligned} U &\longrightarrow \mathcal{L}(X,Y) \\ u &\longmapsto DT(u) \end{aligned}$$

  is continuous, then T is Fréchet differentiable at u and $\mathrm{d}T(u) = DT(u)$.

Proposition 66. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying the growth condition $|f(x,t)| \leq C(1+|t|^\theta)$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $1 \leq \theta \leq \frac{d+2}{d-2}$ (if $d \geq 3$) and $1 \leq \theta < \infty$ (if $d = 2$). Let $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$ and consider the functional

$$\begin{aligned} \Psi : H^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega F(x,u(x))\,\mathrm{d}x \end{aligned}$$

Then, Ψ is well-defined on H^1, it is of class $\mathcal{C}^1$ and its differential is given by:

$$\mathrm{d}\Psi(u)\,h = \int_\Omega f(x,u(x))h(x)\,\mathrm{d}x$$

Proof. We will assume $d \geq 3$, the case $d = 2$ is similar. We have that $|F(x,t)| \leq \tilde{C}(1+|t|^{\theta+1})$. Note that $2 \leq \theta+1 \leq 2^*$, so taking $p = \theta+1$ in Theorem 62 we have that

$$\begin{aligned} \Phi_F : H^1(\Omega) &\longrightarrow L^1(\Omega) \\ u &\longmapsto \textstyle\int_0^u f(x,s)\,\mathrm{d}s \end{aligned}$$

is continuous. Thus, Ψ is well-defined and continuous. Let $h \in H^1(\Omega)$, $t \in (-1,1)$ and consider $g(t) = \Psi(u+th)$. We have:

$$\begin{aligned} \left|\frac{\partial}{\partial t} F(x, u+th)\right| &= |f(x,u+th)h(x)| \\ &\leq C\left(1 + (|u|+|h|)^\theta\right)|h| =: H \end{aligned}$$

By Theorem 62, we know that $(|u|+|h|)^\theta \in L^{2^*/\theta}$ and $|h| \in L^{2^*}$, so by ?? ?? (since $\frac{1}{2^*} + \frac{\theta}{2^*} = \frac{\theta+1}{2^*} \leq 1$) we have that $H \in L^1(\Omega)$. Thus, by ?? we have that g is differentiable and:

$$g'(0) = \int_\Omega f(x,u(x))h(x)\,\mathrm{d}x$$

So $\exists D\Psi(u)$ and

$$D\Psi(u)h = \int_\Omega f(x,u(x))h(x)\,\mathrm{d}x = \langle \Phi_f(u), h\rangle_{L^{p'} \times L^p}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $2 \leq p \leq 2^*$. To prove that $\Psi \in \mathcal{C}^1$, it suffices to show that $\Phi_f \in \mathcal{C}(H^1, L^{p'})$. We have $f(x,t) \leq C(1+|t|^\theta)$, so $\Phi_f : H^1 \to L^{p/\theta}$ is continuous for $\frac{d+2}{d-2} \leq p \leq 2^*$. If $p' \leq p/\theta$, since Ω is bounded, $L^{p/\theta} \hookrightarrow L^{p'}$ is continuous by ?? ??. An easy check shows that if we take $p = 2^*$, and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, these inequality hold. □

Theorem 67 (Without constraints). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying

- $|f(x,t)| \leq C(1+|t|^\theta)$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $1 \leq \theta \leq \frac{d+2}{d-2}$ (if $d \geq 3$) and $1 \leq \theta < \infty$ (if $d = 2$).
- $f(x,t)\operatorname{sgn}(t) \leq C'$ $\forall (x,t) \in \Omega \times \mathbb{R}$.

