Introduction to control theory

1. | Control theory in ODEs

Stability

Definition 1. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If, moreover, $\lim_{s\to\infty} \alpha(s) = \infty$, then α is said to be of *class* $\mathcal{K}^\infty$.

Definition 2. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class* $\mathcal{KL}$ if it is continuous, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

Remark. An example of a function class $\mathcal{K}$ not in $\mathcal{K}^\infty$ is for example $\alpha(s) = \arctan(s)$. Examples of functions of class $\mathcal{KL}$ are for instance $\beta(s,t) = s\mathrm{e}^{-t}$ or $\beta(s,t) = \arctan(s/(t+1))$.

Definition 3. Let $E \subseteq \mathbb{R}^n$ be a neighbourhood of the origin and $V : E \to \mathbb{R}_{\geq 0}$ be a function. We say that V is *positive definite* on E if $\{V = 0\} = \{0\}$. We say that V is *negative definite* on E if $-V$ is positive definite on E.

Lemma 4. Let $E \subseteq \mathbb{R}^n$ be a neighbourhood of the origin and $V : E \to \mathbb{R}_{\geq 0}$ be positive definite on E. Then, for any compact set $K \subseteq E$ with $0 \in \operatorname{Int} K$, there exists $\alpha \in \mathcal{K}$ such that $\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x})$ for all $\mathbf{x} \in K$.

Remark. If V is continuous, then it is uniformly continuous on compact sets, and so we have:

$$|V(\mathbf{x}) - V(\mathbf{y})| \leq \omega(\|\mathbf{x} - \mathbf{y}\|)$$

where ω is a modulus of continuity of V. Then, we can find $\alpha_1 \in \mathcal{K}^\infty$ such that $\alpha_1 \geq \omega$ and so we have an upper bound for $V(x) \leq \alpha_1(\|\mathbf{x}\|)$.

Definition 5. Let $E \subseteq \mathbb{R}^n$ be a neighbourhood of the origin. We defined the *penalized norm* on E as the function:

$$\begin{aligned} \omega_E : E &\longrightarrow \mathbb{R}_{\geq 0} \\ \mathbf{x} &\longmapsto \|\mathbf{x}\| \left(1 + \frac{1}{d(\mathbf{x}, \partial E)}\right) \end{aligned}$$

From now on, we will consider that the system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{1}$$

has an equilibrium point at the origin. We will denote by $\mathbf{X}(\mathbf{x}_0, t)$ a solution of the system with initial condition $\mathbf{X}(\mathbf{x}_0, 0) = \mathbf{x}_0 \in \mathcal{O} \subseteq \mathbb{R}^n$.

Definition 6. The equilibrium $\mathbf{X}(0,t) = 0$ of Eq. (1) is said to be:

- *stable* if $\exists \mu > 0$ and $\alpha \in \mathcal{K}$ such that $\forall \|\mathbf{x}_0\| < \mu$ any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies:
$$\|\mathbf{X}(\mathbf{x}_0, t)\| \leq \alpha(\|\mathbf{x}_0\|) \quad \forall t \geq 0$$
- *attractive* if $\exists \mu > 0$ such that $\forall \|\mathbf{x}_0\| < \mu$ any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies:
$$\lim_{t\to\infty} \|\mathbf{X}(\mathbf{x}_0, t)\| = 0$$
- *asymptotically stable* if $\exists \mu > 0$ and $\beta \in \mathcal{KL}$ such that $\forall \|\mathbf{x}_0\| < \mu$ any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies:
$$\|\mathbf{X}(\mathbf{x}_0, t)\| \leq \beta(\|\mathbf{x}_0\|, t) \quad \forall t \geq 0$$
- *exponentially stable* if $\exists k, \lambda, \mu > 0$ such that $\forall \|\mathbf{x}_0\| < \mu$ any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies:
$$\|\mathbf{X}(\mathbf{x}_0, t)\| \leq k \|\mathbf{x}_0\| \mathrm{e}^{-\lambda t} \quad \forall t \geq 0$$

Moreover, in the last two cases, if μ can be picked as large as we want, then the equilibrium is said to satisfy that property *globally*.

Remark. Note that exponential stability implies asymptotic stability, which implies stability and attractivity. Moreover, it can be seen that asymptotically stability is equivalent to stability and attractivity.

Remark. An equivalent definition for stability is the following: $\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $\|\mathbf{x}_0\| < \delta$ then $\|\mathbf{X}(\mathbf{x}_0, t)\| < \varepsilon$ for all $t \geq 0$.

Definition 7. The equilibrium $\mathbf{X}(0,t) = 0$ of Eq. (1) is said to be unstable if $\exists \varepsilon > 0$ such that $\forall \delta > 0 \ \exists \mathbf{x_0} \in B(\mathbf{0}, \delta)$ and a solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ such that $\|\mathbf{X}(\mathbf{x}_0, t^*)\| \geq \varepsilon$ for some $t^* \geq 0$.

Remark. A solution may be unstable and attractive at the same time. For example, the system

$$\begin{cases} \dot{x} = x^2(y - x) + y^5 \\ \dot{y} = y^2(y - 2x) \end{cases}$$

exhibits the behaviour shown in Fig. 1.

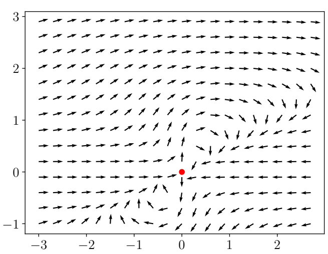

Figure 1: Unstable attractor

Definition 8. We define the *basin of attraction* of the origin as the set $\mathcal{A}$ of all initial conditions $\mathbf{x}_0$ such that the solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies $\lim_{t\to\infty} \mathbf{X}(\mathbf{x}_0, t) = 0$.

Theorem 9. If the origin is asymptotically stable, then its basin of attraction is an open set included in $\mathcal{O}$. Besides, $\exists \beta_{\mathcal{A}} \in \mathcal{KL}$ such that $\forall \mathbf{x}_0 \in \mathcal{A}$, any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ exists for all $t \geq 0$ and satisfies $\omega_{\mathcal{A}}(\|\mathbf{X}(\mathbf{x}_0, t)\|) \leq \beta_{\mathcal{A}}(\|\mathbf{x}_0\|, t)$ for all $t \geq 0$, where $\omega_{\mathcal{A}}$ is the penalized norm of $\mathcal{A}$.

Theorem 10. Assume that $\mathbf{f} \in \mathcal{C}^1$. Then:

1. The zero solution is exponentially stable if and only if the zero solution of the system $\dot{\mathbf{y}} = \mathbf{Df}(\mathbf{0})\mathbf{y}$ is exponentially stable.
2. If $\mathbf{Df}(\mathbf{0})$ has an eigenvalue with positive real part, then the origin is unstable.

Proof.

1. We only do the $\Longleftarrow$) part. So assume the origin is exponentially stable for the system $\dot{\mathbf{y}} = \mathbf{Df}(\mathbf{0})\mathbf{y}$. Then, $\exists k, \lambda > 0$ such that $\|\mathbf{y}(0,t)\| \leq k \|\mathbf{y}_0\| \mathrm{e}^{-\lambda t}$ for all $t \geq 0$, which implies $\mathrm{e}^{\mathbf{Df}(\mathbf{0})t} \leq k\mathrm{e}^{-\lambda t}$ for all $t \geq 0$. Now consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{Df}(\mathbf{0})\mathbf{x} + \Delta\mathbf{f}(\mathbf{x})$, with $\Delta\mathbf{f}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) - \mathbf{Df}(\mathbf{0})\mathbf{x}$. As $f \in \mathcal{C}^1$, $\exists R > 0$ such that $\frac{\|\Delta\mathbf{f}(\mathbf{x})\|}{\|\mathbf{x}\|} \leq \frac{\lambda}{2k}$ for all $\|\mathbf{x}\| \leq R$. Defining $\mu := \frac{R}{2k}$, then if $\|\mathbf{x}_0\| \leq \mu$ we must have that the solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ belongs to $B(\mathbf{0}, R)$ at least on $[0, T)$ for certain $T > 0$. Thus, $\forall t \in [0, T)$ we have $\frac{\|\Delta\mathbf{f}(\mathbf{X}(\mathbf{x}_0,t))\|}{\|\mathbf{X}(\mathbf{x}_0,t)\|} \leq \frac{\lambda}{2k}$, and so using the variations of constants formula:
$$\mathbf{X}(\mathbf{x}_0, t) = \mathrm{e}^{\mathbf{Df}(\mathbf{0})t}\mathbf{x}_0 + \int_0^t \mathrm{e}^{\mathbf{Df}(\mathbf{0})(t-s)} \Delta\mathbf{f}(\mathbf{X}(\mathbf{x}_0, s))\mathrm{d}s$$
Thus:
$$\|\mathbf{X}(\mathbf{x}_0, t)\| \leq k\mathrm{e}^{-\lambda t}\|\mathbf{x}_0\| + \frac{\lambda}{2} \int_0^t \mathrm{e}^{-\lambda(t-s)} \|\mathbf{X}(\mathbf{x}_0, s)\| \,\mathrm{d}s$$
And so:
$$\mathrm{e}^{\lambda t} \|\mathbf{X}(\mathbf{x}_0, t)\| \leq k \|\mathbf{x}_0\| + \frac{\lambda}{2} \int_0^t \mathrm{e}^{\lambda s} \|\mathbf{X}(\mathbf{x}_0, s)\| \,\mathrm{d}s$$
Finally, by **?? ??** we have $\mathrm{e}^{\lambda t} \|\mathbf{X}(\mathbf{x}_0, t)\| \leq k\mathrm{e}^{\frac{\lambda}{2}t} \|\mathbf{x}_0\|$, and so the origin is exponentially stable. □

Remark. In linear dynamics exponentially stability is equivalent to global exponentially stability, which in turn is equivalent to global asymptotic stability which is equivalent to asymptotic stability.

Corollary 11. If $\mathbf{f} \in \mathcal{C}^1$ and $\mathbf{Df}(\mathbf{0})$ has all its eigenvalues with negative real part, then the origin is asymptotically stable.