Let $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$ and for $u \in H_0^1(\Omega)$ consider the functional:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)\,\mathrm{d}x$$

Then, $I \in \mathcal{C}^1(H_0^1, \mathbb{R})$, it is bounded from below and there is $\underline{u} \in H_0^1(\Omega)$ such that $I(\underline{u}) = \min\limits_{u \in H_0^1(\Omega)} I(u)$. Moreover, $\underline{u}$ is a weak solution to the problem:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

Proof. We saw in Theorem 66 that the map $u \mapsto \int_\Omega F(x, u(x))\,\mathrm{d}x$ is of class $\mathcal{C}^1$ and its differential is given by $h \mapsto \int_\Omega f(x, u(x))h(x)\,\mathrm{d}x$. Moreover:

$$\int_\Omega |\boldsymbol{\nabla}(u+h)|^2 - \int_\Omega |\boldsymbol{\nabla} u|^2 = 2\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} h + \mathrm{o}\left(\|h\|_{H_0^1}\right)$$

Since, $u \mapsto \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} h$ is linear and continuous, we have that I is of class $\mathcal{C}^1$ and its differential is given by:

$$\mathrm{d}I(u)\,h = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} h - \int_\Omega f(x, u(x))h(x)\,\mathrm{d}x$$

Integrating the hypothesis on f, we deduce that:

- $|F(x,t)| \leq \tilde{C}(1 + |t|^{\theta+1})$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $1 \leq \theta \leq 2^*$ (if $d \geq 3$) and $1 \leq \theta < \infty$ (if $d = 2$).
- $F(x,t) \leq C'|t|$ $\forall (x,t) \in \Omega \times \mathbb{R}$.

Thus:

$$I(u) \geq \frac{1}{2}\|\boldsymbol{\nabla} u\|_{L^2}{}^2 - C'\int |u| \geq \frac{1}{2}\|\boldsymbol{\nabla} u\|_{L^2}{}^2 - C''\|u\|_{L^2} \geq$$
$$\geq \frac{1}{2}\|\boldsymbol{\nabla} u\|_{L^2}{}^2 - \bar{C}\|\boldsymbol{\nabla} u\|_{L^2} = -\frac{\bar{C}^2}{2} + \frac{1}{2}\left(\|\boldsymbol{\nabla} u\|_{L^2} - \bar{C}\right)^2$$

where we used **?? ??** in the third inequality. So $\inf_{u \in H_0^1(\Omega)} I(u) \geq -\frac{\bar{C}^2}{2} > -\infty$. Thus, I is bounded from below. Moreover, I is *coercive* in the sense that $\lim_{\|u\|_{H_0^1} \to \infty} I(u) = +\infty$.

Now take a minimizing sequence (u_n) for I. Then, $\sup_{n \in \mathbb{N}} I(u_n) < \infty$ and by the coercivity property we have $\sup_{n \in \mathbb{N}} \|u_n\|_{H_0^1} < \infty$. After extraction, we have $u_n \overset{H_0^1}{\rightharpoonup} \underline{u}$ for some $\underline{u} \in H_0^1(\Omega)$ and using Theorem 62 we have a compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 < p < 2^*$, so $u_n \overset{L^p}{\to} \underline{u}$. Using the growth property and taking $p = \theta + 1 < 2^*$ we conclude that $F(\cdot, u_n(\cdot)) \overset{L^{p/\theta}}{\to} F(\cdot, \underline{u}(\cdot))$. So, $\int_\Omega F(x, u_n(x))\,\mathrm{d}x \to \int_\Omega F(x, \underline{u}(x))\,\mathrm{d}x$. On the other hand, since $u_n \overset{H_0^1}{\rightharpoonup} \underline{u}$, we have that $\|\underline{u}\|_{H_0^1} \leq \liminf_{n\to\infty} \|u_n\|_{H_0^1}$. Thus, if $m := \min_{u \in H_0^1(\Omega)} I(u)$, we have:

- $I(\underline{u}) \leq \liminf_{n\to\infty} I(u_n) = m$.
- $I(\underline{u}) \geq m$ because $\underline{u} \in H_0^1(\Omega)$.

So $\underline{u}$ is a minimizer for I. Moreover, this implies that $\int_\Omega |\boldsymbol{\nabla} u_n|^2 \to \int_\Omega |\boldsymbol{\nabla} \underline{u}|^2$, so $u_n \overset{H_0^1}{\to} \underline{u}$. Since, $\underline{u}$ is a minimizer for I, we have that the map $t \mapsto I(\underline{u} + th)$ has a minimum at $t = 0$. Thus, $\forall h \in H_0^1(\Omega)$:

$$\mathrm{d}I(\underline{u})\,h = \int_\Omega \boldsymbol{\nabla} \underline{u} \cdot \boldsymbol{\nabla} h - \int_\Omega f(x, \underline{u}(x))h(x)\,\mathrm{d}x = 0$$

So $\underline{u}$ is a weak solution to the problem of Eq. (4). □

Theorem 68 (Bootstrap). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying the growth condition $|f(x,t)| \leq C(1 + |t|^\theta)$ $\forall (x,t) \in \Omega \times \mathbb{R}$ with $\theta \geq 1$. Assume $\partial\Omega \in \mathcal{C}^2$ and $1 \leq p < \infty$. We have an isomorphism $-\Delta : W^{2,p}(\Omega) \cap W^{1,p} \to L^p(\Omega)$, meaning that for each $g \in L^p(\Omega)$ there exists a unique strong solution $\underline{u}$ of

$$\begin{cases} -\Delta u = g & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

in $W^{2,p}$. Then, $\underline{u} \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ and $\underline{u} \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$.

Proof. Define $g(x) = \Phi_f(\underline{u})(x) = f(x, \underline{u}(x))$. We have that $\underline{u} \in H_0^1(\Omega)$ (because it is a weak solution), so by Theorem 62 $\underline{u} \in L^{2^*}$. Thus, $g \in L^{p_1}$ with $p_1 = \frac{2^*}{\theta}$. So $\underline{u} \in W^{2,p_1}$ and thus $\underline{u} \in L^{q_1}$ with $\frac{1}{q_1} = \frac{1}{p_1} - \frac{2}{d}$ (critical Sobolev embedding). Hence, we get $g \in L^{p_2}$ with $p_2 = \frac{q_1}{\theta}$. We can repeat this process as long as $p_n < \frac{d}{2}$. We study the sequence $a_n = \frac{1}{p_n}$. In the process we have that if $a_n > \frac{2}{d}$, then:

$$a_{n+1} = \theta a_n - \frac{2}{d}\theta$$

with $a_1 = \frac{\theta}{2} - \frac{\theta}{d}$. The fixed point is $r := \frac{2\theta}{d(\theta - 1)}$. So:

$$a_n = r + \theta^n(a_1 - r)$$

But an easy check shows that $a_1 - r < 0$, so $a_n \to -\infty$, which is a contradiction since $a_n > \frac{2}{d}$. Thus, the process stops after a finite number of times, and thus, we get $\underline{u} \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ and $\underline{u} \in \bigcap_{1 \leq p < \infty} W^{2,p}(\Omega)$. □

Nonlinear case with constraints

Theorem 69 (Lagrange multipliers). Let E be a normed space and $I, J \in \mathcal{C}^1(E, \mathbb{R})$. Assume that:

- For some $\mu \in \mathbb{R}$ and all $u \in E$ we have that if $J(u) = \mu$, then $\mathrm{d}J(u) \neq 0$.
- $\exists \underline{u} \in E$ such that $J(\underline{u}) = \mu$ and $I(\underline{u}) = \min_{\substack{u \in E \\ J(u) = \mu}} I(u)$.

Then, $\exists \lambda \in \mathbb{R}$, called *Lagrange multiplier*, such that $\mathrm{d}I(\underline{u}) = \lambda\,\mathrm{d}J(\underline{u})$.

Theorem 70 (Lagrange multipliers in several variables). Let E be a normed space and $I, J_1, \ldots, J_m \in \mathcal{C}^1(E, \mathbb{R})$. Assume that:

- For some $\mu_1, \ldots, \mu_m \in \mathbb{R}$ and all $u \in E$ we have that if $J_i(u) = \mu_i$ for all $i = 1, \ldots, m$, then $\mathrm{d}J_1(u), \ldots, \mathrm{d}J_m(u)$ are linearly independent in E^*.
- $\exists \underline{u} \in E$ such that $J_i(\underline{u}) = \mu_i$ for all $i = 1, \ldots, m$ and $I(\underline{u}) = \min_{\substack{u \in E \\ J_i(u) = \mu_i\ \forall i}} I(u)$.