Theorem 12. Let $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function which is positive definite on $\mathcal{O}$. Then, if
$$D_f^+ V(\mathbf{x}) = \limsup_{t \to 0^+} \frac{V(\mathbf{x} + t\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{t}$$
is non-positive for all $\mathbf{x} \in \mathcal{O}$, then the origin is stable. The function V is called a *Lyapunov function*.

Proof. Since $\mathcal{O}$ is a neighbourhood of the origin $\exists R > 0$ such that $\overline{B(0,R)} \subseteq \mathcal{O}$. Then, since V is continuous and positive definite, $\exists \alpha_1, \alpha_2 \in \mathcal{K}^\infty$ such that $\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$ for all $\mathbf{x} \in B(0,R)$ (by Theorem 4). Let $\mu := {\alpha_2}^{-1}(\alpha_1(R/2))$. Then, any solution with initial conditions $\|\mathbf{x}_0\| < \mu$ belongs to $\overline{B(0,R)}$ at least for $t \in [0,T)$. Now if we consider $v(t) := V(\mathbf{X}(\mathbf{x}_0, t))$, then we have $\dot{v}(t) = D_f^+ V(\mathbf{X}(\mathbf{x}_0, t)) \leq 0$ for all $t \geq 0$. Thus, $\forall t \in [0, T)$ we have:
$$\alpha_1(\|\mathbf{X}(\mathbf{x}_0, t)\|) \leq V(\mathbf{X}(\mathbf{x}_0, t)) = v(t) \leq v(0) = \\ = V(\mathbf{x}_0) \leq \alpha_2(\|\mathbf{x}_0\|)$$
And so $\|\mathbf{X}(\mathbf{x}_0, t)\| \leq \alpha_1^{-1}(\alpha_2(\|\mathbf{x}_0\|)) \leq R/2$ for all $t \in [0, T)$. This mean that in fact $T = \infty$ and so the origin is stable with the function $\alpha := \alpha_1^{-1} \circ \alpha_2$. □

Theorem 13. Let $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function which is positive definite on $\mathcal{O}$. Then, if
$$D_f^+ V(\mathbf{x}) \leq -w(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{O}$$
with $w : \mathcal{O} \to \mathbb{R}_{\geq 0}$ continuous and positive definite, then the origin is globally asymptotically stable.

Proof. As in the previous proof, we define $\mu := \alpha_2^{-1}(\alpha_1(R/2))$ and we get $\dot{v}(t) \leq -w(\mathbf{X}(\mathbf{x}_0, t))$. Since w is continuous and positive definite, $\exists \alpha_3 \in \mathcal{K}^\infty$ such that $\alpha_3(\|\mathbf{x}\|) \leq w(\mathbf{x})$ for all $\mathbf{x} \in \overline{B(0,R)}$. Thus, $\dot{v}(t) \leq -\alpha_3(\|\mathbf{X}(\mathbf{x}_0, t)\|) \leq -\alpha_3({\alpha_2}^{-1}(V(\mathbf{X}(x_0, t))))$. Now, in this case, one can prove that $\exists \beta \in \mathcal{KL}$ such that $v(t) \leq \beta(v(0), t)$ for all $t \geq 0$. But:
$$\alpha_1(\|\mathbf{X}(\mathbf{x}_0, t)\|) \leq V(\mathbf{X}(\mathbf{x}_0, t)) = v(t) \leq \beta(v(0), t) = \\ = \beta(V(\mathbf{x}_0), t) \leq \beta(\alpha_2(\|\mathbf{x}_0\|), t)$$
And thus, $\|\mathbf{X}(\mathbf{x}_0, t)\| \leq {\alpha_1}^{-1}(\beta(\alpha_2(\|\mathbf{x}_0\|), t))$ for all $t \geq 0$, which implies that the origin is globally asymptotically stable since the latter function is of class $\mathcal{KL}$. □

Theorem 14 (Lasaalle's invariance principle). Let K be a compact set contained in $\mathcal{O}$ and let $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function which is positive definite on $\mathcal{O}$ and such that $D_f^+ V(\mathbf{x}) \leq -w(\mathbf{x})$ for all $\mathbf{x} \in K$ with $w : \mathcal{O} \to \mathbb{R}_{\geq 0}$ continuous (not necessarily positive definite). Then, for any solution $\mathbf{X}(\mathbf{x}_0, \cdot)$ with $\mathbf{x}_0 \in K$ and defined on K for all $t \geq 0$, $\exists v^* \in \mathbb{R}_{\geq 0}$ such that $\mathbf{X}(\mathbf{x}_0, t)$ converges to the largest positively invariant set contained in:
$$\{\mathbf{y} \in K : V(\mathbf{y}) = v^* \text{ and } w(\mathbf{y}) = 0\}$$

Remark. If the function V is such that
$$k_1 \|\mathbf{x}\|^n \leq V(\mathbf{x}) \leq k_2 \|\mathbf{x}\|^m$$
and w such that $w(\|\mathbf{x}\|) \geq k_3 \|\mathbf{x}\|^m$, for some $k_1, k_2, k_3 > 0$ and $m, n \in \mathbb{N}$, then the origin is globally exponentially stable.

Theorem 15 (Chetaev's theorem). Let $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function such that:

- $0 \in \partial G$, with $G := \{\mathbf{x} \in \mathcal{O} : V(\mathbf{x}) = 0\}$.

- There exists a neighbourhood U (called Chetaev surface) of the origin such that $D_f^+ V(\mathbf{x}) > 0$ for all $\mathbf{x} \in U \cap G$.

Then, the origin is unstable.

Theorem 16. If the origin is asymptotically stable, then $\forall \varepsilon > 0$ $\{f(\mathbf{x}) : \|\mathbf{x}\| \leq \varepsilon\}$ is a neighbourhood of the origin.

Theorem 17. If the origin is locally asymptotically stable with basin of attraction $\mathcal{A}$, then $\exists \lambda > 0$ and $V \in \mathcal{C}^\infty(\mathcal{A}, \mathbb{R}_{\geq 0})$ positive definite and proper (that is, $\lim_{d(\mathbf{x}, \partial \mathcal{A}) \to 0} V(\mathbf{x}) = \infty$) such that:

$$D_f^+ V(\mathbf{x}) \leq -\lambda V(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{A}$$

Control design and stabilization of equilibrium points

Definition 18. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is said to be *controllable* in time $T > 0$ if $\forall \mathbf{x}_0, \mathbf{x}_T \in \mathcal{O}$ $\exists \mathbf{u} : [0, T] \to \mathbb{R}^p$ such that the solution $\mathbf{X}(\mathbf{x}_0, \cdot, \mathbf{u})$ of the system with initial condition $\mathbf{X}(\mathbf{x}_0, 0, \mathbf{u}) = \mathbf{x}_0$ satisfies $\mathbf{X}(\mathbf{x}_0, T, \mathbf{u}) = \mathbf{x}_T$.

Definition 19. The origin is said to be *asymptotically stabilizable* if there exists $q \in \mathbb{N}$, a neighbourhood $\mathcal{V} \subseteq \mathbb{R}^q$ of the origin and $\boldsymbol{\varphi} : \mathbb{R} \times \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}^q$, $\boldsymbol{\psi} : \mathbb{R} \times \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}^p$ both continuous, such that the origin is an asymptotically stable solution of the system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \dot{\mathbf{u}} = \boldsymbol{\varphi}(t, \mathbf{x}, \boldsymbol{\chi}) \\ \dot{\boldsymbol{\chi}} = \boldsymbol{\psi}(t, \mathbf{x}, \boldsymbol{\chi}) \end{cases} \tag{2}$$

The last two equations are called the *feedback control laws*. If $q = 0$, then the feedback control law is called *static*, whereas if $q > 0$ it is called *dynamic*. Moreover if both $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are independent of t, then the control law is called *stationary* and if $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$ are independent of $\mathbf{x}$, it is called *open-loop control*.

Theorem 20 (Kalmann's theorem). Consider the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$. Then, the system is controllable (or the pair $(\mathbf{A}, \mathbf{B})$ is controllable) if and only if

$$\operatorname{rank} \mathbf{C} := \operatorname{rank} \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix} = n$$

The matrix $\mathbf{C}$ is called the *controllability matrix*.

Theorem 21. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$. Then, the pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ $\exists \mathbf{K} \in \mathbb{R}^{p \times n}$ such that:

$$\sigma(\mathbf{A} + \mathbf{BK}) = \{\lambda_1, \ldots, \lambda_n\}$$

Remark. In practice we pick $\lambda_1, \ldots, \lambda_n \in \{\operatorname{Re} z < 0\}$, and then we look for $\mathbf{K}$ such that $\sigma(\mathbf{A} + \mathbf{BK}) = \{\lambda_1, \ldots, \lambda_n\}$ (for example by using the characteristic polynomial). Note that if $p > 1$, the solution may not be unique.

Theorem 22. Suppose that there exists $q \in \mathbb{N}$, $\boldsymbol{\psi} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$ and $\boldsymbol{\varphi} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ continuous such that $\boldsymbol{\psi}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\varphi}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Assume, moreover, that the system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\chi})) \\ \dot{\boldsymbol{\chi}} = \boldsymbol{\varphi}(\mathbf{x}, \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\chi})) \end{cases}$$

admits $\mathbf{0}$ as an asymptotically stable equilibrium. Then, $\forall \varepsilon > 0$, $\{f(\mathbf{x}, \mathbf{u}) : \|\mathbf{x}\| + \|\mathbf{u}\| \leq \varepsilon\}$ is a neighbourhood of the origin.

Definition 23. Assume that V is a $\mathcal{C}^1$ Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$. We say that V is a *strictly control Lyapunov function (SCLF)* if $\forall \mathbf{x} \neq 0$ $\exists \mathbf{u} \in \mathbb{R}^p$ such that $\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) < 0$. V is a *SCLF continuously at the origin* if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall \mathbf{x} \in B(\mathbf{0}, \delta) \setminus \{0\}$ $\exists \mathbf{u} \in B(\mathbf{0}, \varepsilon)$ such that $\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) < 0$.

Theorem 24. If V is a SCLF continuously at the origin, then for any $T > 0$, there exists a continuous static T-periodic feedback control law asymptotically stabilizing the origin. In addition, if the system is input-affine, that is

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\mathbf{u} \tag{3}$$

there exists a continuous static stationary feedback control law asymptotically stabilizing the origin.

Theorem 25 (Sonntag's theorem). Let V be a SCLF for an input-affine system (Eq. (3)) Then, a stabilizing control law is given by:

$$\boldsymbol{\psi} = \begin{cases} \mathbf{0} & \text{if } L_b V(\mathbf{x}) = 0 \\ -\frac{L_a V(\mathbf{x}) + \sqrt{(L_a V(\mathbf{x}))^2 + |L_b V(\mathbf{x})|^4}}{|L_b V(\mathbf{x})|^2} L_b V(\mathbf{x})^{\mathrm{T}} & \text{otherwise} \end{cases}$$

where $L_a V(\mathbf{x}) := \frac{\partial V}{\partial \mathbf{x}} \mathbf{a}(\mathbf{x})$ and $L_b V(\mathbf{x}) := \frac{\partial V}{\partial \mathbf{x}} \mathbf{b}(\mathbf{x})$.

Remark. This ψ is as smooth as $L_a V$ and $L_b V$ on $\mathbb{R}^n \setminus \{0\}$. And if V is a SCLF continuously at the origin, then $\boldsymbol{\psi}$ is continuous at the origin.

Backstepping

Consider a system of the form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} = \mathbf{u} \end{cases} \tag{4}$$

We would like to construct a SCLF for this system, that is, to find V such that $\forall (\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0})$ $\exists \mathbf{u}$ such that

$$\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{y}) + \frac{\partial V}{\partial \mathbf{y}} \mathbf{u} < 0$$

We define $\boldsymbol{\eta}$ such that

$$\begin{cases} \frac{\partial V}{\partial \mathbf{y}}(\mathbf{x}, \boldsymbol{\eta}(\mathbf{x})) = \mathbf{0} \\ \boldsymbol{\eta}(\mathbf{0}) = \mathbf{0} \end{cases}$$

Lemma 26. If V is a $\mathcal{C}^2$ function and $\boldsymbol{\eta}$ is a locally 1/2-Hölder continuous function, then $W(\mathbf{x}) := V(\mathbf{x}, \boldsymbol{\eta}(\mathbf{x}))$ is a SCLF for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{v})$.

Finally we consider

$$V(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \boldsymbol{\eta}(\mathbf{x})) + \int_{\boldsymbol{\eta}(\mathbf{x})}^{\mathbf{y}} \boldsymbol{\varphi}(\mathbf{x}, \mathbf{s}) \mathrm{d}s$$

with $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \iff \mathbf{y} = \boldsymbol{\eta}(\mathbf{x})$. Then, this V is a SCLF for the system of Eq. (4).

Remark. Usually we take $\boldsymbol{\varphi}(\mathbf{x},\mathbf{y}) = \mathbf{y} - \boldsymbol{\eta}(\mathbf{x})$ and consider

$$V(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\boldsymbol{\eta}(\mathbf{x})) + \frac{1}{2}\|\mathbf{y} - \boldsymbol{\eta}(\mathbf{x})\|^2 \tag{5}$$

In practice, we first look for a SCLF W for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{v})$, and then we find $v = \boldsymbol{\eta}(\mathbf{x})$ such that $\dot{W} < 0$. Finally, we construct V as in Eq. (5). And we could iterate this process.

Remark. This method is only valid for systems in *strict-feedback form*, that is, systems of the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2) \\ \dot{x}_2 = f_2(x_1,x_2,x_3) \\ \quad\vdots \\ \dot{x}_{n-1} = f_{n-1}(x_1,\ldots,x_n) \\ \dot{x}_n = f_n(x_1,\ldots,x_n,u) \end{cases}$$

2. | Control theory in PDEs

From what follows $\mathbf{x}$ will denote the state variable whose values are in a Hilbert space $\mathcal{X}$, and $\mathbf{u}$ will denote the control variable whose values are in a Hilbert space $\mathcal{U}$.

Classical problems

Definition 27 (Exact controllability). Let $T > 0$. The *exact controllability* of a system is said to be achieved if, for any initial condition $\mathbf{x}_0$ and any final condition $\mathbf{x}_T$, there exists a control $\mathbf{u} : [0,T] \to \mathcal{U}$ such that the solution $\mathbf{X}(\mathbf{x}_0,\cdot,\mathbf{u})$ of the system with initial condition $\mathbf{X}(\mathbf{x}_0,0,\mathbf{u}) = \mathbf{x}_0$ satisfies $\mathbf{X}(\mathbf{x}_0,T,\mathbf{u}) = \mathbf{x}_T$.

Definition 28 (Approximate controllability). Let $T > 0$, $\varepsilon > 0$. The *approximate controllability* of a system is said to be achieved if, for any initial condition $\mathbf{x}_0$ and any final condition $\mathbf{x}_T$, there exists a control $\mathbf{u} : [0,T] \to \mathcal{U}$ such that the solution $\mathbf{X}(\mathbf{x}_0,\cdot,\mathbf{u})$ of the system satisfies $\|\mathbf{X}(\mathbf{x}_0,T,\mathbf{u}) - \mathbf{x}_T\| < \varepsilon$.

Definition 29 (Null controllability). Let $T > 0$. The *null controllability* of a system is said to be achieved if, for any initial condition $\mathbf{x}_0$, there exists a control $\mathbf{u} : [0,T] \to \mathcal{U}$ such that the solution $\mathbf{X}(\mathbf{x}_0,\cdot,\mathbf{u})$ of the system satisfies $\mathbf{X}(\mathbf{x}_0,T,\mathbf{u}) = \mathbf{0}$.

Lemma 30. Consider a linear reversible system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. Then, the system is exactly controllable if and only if it is null controllable.

Proof. The implication to the right is clear. Now assume it is null controllable. Let $T > 0$ and $x_0, x_T \in \mathbb{R}^n$. Since the system is reversible we can first solve for $\overline{\mathbf{x}}$

$$\begin{cases} \dot{\overline{\mathbf{x}}} = \mathbf{A}\overline{\mathbf{x}} \\ \overline{\mathbf{x}}(T) = \mathbf{x}_T \end{cases}$$

Now we solve the null controllability problem with initial state $\mathbf{x}_0 - \overline{\mathbf{x}}(0)$. Thus, we find $\mathbf{u}$ such that $\mathbf{x}$ satisfies

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{x}(0) = \mathbf{x}_0 - \overline{\mathbf{x}}(0) \end{cases}$$

and so $\mathbf{x}(T) = 0$. Now consider $\widehat{\mathbf{x}} := \overline{\mathbf{x}} + \mathbf{x}$. Then, $\widehat{\mathbf{x}}$ satisfies

$$\begin{cases} \dot{\widehat{\mathbf{x}}} = \mathbf{A}\widehat{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \widehat{\mathbf{x}}(0) = \mathbf{x}_0 \\ \widehat{\mathbf{x}}(T) = \mathbf{x}_T \end{cases}$$

□

Definition 31 (Feedback stabilization). Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, the *feedback stabilization process* consists in finding an operator $K : \mathcal{X} \to \mathcal{U}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{x}$ has a stable (or asymptotically stable) equilibrium at the origin.

Definition 32 (Optimal control). Let J be a cost function, $J = J(\mathbf{x},\mathbf{u},\mathbf{x}(T))$. The *optimal control problem* consists in finding $\mathbf{u} : [0,T] \to \mathcal{U}$ such that J is minimized, where $\mathbf{x}$ satisfies $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with $\mathbf{x}(0) = \mathbf{x}_0$.

Interior control for the heat equation

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded regular domain (i.e. connected) and $\omega \subseteq \Omega$ be a non-empty open subset. We consider the control system:

$$\begin{cases} \partial_t v - \Delta v = \mathbf{1}_\omega u & \text{in } [0,T]\times\Omega \\ v = 0 & \text{in } [0,T]\times\partial\Omega \\ v = v_0 & \text{in } \Omega \end{cases} \tag{6}$$

Theorem 33 (Strong solutions). Let $f \in L^2((0,T);\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then, the Cauchy problem

$$\begin{cases} \partial_t v - \Delta v = f & \text{in } [0,T]\times\Omega \\ v = 0 & \text{in } [0,T]\times\partial\Omega \\ v = v_0 & \text{in } \Omega \end{cases} \tag{7}$$

has a unique solution

$$v \in \mathcal{C}^0([0,T];H_0^1(\Omega)) \cap L^2((0,T);H^2(\Omega)\cap H_0^1(\Omega))$$

Proof. We start from uniqueness. Let $(e_i)_{i\in\mathbb{N}}$ be a Hilbert basis of $L^2(\Omega)$ from the eigenvectors of the Laplacian operator:

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega \\ e_i = 0 & \text{in } \partial\Omega \end{cases}$$

and $\varphi \in \mathcal{D}((0,T)\times\Omega)$ be a test function. Then, we have

$$-\int_0^T\int_\Omega v\partial_t\varphi + \int_0^T\int_\Omega \nabla v\nabla\varphi = \int_0^T\int_\Omega f\varphi$$

In particular for $\varphi = \rho(t)\psi_{n,i}(x)$ with $\rho \in D(0,T)$ and $\psi_{n,i} \overset{H^1}{\to} e_i$ (here we use the fact that $H_0^1 = \overline{\mathcal{D}(\Omega)}^{H^1}$). Thus, we arrive at:

$$-\int_0^T\int_\Omega v\rho' e_i + \int_0^T\int_\Omega \rho\nabla v\nabla e_i = \int_0^T\int_\Omega f\rho e_i$$

Decomposing $v = \sum_{i\in\mathbb{N}} v_i e_i$ and $f = \sum_{i\in\mathbb{N}} f_i e_i$, we get:

$$-\int_0^T v_i\rho' + \lambda_i\int_0^T \rho v_i = \int_0^T f_i\rho$$

which in the sense of $D^*(0,T)$ gives $v_i' + \lambda_i v_i = f_i$, which has solution:

$$v_i(t) = \mathrm{e}^{-\lambda_i t} v_i(0) + \int_0^t \mathrm{e}^{-\lambda_i (t-s)} f_i(s) \mathrm{d}s$$