Then, $\exists \lambda_1, \ldots, \lambda_m \in \mathbb{R}$, called *Lagrange multipliers*, such that:

$$\mathrm{d}I(\underline{u}) = \lambda_1\,\mathrm{d}J_1(\underline{u}) + \cdots + \lambda_m\,\mathrm{d}J_m(\underline{u})$$

Proposition 71 (**Aplication**)**.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function defined by $f(x,t) = |t|^\theta \operatorname{sgn}(t)$, with $1 \le \theta \le \frac{d+2}{d-2}$ and define the following functionals in $E = H_0^1(\Omega)$:

$$I(u) = \frac{1}{2}\int_\Omega |\boldsymbol{\nabla} u|^2 \qquad J(u) = \int_\Omega F(x,u)\,\mathrm{d}x$$

with $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$. Then, $\tilde{u} = \underline{u}/t$ is a weak solution to the problem:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $\underline{u}$ is the minimizer of the problem $\min\limits_{\substack{u \in H_0^1(\Omega) \\ J(u)=1}} I(u)$.

Proof. We will solve first a much simpler problem:

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

with $\lambda > 0$. Denote by m the minimizer of I under $J(u) = 1$. Since $F(x,t) = \frac{|t|^{\theta+1}}{\theta+1}$, under $J(u) = 1$ we have that $\|u\|_{L^{\theta+1}}{}^{\theta+1} = \theta + 1$. Now, since $\theta + 1 \le 2^*$, we have a continuous embedding $H_0^1(\Omega) \hookrightarrow L^{\theta+1}(\Omega)$, so $\frac{1}{2}\|\boldsymbol{\nabla} u\|_{L^2}{}^2 \ge C\|u\|_{L^{\theta+1}}{}^2 \ge K > 0$. Thus, $m \ge K > 0$. Now, take a minimizing sequence (u_n) for I. Since u_n is bounded in H_0^1, after extraction we have $u_n \overset{H_0^1}{\rightharpoonup} \underline{u}$ for some $\underline{u} \in H_0^1(\Omega)$. Moreover, $u_n \overset{L^{\theta+1}}{\to} \underline{u}$ by compact embedding. Thus, $1 = J(u_n) \to J(\underline{u})$. So $J(\underline{u}) = 1$ and since $m = \liminf\limits_{n\to\infty} I(u_n) \ge I(\underline{u})$ and $I(\underline{u}) \ge m$, we have that $\underline{u}$ is a minimizer for I under $J(u) = 1$. Now, we know that I, J are of class $\mathcal{C}^1$ on $H_0^1(\Omega)$ and

$$\mathrm{d}J(u)\,h = \int_\Omega |u|^{\theta-1} uh$$

If $J(u) = 1$, then $\mathrm{d}J(u)\,u = \theta + 1 \ne 0$. So there is a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\mathrm{d}I(u) = \lambda\,\mathrm{d}J(u)$, that is:

$$\int_\Omega \boldsymbol{\nabla}\underline{u} \cdot \boldsymbol{\nabla} h = \lambda \int_\Omega |\underline{u}|^{\theta-1}\underline{u}h$$

Whence $\underline{u}$ is a weak solution of Eq. (6). Note that taking $h = \underline{u}$ we deduce that $\lambda > 0$.
Now take $t = \lambda^{-\frac{1}{\theta-1}} > 0$ and $\tilde{u} = t^{-1}\underline{u}$. Then:

$$-\Delta(\tilde{u}t) = \lambda t^{\theta-1}|\tilde{u}|^{\theta-1}\tilde{u}t \iff -\Delta\tilde{u} = |\tilde{u}|^{\theta-1}\tilde{u} = f(x,\tilde{u})$$

So $\tilde{u}$ is a weak solution in $H_0^1(\Omega)$ of Eq. (5) and $\tilde{u} \ne 0$ because $\frac{1}{\theta+1}\int_\Omega |\tilde{u}|^{\theta+1} = \lambda^{\frac{\theta+1}{\theta-1}} > 0$. □

Remark. In general, it suffices to have $f : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodory with:

- $|f(x,t)| \le C(1 + |t|^\theta)\ \forall (x,t) \in \Omega \times \mathbb{R}$ with $1 \le \theta \le \frac{d+2}{d-2}$ (if $d \ge 3$) and $1 \le \theta < \infty$ (if $d = 2$).
- $f(x,t)t \ge C' \min\limits_{2 \le \alpha,\beta \le \theta+1} \{|t|^\alpha, |t|^\beta\}\ \forall (x,t) \in \Omega \times \mathbb{R}$.