So we have uniqueness and a formula:

$$\begin{aligned} v(t,x) &= \sum_{i\in\mathbb{N}} \mathrm{e}^{-\lambda_i t} v_i(0) e_i(x) + \sum_{i\in\mathbb{N}} \int_0^t \mathrm{e}^{-\lambda_i (t-s)} f_i(s) e_i(x) \mathrm{d}s \\ &=: v_a(t,x) + v_b(t,x) \end{aligned} \tag{8}$$

For the existence, it suffices to check that the solution in Eq. (8) belongs to the desired space. We first check that $v \in \mathcal{C}^0([0,T]; H_0^1(\Omega))$. We have:

$$\begin{aligned} \|v_a\|^2_{L^\infty(0,T;H_0^1(\Omega))} &= \sup_{t\in[0,T]} \|v^a(t,\cdot)\|^2_{H_0^1(\Omega)} = \\ &= \sup_{t\in[0,T]} \int_\Omega \sum_{i\in\mathbb{N}} \mathrm{e}^{-2\lambda_i t} |v_i(0)|^2 \|\boldsymbol{\nabla} e_i\|^2 = \\ &= \sup_{t\in[0,T]} \sum_{i\in\mathbb{N}} \lambda_i \mathrm{e}^{-2\lambda_i t} |v_i(0)|^2 \le \sum_{i\in\mathbb{N}} \lambda_i |v_i(0)|^2 = \\ &= \|v_0\|^2_{H_0^1(\Omega)} \end{aligned}$$

On the other hand:

$$\begin{aligned} \|v_b(t,\cdot)\|^2_{H_0^1(\Omega)} &= \sum_{i\in\mathbb{N}} \lambda_i \left(\int_0^t \mathrm{e}^{-\lambda_i(t-s)} f_i(s) \mathrm{d}s \right)^2 = \\ &= \sum_{i\in\mathbb{N}} \left(\int_0^t \mathrm{e}^{-\lambda_i(t-s)} f_i(s) \sqrt{\lambda_i} \mathrm{d}s \right)^2 \le \sum_{i\in\mathbb{N}} \|f_i\|^2_{L^2(0,T)} = \\ &= \|f\|^2_{L^2((0,T);L^2(\Omega))} \end{aligned}$$

where we have used the fact that the penultimate term can be written as a convolution and then we use ?? ?? $\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$ with $1/p + 1/q = 1 + 1/r$, in the case $p = q = 2$ and $r = \infty$. Finally, we prove $v \in L^2((0,T); H^2(\Omega))$. Indeed:

$$\begin{aligned} \|v_a(t,\cdot)\|^2_{L^2(0,T;H^2)} &= \int_0^T \sum_{i\in\mathbb{N}} \lambda_i^2 \mathrm{e}^{-2\lambda_i t} |v_i(0)|^2 \mathrm{d}t = \\ &= \sum_{i\in\mathbb{N}} \lambda_i |v_i(0)|^2 \int_0^T \lambda_i \mathrm{e}^{-2\lambda_i t} \mathrm{d}t \le C \|v_0\|^2_{H_0^1(\Omega)} \end{aligned}$$

because the latter term in the penultimate equality is bounded. Moreover:

$$\begin{aligned} \|v_b(t,\cdot)\|^2_{H_0^1} &= \int_0^T \sum_{i\in\mathbb{N}} \lambda_i^2 \left(\int_0^t \mathrm{e}^{-\lambda_i(t-s)} f_i(s) \mathrm{d}s \right)^2 \mathrm{d}t = \\ &= \int_0^T \sum_{i\in\mathbb{N}} \left(\int_0^t \mathrm{e}^{-\lambda_i(t-s)} f_i(s) \lambda_i \mathrm{d}s \right)^2 \mathrm{d}t \le \\ &\le \int_0^T \sum_{i\in\mathbb{N}} \|f_i\|^2_{L^2(0,T)} \mathrm{d}t \le T \|f\|^2_{L^2((0,T);L^2(\Omega))} \end{aligned}$$

again by ?? ??. □

Theorem 34 (Weak solutions). Let $f \in L^2((0,T); H^{-1}(\Omega))$ and $v_0 \in L^2(\Omega)$. Then, the Cauchy problem of Eq. (7) has a unique solution

$$v \in \mathcal{C}^0((0,T); L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$$

We consider now the dual problem of Eq. (6):

$$\begin{cases} -\partial_t \theta - \Delta\theta = 0 & \text{in } [0,T]\times\Omega \\ \theta = 0 & \text{in } [0,T]\times\partial\Omega \\ \theta(T) = \theta_T & \text{in } \Omega \end{cases} \tag{9}$$

Proposition 35. Let $u \in L^2((0,T)\times\Omega)$, $v_0 \in L^2(\Omega)$ and v the corresponding solution of Eq. (6). Then, the solution θ of Eq. (9) with $\theta_T \in L^2(\Omega)$ satisfies:

$$\langle \theta, v\rangle_{L^2(\Omega)} \Big|_0^T = \int_0^T \int_\Omega \mathbf{1}_\omega u\theta$$

Proof. We can suppose that all functions are smooth (otherwise we replace them by a linear combination of e_i and pass to the limit using the fact that $(v_0, f) \mapsto v$ is continuous from $L^2(\Omega) \times L^2((0,T)\times\Omega) \to \mathcal{C}^0([0,T]; L^2(\Omega)) \cap L^2((0,T); L^2(\Omega)))$. Now, multiplying Eq. (6) by θ and integrating we get:

$$\begin{aligned} \int_0^T \int_\Omega \mathbf{1}_\omega u\theta &= \int_0^T \int_\Omega \partial_t v\theta + \int_0^T \int_\Omega \nabla v \nabla\theta = \\ &= \int_\Omega v\theta \Big|_0^T - \int_0^T \int_\Omega \partial_t \theta v + \int_0^T \int_\Omega \nabla v \nabla \theta = \int_\Omega v\theta \Big|_0^T \end{aligned}$$

□

Definition 36 (Observability inequality). We will say that the dual problem Eq. (9) satisfies the *finite-time observability inequality* if $\exists C > 0$ such that $\forall \theta_T \in L^2(\Omega)$ the solution θ satisfies:

$$\|\theta(0)\|^2_{L^2(\Omega)} \le C \int_0^T \int_\omega \theta^2$$

Proposition 37. If the dual problem Eq. (9) is finite-time observable, then the control problem Eq. (6) is null controllable.

Proof. Note that the null controllability condition is equivalent to $\forall\theta_T \in L^2(\Omega)$ we have (by Theorem 35):

$$-\langle \theta(0), v_0\rangle_{L^2(\Omega)} = \int_0^T \int_\omega u\theta$$

Now let's define:

$$B := \{\mathbf{1}_\omega \theta : \theta \text{ solution of Eq. (9) for some } \theta_T \in L^2(\Omega)\}$$

$$A := \overline{B}^{L^2((0,T)\times\omega)}$$

We equip A with the norm $\|\cdot\|_{L^2((0,T)\times\omega)}$. Now consider:

$$\begin{array}{rcl}\Phi : L^2(\Omega) & \longrightarrow & L^2((0,T)\times\omega) \\ \theta_T & \longmapsto & \mathbf{1}_\omega\theta\end{array}$$

Note that $\overline{\operatorname{im}\Phi} = A$. Now, to any $\phi \in \operatorname{im}(\Phi)$ we could a priori associate several θ_T, but all of them would generate the same $\theta(0)$ due to the observability condition. So we may consider the map:

$$\begin{array}{rcl}\operatorname{im}(\Phi) & \longrightarrow & L^2(\Omega) \\ \mathbf{1}_\omega\theta & \longmapsto & \theta(0)\end{array}$$

which is continuous by the observability condition. Now, extending the map to A by uniform continuity, we get that

$$\begin{array}{rcl}\ell : \; A & \longrightarrow & \mathbb{R} \\ \mathbf{1}_\omega\theta & \longmapsto & -\langle\theta(0), v_0\rangle_{L^2(\Omega)}\end{array}$$

is a continuous linear form (by composition). We conclude now with **?? ??** since A is a Hilbert space. □

Proposition 38 (1D observability inequality). Let $\Omega = (0,1)$, $\omega = (a,b)$ and $T > 0$. Then, $\exists C > 0$ such that $\forall\theta_T \in L^2(0,1)$ we have:

$$\|\theta(0)\|_{L^2(0,1)} \leq C\,\|\theta\|_{L^2((0,T)\times\omega)}$$

Proof. Let $w(t,x) := \theta(T-t,x)$ so that w satisfies:

$$\begin{cases}\partial_t w - \partial_{xx} w = 0 & \text{in } [0,T]\times(0,1) \\ w(t,0) = w(t,1) = 0 & \\ w(0,x) = w_0(x) & \text{in } (0,1)\end{cases} \tag{10}$$

We want to prove that

$$\|w(T)\|_{L^2(0,1)} \leq C\,\|w\|_{L^2((0,T)\times(a,b))}$$

From Theorem 39 between t_1 and t_0 we have:

$$\|w(t_0)\|_{L^\infty(0,1)} \leq C e^{\frac{D}{t_1-t_0}}\,\|w(t_1)\|^{1-\delta}_{L^\infty(0,1)}\,\|w(t_0)\|^{\delta}_{L^\infty(a,b)}$$

We repeat that in the interval (t_2,t_1) and we get:

$$\|w(t_0)\|_{L^\infty(0,1)} \leq C^{2-\delta} e^{\frac{D}{t_1-t_0}+\frac{D(1-\delta)}{t_2-t_1}}\,\|w(t_2)\|^{(1-\delta)^2}_{L^\infty(0,1)}\cdot$$
$$\cdot\,\|w(t_1)\|^{(1-\delta)\delta}_{L^\infty(0,1)}\,\|w(t_0)\|^{\delta}_{L^\infty(a,b)}$$

Repeating the argument we get each time an extra power $1-\delta$ in $e^{D/(t_{n+1}-t_n)}$. So we would like to have for example $t_{n+1}-t_n = \alpha\left(1-\frac{\delta}{2}\right)^n$ $\sum_{n\in\mathbb{N}}\left(1-\frac{\delta}{2}\right)^n = \frac{2}{\delta}$ so we let $t_0 = T$ and $t_{n+1} = t_n - \frac{\delta}{2}T\left(1-\frac{\delta}{2}\right)^n$. We conclude arguing by induction and passing to the limit:

$$\|w(0)\|_{L^\infty(0,1)} \leq C\,\|w\|_{L^\infty((0,T)\times(a,b))}$$

Now to prove the L^2 inequality, for the left hand side we have $\|w(0)\|_{L^2(0,1)} \leq \|w(0)\|_{L^\infty(0,1)}$ and for the right hand side we use 40 Interior regulariy. □

Lemma 39. Using the hypotheses and notation of the previous proposition, we have that $\exists C, D > 0$ and $\delta > 0$ such that $\forall w_0$ we have:

$$\|w(T)\|_{L^\infty(0,1)} \leq C e^{D/T}\,\|w_0\|^{1-\delta}_{L^\infty(0,1)}\,\|w(T)\|^{\delta}_{L^\infty(a,b)}$$

Lemma 40 (Interior regulariy). Let w be a solution of the heat equation ($w \in \mathcal{C}^0([0,T];H^1_0(0,1)) \cap L^2((0,T);H^2(0,1)\cap H^1_0(0,1))$). Then:

$$\|w\|_{L^\infty([T/2,T]\times[a,b])} \leq C\,\|w\|_{L^2([T/4,T]\times[c,d])}$$

for all $0 < c < a < b < d < 1$.

Proof. Let w be a solution. We introduce a *cut-off* function $\varphi_1 \in \mathcal{C}^\infty$ with $\varphi_1 = 0$ outside $[T/4,T]\times[c,d]$ and $\varphi_1 = 1$ in $[T/3,T]\times[\mu,\nu]$, with $c < \mu < a$ and $b < \nu < d$. We look at $w_1 := w\varphi_1$. We have:

$$\begin{cases}\partial_t w_1 - \partial_{xx} w_1 = w\partial_t\varphi_1 - 2\partial_x w\partial_x\varphi_1 - w\partial_{xx}\varphi_1 \\ w_1|_{t=0} = 0 \\ w_1|_{[0,T]\times\partial\Omega} = 0\end{cases}$$

Let's study the right hand side. We have:

- $\|w\partial_t\varphi_1\|_{L^2} \leq C\,\|w\|_{L^2} \Longrightarrow \|w\partial_t\varphi_1\|_{L^2,H^{-1}} \leq C\,\|w\|_{L^2}$
- $\|w\partial_{xx}\varphi_1\|_{L^2} \leq C\|w\|_{L^2} \Longrightarrow \|w\partial_{xx}\varphi_1\|_{L^2,H^{-1}} \leq C\|w\|_{L^2}$
- $\|\partial_x w\partial_x\varphi_1\|_{L^2,H^{-1}} \leq C\,\|w\|_{L^2}$

By the theorem of existence of weak solutions we get $\|w_1\|_{L^2,H^1_0} \leq C\,\|w\|_{L^2}$. Now we introduce a second *cut-off* function φ_2 with $\varphi_2 = 0$ outside $[T/3,T]\times[\mu,\nu]$ and $\varphi_2 = 1$ in $[T/2,T]\times[a,b]$. We define $w_2 := w_1\varphi_2$. We have similar calculations to the previous ones but with $w \in L^2([0,T];H^1(\mu,\nu))$ and so:

- $\|w\partial_t\varphi_2\|_{L^2} \leq C\,\|w\|_{L^2}$
- $\|w\partial_{xx}\varphi_2\|_{L^2} \leq C\,\|w\|_{L^2}$
- $\|\partial_x w\partial_x\varphi_2\|_{L^2} \leq C\,\|w\|_{L^2}$

Using the theorem of existence of weak solutions we get:

$$\begin{aligned}\|w_2\|_{L^\infty;H^1} &\leq C\,\|w\|_{L^2([T/3,T];H^1(\mu,\nu))} \\ &\leq C\,\|w\|_{L^2([T/4,T];H^1(c,d))}\end{aligned}$$

where the last inequality follows from the first step. □

Boundary control for the wave equation

In this section $\Omega \subseteq \mathbb{R}^n$ is a bounded regular domain and $\Sigma \subseteq \partial\Omega$ is a non-empty open subset. We are interested in studying the control system:

$$\begin{cases}\partial_{tt} v - \Delta v = 0 & \text{in } [0,T]\times\Omega \\ v = \mathbf{1}_\Sigma u & \text{in } [0,T]\times\partial\Omega \\ (v,\partial_t v)|_{t=0} = (v_0,v_1) & \text{in } \Omega\end{cases} \tag{11}$$

Theorem 41 (Weak solutions). We consider the problem

$$\begin{cases}\partial_{tt} v - \Delta v = f & \text{in } [0,T]\times\Omega \\ v = 0 & \text{in } [0,T]\times\partial\Omega \\ (v,\partial_t v)|_{t=0} = (v_0,v_1) & \text{in } \Omega\end{cases} \tag{12}$$

with $(v_0,v_1) \in H^1_0(\Omega)\times L^2(\Omega)$ and $f \in L^1((0,T);L^2(\Omega))$. Then, the problem has a unique solution $v \in \mathcal{C}^0([0,T];H^1_0(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ with:

$$\|v\|_{L^\infty([0,T];H^1_0(\Omega))} + \|\partial_t v\|_{L^\infty([0,T];L^2(\Omega))} \leq$$
$$\leq C\left(\|v_0\|_{H^1_0} + \|v_1\|_{L^2} + \|f\|_{L^2((0,T);L^2(\Omega))}\right)$$

Theorem 42 **(Strong solutions).** Consider the problem Eq. (12) with $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_1 \in H_0^1(\Omega)$ and $f \in L^1((0,T); H^1(\Omega))$. Then, the problem has a unique solution $v \in \mathcal{C}^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; H_0^1(\Omega))$ with:

$$\|v\|_{L^\infty([0,T];H^2(\Omega)\cap H_0^1(\Omega))} + \|\partial_t v\|_{L^\infty([0,T];H^1(\Omega))} \leq$$
$$\leq C\left(\|v_0\|_{H^2\cap H_0^1} + \|v_1\|_{H_0^1} + \|f\|_{L^1((0,T);H^1(\Omega))}\right)$$

Proof. We proceed as for the heat equation. Consider the Hilbert basis $(e_i)_{i\in\mathbb{N}}$ of $L^2(\Omega)$ from the eigenvectors of the Laplacian operator of eigenvalues λ_i. Let $v_0 = \sum_{i\in\mathbb{N}} a_i(t)e_i$, $v_1 = \sum_{i\in\mathbb{N}} b_i(t)e_i$ and $f = \sum_{i\in\mathbb{N}} f_i e_i$. We look for a solution of the form $v(t,x) = \sum_{i\in\mathbb{N}} y_i(t)e_i$. Taking test functions $\varphi(t)\psi(x)$ and letting $\psi \to e_i$ we get (in the sense of distributions in time):

$$\begin{cases} y_i'' + \lambda_i y_i = f_i \\ y_i(0) = a_i, y_i'(0) = b_i \end{cases}$$

So the solution must be of the form $v(t,x) = \sum_{i\in\mathbb{N}} y_i(t)e_i$ with:

$$y_i(t) = a_i \cos\sqrt{\lambda_i}t + b_i\frac{\sin(\sqrt{\lambda_i}t)}{\sqrt{\lambda_i}} + \int_0^t \frac{\sin(\sqrt{\lambda_i}(t-s))}{\sqrt{\lambda_i}} f_i(s)\mathrm{d}s$$

This gives uniqueness and existence if the sums are finite. To check that the solution belongs to the desired space we proceed as in the heat equation case. □

Theorem 43 **(Hidden regularity).** For a regular solution, we have for some $C > 0$:

$$\int_0^T\int_{\partial\Omega} |\partial_{\mathbf{n}} v|^2 \leq C(1+T)\left[\|v_0\|^2_{H_0^1} + \|v_1\|^2_{L^2} + \|f\|^2_{L^1(0,T;L^2(\Omega))}\right]$$

Proof. Suppose v regular. We will use a multiplier method. Let $q : \overline{\Omega} \to \mathbb{R}^n$ be a smooth vector field. We multiply the equation by $(\mathbf{q}\cdot\boldsymbol{\nabla})v$ and integrate (we denote $Q := [0,T]\times\Omega$ and $\Sigma_T := [0,T]\times\partial\Omega$). On the one hand:

$$\iint_Q \partial_{tt} v(\mathbf{q}\cdot\boldsymbol{\nabla})v = \int_\Omega \partial_t v(\mathbf{q}\cdot\boldsymbol{\nabla})v\Big|_0^T - \iint_Q \partial_t v(\mathbf{q}\cdot\boldsymbol{\nabla})\partial_t v =$$
$$= \int_\Omega \partial_t v(\mathbf{q}\cdot\boldsymbol{\nabla})v\Big|_0^T - \iint_Q (\mathbf{q}\cdot\boldsymbol{\nabla})\frac{(\partial_t v)^2}{2} =$$
$$= \int_\Omega \partial_t v(\mathbf{q}\cdot\boldsymbol{\nabla})v\Big|_0^T + \iint_Q \frac{(\partial_t v)^2}{2}\,\mathbf{div}\,\mathbf{q}$$

where in the last equality we have used integration by parts, the ?? ?? and the fact that $v_t|_{\partial\Omega} = 0$. On the other hand:

$$\iint_Q \Delta v(\mathbf{q}\cdot\boldsymbol{\nabla})v = \int_{\Sigma_T} \partial_{\mathbf{n}} v(\mathbf{q}\cdot\boldsymbol{\nabla})v - \iint_Q \boldsymbol{\nabla}v\cdot\boldsymbol{\nabla}((\mathbf{q}\cdot\boldsymbol{\nabla})v)$$