Remark. If J is not homogeneous we cannot proceed as in the proof. But in this case we use the *Nehari manifold method*.

Proposition 72 (**Nehari manifold method**)**.** Let f be as in 71 Aplication with the additional assumptions that:

- $t \mapsto f(\cdot,t)$ is $\mathcal{C}^1$ with a growth condition $\left|\frac{\partial f}{\partial t}\right| \le C(1 + |t|^{\theta-1})$.
- $f(x,t)t < \partial_t f(x,t)t^2\ \forall (x,t) \in \Omega \times \mathbb{R}^*$.

Let $\mathcal{N} := \{u \in H_0^1(\Omega) \setminus \{0\} : J(u) = 0\}$, where:

$$I(u) = \frac{1}{2}\int_\Omega |\boldsymbol{\nabla} u|^2 - \int_\Omega F(x,u)\,\mathrm{d}x$$
$$J(u) = \mathrm{d}I(u)\,u = \int_\Omega |\boldsymbol{\nabla} u|^2 - \int_\Omega f(x,u)u$$

Then, if $\underline{u} \in \mathcal{N}$ is a minimizer of I under $J(u) = 0$, then $\mathrm{d}I(\underline{u}) = 0$ and so $\underline{u}$ is a weak solution to Eq. (5).

Proof. We have that:

$$\mathrm{d}J(u)\,h = 2\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} h - \int_\Omega [\partial_u f(x,u)u + f(x,u)]h$$

Thus, if $u \in \mathcal{N}$, we have:

$$\mathrm{d}J(u)\,u = \int_\Omega [f(x,u)u - \partial_u f(x,u)u^2] < 0$$

because $J(u) = 0$ and at the end we used one of the extra hypothesis on f. Now assume $\underline{u} \in \mathcal{N}$ and $I(\underline{u}) = \min\limits_{\substack{u \in H_0^1(\Omega) \\ J(u)=0}} I(u)$. Then, $\exists \lambda \in \mathbb{R}$ such that $\mathrm{d}I(\underline{u}) = \lambda\,\mathrm{d}J(\underline{u})$. Thus:

$$\int_\Omega [\boldsymbol{\nabla}\underline{u} \cdot \boldsymbol{\nabla} h - f(x,\underline{u})h] = \lambda \int_\Omega [\partial_u f(x,\underline{u})\underline{u} + f(x,\underline{u}) - f(x,\underline{u})]h$$

Moreover, $\int_\Omega |\boldsymbol{\nabla}\underline{u}|^2 = \int_\Omega f(x,\underline{u})\underline{u}$. So taking $h = \underline{u}$ we get:

$$0 = \lambda \int_\Omega [f\underline{u} - \partial_u f\underline{u}^2]$$

which implies $\lambda = 0$ because of the extra hypothesis on f. □

Mountain pass method

Our goal in this section is again find a nonzero weak solution in $H_0^1(\Omega)$ to Eq. (5).

Definition 73. Let E be a Banach space and $I \in \mathcal{C}^1(E,\mathbb{R})$. We say that I satisfies the *Palais-Smale condition at level c* if every sequence (u_n) in E, such that $I(u_n) \to c$ and $\mathrm{d}I(u_n) \to 0$ in E^*, has a convergent subsequence (that is, is precompact).

Theorem 74 (Ambrosetti-Rabinowitz theorem). Let E be a Banach space and $I \in \mathcal{C}^1(E, \mathbb{R})$. Assume that $\exists a \neq b \in E$ such that

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > \max\{I(a), I(b)\}$$

with

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = a, \gamma(1) = b\}$$

Then, there is a sequence (u_n) in E such that $I(u_n) \to c$ and $\mathrm{d}I(u_n) \to 0$ in E^*. Such a sequence is called a *Palais-Smale sequence.*

Corollary 75 (Mountain pass theorem). Let E be a Banach space and $I \in \mathcal{C}^1(E, \mathbb{R})$. Assume that $\exists a \neq b \in E$ such that

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > \max\{I(a), I(b)\}$$

with

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = a, \gamma(1) = b\}$$

If, moreover, I satisfies the Palais-Smale condition at level c, then $\exists u_* \in E$ such that $I(u_*) = c$ and $\mathrm{d}I(u_*) = 0$.