Notice that since $v = 0$ on Σ_T, the tangential derivatives of v are zero, and so on Σ_T we have $(\mathbf{q}\cdot\boldsymbol{\nabla})v = (\mathbf{q}\cdot\mathbf{n})\partial_{\mathbf{n}} v$. The second term can be written as:

$$\iint_Q \boldsymbol{\nabla}v\cdot\boldsymbol{\nabla}((\mathbf{q}\cdot\boldsymbol{\nabla})v) = \iint_Q \partial_k v\partial_k(q_i\partial_i v) =$$
$$= \iint_Q \partial_k v\partial_k q_i\partial_i v + \iint_Q \partial_k v q_i \partial_{ki} v =$$
$$= \iint_Q \partial_k v\partial_k q_i\partial_i v + \iint_Q q_i\partial_i\left(\frac{(\partial_k v)^2}{2}\right) =$$
$$= \iint_Q \partial_k v\partial_k q_i\partial_i v + \iint_Q (\mathbf{q}\cdot\boldsymbol{\nabla})\frac{\|\boldsymbol{\nabla}v\|^2}{2} =$$
$$= \iint_Q \partial_k v\partial_k q_i\partial_i v - \iint_Q \frac{\|\boldsymbol{\nabla}v\|^2}{2}\,\mathbf{div}\,\mathbf{q} + \iint_{\Sigma_T} \frac{\|\boldsymbol{\nabla}v\|^2}{2}\mathbf{q}\cdot\mathbf{n}$$

Finally grouping all terms we get:

$$\frac{1}{2}\iint_{\Sigma_T} (\mathbf{q}\cdot\mathbf{n})(\partial_{\mathbf{n}} v)^2 = \iint_Q \partial_k v\partial_k q_i\partial_i v - \frac{1}{2}\iint_Q \|\boldsymbol{\nabla}v\|^2\,\mathbf{div}\,\mathbf{q} +$$
$$+ \int_\Omega \partial_t v(\mathbf{q}\cdot\boldsymbol{\nabla})v\Big|_0^T + \iint_Q \frac{(\partial_t v)^2}{2}\,\mathbf{div}\,\mathbf{q} - \iint_Q f(\mathbf{q}\cdot\boldsymbol{\nabla})v \lesssim$$
$$\lesssim a^2 + a^2 + ab + b + ac$$

with $a := \|v\|_{L^\infty([0,T];H_0^1(\Omega))}$, $b := \|\partial_t v\|_{L^\infty([0,T];L^2(\Omega))}$ and $c := \|f\|_{L^1((0,T);L^2(\Omega))}$. Here we used that $\|\boldsymbol{\nabla}v\| = |\partial_{\mathbf{n}} v|$, because $v = 0$ on Σ_T. We conclude choosing $\mathbf{q}$ a regular extension of the unit normal vector field to Ω. □

Now, we want to define weak solutions of Eq. (11). To do so, we take as test functions solutions of:

$$\begin{cases} \partial_{tt}\theta - \Delta\theta = f & \text{in } [0,T]\times\Omega \\ \theta = 0 & \text{in } [0,T]\times\partial\Omega \\ (\theta,\partial_t\theta)|_{t=T} = (0,0) & \text{in } \Omega \end{cases} \tag{13}$$

Definition 44 **(Transposition solution).** Let $(v_0, v_1, u) \in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Sigma_T)$. We call *transposition solution* of Eq. (11) a function $v \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H^{-1}(\Omega))$ such that for any $f \in L^1((0,T); L^2(\Omega))$ we have:

$$\iint_Q vf = -\int_\Omega \partial_t\theta(0)v(0) + \int_\Omega \theta(0)v_1 - \int_{\Sigma_T} u\partial_{\mathbf{n}}\theta \tag{14}$$

where θ is the solution of Eq. (13) associated to f.

Remark. Any regular solution is a transposition solution.

Theorem 45. For any $(v_0, v_1, u) \in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Sigma_T)$, there exists a unique transposition solution of Eq. (11).

Proof. We would like to prove that the right hand side of Eq. (14) is a continuous linear form on $L^1((0,T); L^2(\Omega))$. If so then $\exists! v \in [L^1((0,T); L^2(\Omega))]^* = L^\infty([0,T]; L^2(\Omega))$ such that the equation is true $\forall f \in L^1((0,T); L^2(\Omega))$. We have that

$$\begin{array}{ccc} L^1((0,T); L^2(\Omega)) & \longrightarrow & \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)) \\ f & \longmapsto & \theta \end{array}$$

is continuous, and so is $f \mapsto \int_\Omega \partial_t\theta(0)v(0) = \langle \partial_t\theta(0), v_0\rangle_{L^2\times L^2}$, because $\partial_t\theta(0) \in L^2(\Omega)$ and $v(0) \in L^2(\Omega)$. Similarly, since $f \to \theta(0) \in H_0^1(\Omega)$ is continuous, then so is $f \mapsto \int_\Omega \theta(0)v_1 = \langle \theta(0), v_1\rangle_{H_0^1\times H^{-1}}$. Finally, by 43 Hidden regularity we have that $f \mapsto \partial_{\mathbf{n}}\theta|_{\Sigma_T} \in L^2$ is continuous, and so is $f \mapsto \int_{\Sigma_T} u\partial_{\mathbf{n}}\theta = \langle u, \partial_{\mathbf{n}}\theta\rangle_{L^2\times L^2}$. □

Proposition 46. Consider a transposition solution v of Eq. (11), with $v_0 \in L^2(\Omega)$, $v_1 \in H^{-1}(\Omega)$ and $u \in L^2((0,T)\times\Sigma)$. Let θ be a solution of

$$\begin{cases} \partial_{tt}\theta - \Delta\theta = 0 & \text{in } [0,T]\times\Omega \\ \theta = 0 & \text{in } [0,T]\times\partial\Omega \\ (\theta,\partial_t\theta)|_{t=T} = (\theta_T^0,\theta_T^1) & \text{in } \Omega \end{cases} \tag{15}$$

Then:

$$\left[\langle \partial_t v, \theta\rangle_{H^{-1}\times H_0^1} - \langle v, \partial_t\theta\rangle_{L^2\times L^2}\right]\Big|_0^T = \int_0^T\int_\Sigma u\partial_{\mathbf{n}}\theta$$

Proof. It is sufficient to prove it for regular solutions and then pass to the limit using:

$$\|\theta\|_{L^\infty,H_0^1} + \|\partial_t\theta\|_{L^\infty,L^2} \lesssim \|\theta_T^0\|_{H_0^1} + \|\theta_T^1\|_{L^2}$$
$$\|v\|_{L^\infty,L^2} + \|\partial_t v\|_{L^\infty,H^{-1}} \lesssim \|v_0\|_{L^2} + \|v_1\|_{H^{-1}} + \|u\|_{L^2((0,T)\times\Sigma)}$$

Now, multiplying the equation of v by θ and integrating we get:

$$0 = \int_0^T\int_\Omega (\partial_{tt}v - \Delta v)\theta = \int_\Omega \partial_t v\theta\Big|_0^T - \int_0^T\int_\Omega \partial_t v\partial_t\theta +$$
$$+ \int_0^T\int_\Omega \nabla v\nabla\theta = \int_\Omega \partial_t v\theta\Big|_0^T - \int_\Omega v\partial_t\theta\Big|_0^T + \int_0^T\int_\Omega v\partial_{tt}\theta -$$
$$- \int_0^T\int_\Omega v\Delta\theta + \int_0^T\int_{\partial\Omega} v\partial_{\mathbf{n}}\theta$$

□

Remark. Exact controllability is equivalent to exact controllability starting from $(0,0)$ (due to superposition principle).

Definition 47 (Observability inequality). We say that Eq. (15) is *exactly observable* in time T from Σ if $\exists C > 0$ such that for any solution θ of Eq. (15) we have:

$$\|\theta(T)\|_{H_0^1} + \|\partial_t\theta(T)\|_{L^2} \le C\|\partial_{\mathbf{n}}\theta\|_{L^2(\Sigma_T)}$$

Remark. Note the difference with the final-time observability for the dual heat equation:

$$\|\theta(0)\|_{L^2} \le C\|\theta\|_{L^2((0,T)\times\omega)}$$

Proposition 48. If the dual problem Eq. (15) is exactly observable in time T from Σ, then the control problem Eq. (11) is exactly controllable in time T from Σ.

Proof. Suppose Eq. (15) is exactly observable. We make the choice to find u of the form $u = \partial_{\mathbf{n}}\tilde\theta$ for some $\tilde\theta$ solution of Eq. (15) (in order to put the problem in the standard Riesz's form). Consider now $E := H_0^1(\Omega)\times L^2(\Omega)$ equipped with the norm $\|(\theta_0,\theta_1)\|_E := \|\partial_{\mathbf{n}}\theta_0\|_{L^2(\Sigma_T)}$, where θ is the solution of Eq. (15). This is an equivalent norm to the standard one:

$$\|(\theta_0,\theta_1)\|_E \gtrsim \|\theta_0\|_{H_0^1} + \|\theta_1\|_{L^2} \text{ (36 Observability inequality)}$$
$$\|(\theta_0,\theta_1)\|_E \lesssim \|\theta_0\|_{H_0^1} + \|\theta_1\|_{L^2} \text{ (43 Hidden regularity)}$$

E is Hilbert with this norm. Now, given $(\hat v_0,\hat v_1) \in E$, the left hand side is a continuous linear form on $(\theta(T),\partial_t\theta(T)) \in E$. So $\exists(\overline\theta_0,\overline\theta_1) \in E$ such that with $\overline\theta$ the corresponding solution of Eq. (15) one has: $\forall(\theta(T),\partial_t\theta(T)) \in E$ with the corresponding solution θ we have:

$$\langle \hat v_1, \theta(T)\rangle_{H^{-1}\times H_0^1} - \langle \hat v_0, \partial_t\theta(T)\rangle_{L^2\times L^2} = \int_{\Sigma_T}\partial_{\mathbf{n}}\overline\theta\partial_{\mathbf{n}}\theta$$

So we take $u := \partial_{\mathbf{n}}\overline\theta$. □

Theorem 49 (Bardos, Lebeau, Rauch). The system is exactly controllable (or the dual observable) if and only if any ray of geometrical optics in Ω (at speed 1) intersects Σ between times 0 and T.

Remark. If $\Sigma = \partial\Omega$, then the system is controllable of $T > \operatorname{diam}(\Omega)$.

3. | Abstract systems

Basic definitions

Definition 50. Let X, Y be Banach. A *bounded operator* is a couple $(D(A), A)$ where $D(A) \subseteq X$ is a subspace and $A : D(A) \to Y$ is a continuous linear operator. An *unbounded operator* is a couple $(D(A), A)$ where $D(A) \subseteq X$ is a subspace and $A : D(A) \to Y$ is a linear operator.

Remark. Usually we will omit specifying the domain $D(A)$.

Remark. Note that bounded operators are also unbounded operators.

Definition 51. Let A be an unbounded operator between Banach spaces X and Y. A is *densely defined* if $\overline{D(A)} = X$. A is *closed* if $\operatorname{graph}(A) = \{(x,y) \in D(A)\times Y : y = Ax\}$ is closed.

Form now on we will assume X, Y are Hilbert.

Definition 52. Let $(D(A), A)$ be a densely unbounded operator. We define the *adjoint operator* $(D(A^*), A^*)$ by:

$$D(A^*) = \{y \in Y^* : \exists c > 0 \text{ with } |\langle y, Ax\rangle_{Y^*\times Y}| \le c\|x\|_X \ \forall x \in D(A)\}$$

and $\forall y \in D(A^*)$ $\forall x \in D(A)$, A^*y is given in order to satisfy:

$$\langle A^*y, x\rangle_{X^*\times X} = \langle y, Ax\rangle_{Y^*\times Y}$$

Semigroups

From now on we will assume $X = Y$.

Definition 53. A one-parameter family of unbounded operators $(T(t))_{t\geq 0}$ is a *semigroup of operators* if:

- $T(0) = \mathrm{id}$
- $T(t+s) = T(t) \circ T(s)\ \forall t, s \geq 0$

Definition 54. A semigroup of operators $(T(t))_{t\geq 0}$ is called

- *uniformly continuous* if $\|T(t) - \mathrm{id}\| \xrightarrow[t\to 0^+]{} 0$.
- *strongly continuous* if $\forall x \in X\ \|T(t)x - x\| \xrightarrow[t\to 0^+]{} 0$.

Definition 55. Let $(T(t))_{t\geq 0}$ be a semigroup of operators. We call *infinitesimal generator* of $(T(t))_{t\geq 0}$ the unbounded operator $(D(A), A)$ where

$$D(A) = \left\{ x \in X : \exists \lim_{t\to 0^+} \frac{T(t)x - x}{t} \right\}$$

and $\forall x \in D(A)$ we define:

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

Proposition 56. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of operators. Then:

1. $\forall x \in X$, $t \mapsto T(t)x$ is continuous.
2. $\forall x \in X$ and all $t \geq 0$:
$$\int_0^t T(s)x\mathrm{d}s \in D(A) \text{ and } T(t)x - x = A\left(\int_0^t T(s)x\mathrm{d}s\right)$$
3. $\forall x \in D(A)$, $\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = AT(t)x = T(t)Ax$.
4. $\forall x \in D(A)$ and all $t, s \geq 0$:
$$T(t)x - T(s)x = \int_s^t AT(r)x\mathrm{d}r = \int_s^t T(r)Ax\mathrm{d}r$$
5. $\exists \alpha, C > 0$ such that $\|T(t)\| \leq C\mathrm{e}^{\alpha t}$.

Proof.

1. Take $x \in X$, $s, t \geq 0$ and $y := T(s)x$. Then:
$$\|T(t)x - T(s)x\| = \|T(t-s)T(s)x - T(s)x\| = \\ = \|T(t-s)y - y\| \xrightarrow[t\to s^+]{} 0$$
because of the strong continuity of the semigroup and Theorem 53.
2. We have:
$$\frac{T(h) - \mathrm{id}}{h}\int_0^t T(s)x\mathrm{d}s = \frac{1}{h}\int_0^t T(s+h)x\mathrm{d}s - \frac{1}{h}\int_0^t T(s)x\mathrm{d}s$$
$$= \frac{1}{h}\int_0^{t+h} T(s)\,\mathrm{d}s - \frac{1}{h}\int_0^t T(s)\mathrm{d}s =$$
$$= \frac{1}{h}\int_t^{t+h} T(s)\mathrm{d}s - \frac{1}{h}\int_0^h T(s)\mathrm{d}s \xrightarrow[h\to 0^+]{} T(t)x - x$$
3. Let $x \in D(A)$ then the following limit
$$\lim_{h\to 0^+} \frac{T(t+h)x - T(t)x}{h} = \lim_{h\to 0^+} T(t)\frac{T(h)x - x}{h} = \\ = T(t)Ax$$
exists because of $x \in D(A)$. Moreover using the properties of the semigroup we have it is also equal to $AT(t)x$. Now assume $h \to 0^-$ (so $h < 0$). Then:
$$\lim_{h\to 0^-} \frac{T(t+h)x - T(t)x}{h} = T(t+h)\frac{T(-h)x - x}{-h}$$
which exists and is equal to $AT(t)x$ because $x \in D(A)$.
4. We prove it for $s = 0$, and then the general case follows by the linearity of the integral. But then by Item 56-2:
$$T(t)x - x = A\int_0^t T(s)x\mathrm{d}s = \int_0^t AT(s)x\mathrm{d}s$$
and the exchange of the limit and the integral is justified by the existence of both limits.

□

Theorem 57.

1. If A is the infinitesimal generator of a strongly continuous semigroup, then it is closed and densely defined.
2. If $(T(t))_{t\geq 0}$, $(S(t))_{t\geq 0}$ are two strongly continuous semigroups with the same infinitesimal generator, then $T(t) = S(t)\ \forall t \geq 0$.
3. $\forall x_0 \in D(A)$, there exists a unique solution $x \in \mathcal{C}^0([0, +\infty), D(A)) \cap \mathcal{C}^1([0, +\infty), X)$ of $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t)$ with $x(0) = x_0$, and it is given by:
$$x(t) = T(t)x_0$$
Moreover, $\forall f \in \mathcal{C}^1([0, T], X)$, there exists a unique solution $x \in \mathcal{C}^0([0, T], D(A)) \cap \mathcal{C}^1([0, T], X)$ of $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + f(t)$ with $x(0) = x_0$, and it is given by:
$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)\mathrm{d}s$$
This last formula is called the *variation of constants formula* or *Duhamel's formula.*

Remark. Duhamel's formula is still valid even if $x_0 \in X$ and $f \in L^1((0, T), X)$. In this case, the resulting solution $x(t)$ is called *mild solution.* Any mild solution is a limit of classical solutions.

Theorem 58. Suppose X is reflexive. Then, if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with infinitesimal generator A, then A^* is the infinitesimal generator of the adjoint semigroup $(T(t)^*)_{t\geq 0}$.

Remark. We will denote by $T(t)^*$ also by $T^*(t)$.

Definition 59. A semigroup $(T(t))_{t\geq 0}$ is called *contraction* if $\|T(t)\| \leq 1 \; \forall t \geq 0$.

Definition 60. Let $(D(A), A)$ be an unbounded operator. The *resolvent set* of A is the set $\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective}\}$. Given $\lambda \in \rho(A)$, the *resolvent operator* is the operator $R_\lambda(A) := (\lambda I - A)^{-1}$.

Theorem 61 (Hille-Yosida). Let $(D(A), A)$ be an operator closed and densely defined. Then, it is the infinitesimal generator of a contraction semigroup if and only if:

$$(0, \infty) \subseteq \rho(A) \text{ and } \forall \lambda > 0, \|R_\lambda(A)\| \leq \frac{1}{\lambda}$$

Corollary 62. Let $(D(A), A)$ be an operator closed and densely defined. $(D(A), A)$ is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq \mathrm{e}^{ct} \; \forall t \geq 0$ if and only if:

$$(c, \infty) \subseteq \rho(A) \text{ and } \forall \lambda > c, \|R_\lambda(A)\| \leq \frac{1}{\lambda - c}$$

Definition 63. An operator $(D(A), A)$ is called *dissipative* if $\forall x \in D(A)$, $\langle Ax, x\rangle \leq 0$.

Theorem 64 (Lümmer-Phillips). Let $(D(A), A)$ be an operator closed and densely defined. Then:

1. If A is dissipative and $\exists \lambda_0 > 0$ such that $\operatorname{im}(\lambda_0 I - A) = R$, then $\forall \lambda > 0$, $\operatorname{im}(\lambda I - A) = R$ and A generates a semigroup of contractions.

2. If A generates a semigroup of contractions, then $\operatorname{im}(\lambda I - A) = X \; \forall \lambda > 0$.

Corollary 65. Let $(D(A), A)$ be an operator closed and densely defined. If A and A^* are both dissipative, then A generates a semigroup of contractions.

Applications to control theory

In this section we consider the control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{in } [0, T] \\ x(0) = x_0 \end{cases} \tag{16}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ and B is a bounded operator, with:

$$S(t) : L^2(\Omega) \to L^2(\Omega) \qquad B : L^2(\omega) \to L^2(\Omega)$$

We will first consider the interior control in a region $\omega \subseteq \Omega$. We will denote by F_T the operator:

$$\begin{array}{rccc} F_T : L^2(0, T; L^2(\omega)) & \longrightarrow & L^2(\Omega) \\ u & \longmapsto & \int_0^T S(T-s)Bu(s)\mathrm{d}s \end{array}$$

Remark. Note that exact controllability at time T is equivalent to controllability starting from 0 which in turn is equivalent to the surjectivity of F_T; approximate controllability at time T is equivalent to $\operatorname{im} F_T$ being dense in $L^2(\Omega)$, and null controllability at time T is equivalent to $\operatorname{im} F_T \supseteq \operatorname{im} S(T)$.

Theorem 66. Let $S : H_1 \to H$ and $T : H_2 \to H$ be bounded linear operators between Hilbert spaces. Then, $\operatorname{im}(S) \subseteq \operatorname{im}(T)$ if and only if $\exists c > 0$ such that $\forall x \in H$, $\|S^* x\|_{H_1} \leq c \|T^* x\|_{H_2}$.