Proposition 76. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory satisfying:

- $f(x,t)t \geq pF(x,t)\ \forall (x,t) \in \Omega \times \mathbb{R}$ (*superquadradicity condition*).
- $f(x,t)t \leq \overline{C}|t|^{p_1}$ for $|t| \geq 1$.
- $f(x,t)t \geq \underline{C}|t|^{p_2}$ for $|t| \leq 1$.

with $2 < p, p_1, p_2 < 2^*$ and $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$. Consider the functional:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)\,\mathrm{d}x$$

Then, $\exists \underline{u} \in H_0^1(\Omega)$ such that $I(\underline{u}) = \min_{u \in H_0^1(\Omega)} I(u)$ and $\underline{u}$ is a weak solution to Eq. (5).

Proof. First of all, note that the thrid hypothesis on f implies that $F(x,t) \geq 0$ for $|t| \leq 1$. From the superquadradicity condition, for $t > 0$, the function $|t|^{-p}F(x,t)$ is nondecreasing (the derivative is nonnegative). So, for $0 \leq t \leq 1$ we have $F(x,t) \leq |t|^p F(x,1)$. Similarly, for $-1 \leq t \leq 0$ the function is nonincreasing and so we have $F(x,t) \leq |t|^p F(x,-1)$. Using the upper estimate we get, for $|t| \geq 1$, $F(x,t) \leq \overline{\overline{C}}|t|^{p_1}$ and so $|F(x,t)| \leq C'(|t|^p + |t|^{p_1})\ \forall t$. So:

$$\int_\Omega F(x,u) \leq C'\left(\|u\|_{L^p}^p + \|u\|_{L^{p_1}}^{p_1}\right)$$
$$\leq C''\left(\|\nabla u\|_{L^2}^p + \|\nabla u\|_{L^2}^{p_1}\right)$$

by **?? ??** and ?????. And thus:

$$I(u) \geq \|\nabla u\|_{L^2}^2\left(\frac{1}{2} - C''\left[\|\nabla u\|_{L^2}^{p-2} + \|\nabla u\|_{L^2}^{p_1-2}\right]\right)$$
$$\geq \frac{1}{4}\|u\|_{H_0^1}^2 \tag{7}$$

for $\|u\|_{H_0^1} \leq r$ with $r > 0$ small enough. Now, take $u_1 \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda > 0$ to be chosen later. From the previous reasoning, we have $F(x,t) \geq F(x,1)|t|^p$ for $t \geq 1$ and $F(x,t) \geq F(x,-1)|t|^p$ for $t \leq -1$. So, $F(x,t) \geq K|t|^p \geq 0$ for some $K > 0$ and all $|t| \geq 1$. Now, since $u_1 \neq 0$ $\exists \varepsilon > 0$ such that $\int_\Omega |u_1|^p \mathbf{1}_{\{|u_1| \geq \varepsilon\}} > 0$. So for $\lambda \geq \frac{1}{\varepsilon}$ we have:

$$I(\lambda u_1) \leq \frac{\lambda^2}{2}\|\nabla u_1\|_{L^2}^2 - \int_\Omega F(x, \lambda u_1)\mathbf{1}_{\{|u_1| \geq \varepsilon\}} \leq$$
$$\leq \frac{\lambda^2}{2}\|\nabla u_1\|_{L^2}^2 - K\lambda^p\int_\Omega |u_1|^p \mathbf{1}_{\{|u_1| \geq \varepsilon\}} =$$
$$= A\lambda^2 - B\lambda^p \xrightarrow{\lambda \to \infty} -\infty$$

where the first inequality follows from the fact that $F(x,t) \geq 0$ for $|t| \leq 1$. So we may choose $\lambda = \lambda_1 > 0$ such that $I(\lambda_1 u_1) \leq 0$ and given the previous $r > 0$ we choose u_1 with $\|\lambda_1 u_1\|_{H_0^1} > r$. Now let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \lambda_1 u_1\}$$

and define $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$. Take $\gamma \in \Gamma$. Since $\|\gamma(0)\|_{H_0^1} = 0$ and $\|\gamma(1)\|_{H_0^1} > r$, $\exists t_0 \in (0,1)$ such that $\|\gamma(t_0)\|_{H_0^1} = r$. So by Eq. (7) we have:

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \frac{r^2}{4} > 0 = \max\{I(0), I(\lambda_1 u_1)\}$$

In order to use 75 Mountain pass theorem it's missing to check that I satisfies the Palais-Smale condition at level c. Let (u_n) be a Palais-Smale sequence at level c. We then have:

$$\begin{cases} I(u_n) \to c \\ \mathrm{d}I(u_n) \xrightarrow{H^{-1}} 0 \end{cases}$$

The second equation implies that $\mathrm{d}I(u_n)\, u_n \to 0$ and thus:

$$\begin{cases} \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_\Omega F(x,u_n)\,\mathrm{d}x = c + \mathrm{o}(1) \\ \int_\Omega |\nabla u_n|^2 - \int_\Omega f(x,u_n)u_n\,\mathrm{d}x = \mathrm{o}\left(\|u_n\|_{H_0^1}\right) \end{cases}$$

From here subtracting the first equation (multiplied by p) to the second one, we have:

$$\left(1 - \frac{p}{2}\right)\int_\Omega |\nabla u_n|^2\,\mathrm{d}x + \int_\Omega [pF(x,u_n) - f(x,u_n)u_n]\,\mathrm{d}x =$$
$$= -pc + \mathrm{o}(1) + \mathrm{o}\left(\|u_n\|_{H_0^1}\right)$$

By hypothesis the second term is negative, so:

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \leq c + \mathrm{o}(1) + \mathrm{o}\left(\|u_n\|_{H_0^1}\right)$$

So $\exists K > 0$ such that $\|u_n\|_{H_0^1} \leq K\ \forall n \in \mathbb{N}$. So after extracting a subsequence we have $u_n \overset{H_0^1}{\rightharpoonup} u$ for some

$u \in H_0^1(\Omega)$ and by compact embedding $u_n \xrightarrow{L^p} u$ for all $1 \leq p < 2^*$. But f is Carathéodory with growth condition $|f(x,t)| \leq C(1+|t|^\theta)$ with $1 \leq \theta = p_1 - 1 < \frac{d+2}{d-2}$. So $f(x,u_n) \xrightarrow{L^q} f(x,u)$ for all $1 \leq q < \frac{2^*}{\theta}$. Now, by duality, the continuous embedding $H_0^1 \hookrightarrow L^{2^*}$ gives $L^{\hat{q}} \hookrightarrow H^{-1}$ with $\frac{1}{\hat{q}} + \frac{1}{2^*} = 1$. An easy computation shows that:

$$\hat{q} = \frac{2d}{d-2} \implies \theta\hat{q} < 2^* \implies \hat{q} < \frac{2^*}{\theta}$$

So $f(x,u_n) \xrightarrow{H^{-1}} f(x,u)$. Now since

$$\mathrm{d}I(u_n)\,h = \langle -\Delta u_n - f(x,u_n), h\rangle_{H^{-1}\times H_0^1}$$

we have $-\Delta u_n = f(x,u_n) + r_n$ with $r_n = \mathrm{d}I(u_n) \xrightarrow{H^{-1}} 0$. Thus, $-\Delta u_n \xrightarrow{H^{-1}} f(x,u)$ (and so $u_n \xrightarrow{H_0^1} (-\Delta)^{-1} f(x,u)$) and since $u_n \xrightharpoonup{H_0^1} u$ implies $-\Delta u_n \xrightharpoonup{H^{-1}} -\Delta u$, we have $-\Delta u = f(x,u)$. This implies that in fact $u_n \xrightarrow{H_0^1} u$ and so $I(u) = \lim_{n\to\infty} I(u_n) = c$. $\square$