Proof.

$\Longrightarrow$) If $\operatorname{im}(S) \subseteq \operatorname{im}(T)$, then $\forall x \in H_1 \; \exists! y \in \ker(T)^\perp$ such that $Sx = Ty$.

- *Existence*: for $x \in H_1$, we find $y \in H_2$ such that $Sx = Ty$ and we define $z := \pi_{\ker(T)^\perp} y$, where $\pi_{\ker(T)^\perp}$ is the orthogonal projection on $\ker(T)^\perp$. Then, $z - y \in (\ker(T)^\perp)^\perp = \ker(T)$, and so $T(z - y) = 0$, and thus $Tz = Ty = Sx$.
- *Uniqueness*: if $Sx = Ty_1 = Ty_2$, then $T(y_1 - y_2) = 0$, and so $y_1 - y_2 \in \ker(T)$, but also $y_1, y_2 \in \ker(T)^\perp$ (by hypothesis). Thus, $y_1 = y_2$.

We define $G : H_1 \to \ker(T)^\perp \subset H_2$ such that to $x \in H_1$ we associate the unique $y \in \ker(T)^\perp$ such that $Sx = Ty$. We have that G is linear. To see that it is continuous we used the **?? ??**. We need to prove that if $x_n \xrightarrow{H_1} x$ and $Gx_n \xrightarrow{H_2} y$, then $Gx = y$. We have that $G(x_n) \in \ker(T)^\perp \; \forall n$ and $\ker(T)^\perp$ is closed, so $y \in \ker(T)^\perp$. We have that $T(G(x_n)) = Sx_n \; \forall n$. Taking the limit and using the continuity of S and T we get $Ty = Sx$, which by uniqueness implies $y = Gx$. So $S = TG$, and thus $S^* = G^* T^*$, with G^* continuous. Thus, $\forall x \in H$:

$$\|S^* x\|_{H_1} = \|G^* T^* x\|_{H_1} \leq \|G^*\|_{\mathcal{L}(H_2, H_1)} \|T^* x\|_{H_2}$$

$\Longleftarrow$) We will prove that there exists an operator $D : H_2 \to H_1$ such that $S^* = DT^*$. If $y \in \operatorname{im}(T^*)$, let $x \in H$ be such that $y = T^* x$, and then we define $Dy := S^* x$. This definition is independent of the choice of x because if $x_1, x_2 \in H$ are such that $y = T^* x_1 = T^* x_2$, then $T^*(x_1 - x_2) = 0$, and so (by hypotheses) $S^*(x_1 - x_2) = 0$, and thus $S^* x_1 = S^* x_2$. So $D : \operatorname{im}(T^*) \to H_1$ is well-defined, it is linear and continuous:

$$\|Dy\|_{H_1} = \|S^* x\|_{H_1} \leq c \|T^* x\|_{H_2} = c \|y\|_{H_2}$$

So D can be uniquely extended as a continuous linear map on $\overline{\operatorname{im}(T^*)}$. We decide to set $D|_{\operatorname{im}(T^*)^\perp} = 0$ and we get a continuous linear map $D : H_2 \to H_1$ such that $S^* = DT^*$. Taking adjoints we get $S = TD^*$, so $\operatorname{im}(S) \subseteq \operatorname{im}(T)$.

□

Theorem 67. Let $A : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces. Then:

1. $\operatorname{im}(A)$ is dense $\iff \ker(A^*) = \{0\}$.

2. $\operatorname{im}(A) = H_2 \iff \exists c > 0$ such that $\forall x \in H_2$, $\|x\|_{H_2} \leq c\,\|A^*x\|_{H_1}$.

Proof.

1. Recall that $\operatorname{im}(A)^\perp = \ker(A^*)$:

$$\begin{aligned}\overline{\operatorname{im}(A)} = H_2 &\iff (\operatorname{im}(A)^\perp)^\perp = H_2\\ &\iff \operatorname{im}(A)^\perp = \{0\}\\ &\iff \ker(A^*) = \{0\}\end{aligned}$$

2. Use Theorem 66 with $H_1 = H$ and $S = \mathrm{id}_H$.

□

Proposition 68. The adjoint of F_T is given by:

$$\begin{aligned} F_T^* : L^2(\Omega) &\longrightarrow \quad L^2(0,T;L^2(\omega))\\ y_T \quad &\longmapsto s \mapsto B^*S^*(T-s)y_T\end{aligned}$$

Proof. Let $y_T \in L^2(\Omega)$ and $u \in L^2(0,T;L^2(\omega))$. Then:

$$\begin{aligned}\langle F_T^* y_T, u\rangle_{L^2(0,T;L^2(\omega))} &= \int_0^T \langle B^*S^*(T-s)y_T, u(s)\rangle_{L^2(\omega)}\mathrm{d}s\\ &= \int_0^T \langle S^*(T-s)y_T, Bu(s)\rangle_{L^2(\Omega)}\mathrm{d}s\\ &= \int_0^T \langle y_T, S(T-s)Bu(s)\rangle_{L^2(\Omega)}\mathrm{d}s\\ &= \langle y_T, \int_0^T S(T-s)Bu(s)\mathrm{d}s\rangle_{L^2(\Omega)}\\ &= \langle y_T, F_T u\rangle_{L^2(\Omega)}\end{aligned}$$

□

Theorem 69. Consider the control system Eq. (16) and its dual system:

$$\begin{cases}-\dot{x}(t) = A^*x(t) & \text{in } [0,T]\\ x(T) = y_T\end{cases} \tag{17}$$

Then:

1. The system Eq. (16) is exactly controllable at time T if and only if the system Eq. (17) is final time observable by means of B^*, that is, if $\exists c > 0$ such that for all solution y of Eq. (17) we have:

$$\|y_T\|_H \leq c\,\|B^*x\|_{L^2(0,T;L^2(\omega))}$$

2. The system Eq. (16) is approximately controllable at time T if and only if the system Eq. (17) satisfies the unique continuation property, that is, if x solution of Eq. (17) satisfies $B^*x(t) = 0$ $\forall t \in [0,T]$ then $y_T = 0$, i.e. $x = 0$.

3. The system Eq. (16) is null controllable at time T if and only if the system Eq. (17) is initial time observable: $\exists c > 0$ such that for all solution x of Eq. (17) we have:

$$\|x(0)\|_H \leq c\,\|B^*x\|_{L^2(0,T;L^2(\omega))}$$

4. | Backstepping for boundary control in PDEs

Backstepping consists in transforming a system into another one, called *target system*, which has the desired stability properties. In order to study, we will be considering the following reaction-diffusion equation:

$$\begin{cases}\partial_t x = x_{zz} + \lambda x & \text{in } (0,T)\times(0,1)\\ x(t,0) = 0 & \text{in } (0,T)\\ x(t,1) = u(t) & \text{in } (0,T)\end{cases} \tag{18}$$

and we assume that $\lambda > 0$ is large enough such that the system is unstable (the eigenvalues are of the form $\lambda - n^2\pi^2$ with $n \in \mathbb{N}$).
The first step is to choose a target system such that the origin is exponentially (or asymptotically) stable. We will consider the following target system:

$$\begin{cases}\partial_t w = w_{zz} & \text{in } (0,T)\times(0,1)\\ w(t,0) = 0 & \text{in } (0,T)\\ w(t,1) = 0 & \text{in } (0,T)\end{cases} \tag{19}$$

Proposition 70. The system Eq. (19) is exponentially stable for the L^2 norm.

Proof. We need to find a Lyapunov functional V such that $\dot{V} \leq -\alpha V$ for some $\alpha > 0$. We take $V(t) = \int_0^1 w(t,z)^2\mathrm{d}z$. Then:

$$\begin{aligned}\dot{V} = 2\int_0^1 w(t,z)w_t(t,z)\mathrm{d}z &= 2\int_0^1 w(t,z)w_{zz}(t,z)\mathrm{d}z\\ &= -2\int_0^1 w_z(t,z)^2\mathrm{d}z \leq -\alpha V\end{aligned}$$

for some $\alpha > 0$, due to **??** **??**. □

Next step is to find a backstepping transformation $w = T(x)$ and the invertible operator T^{-1}. We will consider the following transformation:

$$w(z,t) = x(z,t) - \int_0^z K(z,y)x(y,t)\mathrm{d}y \tag{20}$$

where K is a kernel yet to be determined.

Proposition 71. Let $f : [a,b] \to \mathbb{C}$ be continuous and $K : [a,b]^2 \to \mathbb{C}$ be a bounded function. Then, the integral equation:

$$f(t) = \varphi(t) - \int_a^b K(t,s)\varphi(s)\mathrm{d}s, \qquad t \in [a,b]$$

admits a unique solution $\varphi \in \mathcal{C}([a,b])$. Furthermore, there exists $\ell : [a,b]^2 \to \mathbb{C}$ bounded such that:

$$\varphi(t) = f(t) - \int_a^t \ell(t,s)f(s)\mathrm{d}s, \qquad t \in [a,b]$$

Thus, our transformation is in Eq. (20) is invertible. Finally, we need to define our control law. Imposing $w_z(1,t)=0$ in Eq. (20) we get:

$$0=w_z(1,t)=x_z(1,t)-K(1,1)x(1,t)-\int_0^1 K_z(1,y)x(y,t)\mathrm{d}y$$

$$u(t)=K(1,1)x(1,t)+\int_0^1 K_z(1,y)x(y,t)\mathrm{d}y$$

So, we are left to find if a suitable K exists. Recall that the condition $w(1,t)=0$ is automatically satisfied. We need to make use of the PDE of w. Using Eq. (20) to compute w_t and w_{zz}, and equating both equations it suffices to find K such that:

$$\begin{cases} -2\frac{\mathrm{d}}{\mathrm{d}z}K(z,z)=\lambda \implies K(z,z)=-\frac{\lambda}{2}z \\ K_{zz}=K_{yy}+\lambda K \\ K(z,0)=0 \end{cases}$$

which has a unique solution given by:

$$K(z,y)=-\lambda y\frac{I_1(\sqrt{\lambda(z^2-y^2)})}{\sqrt{\lambda(z^2-y^2)}}$$

where I_1 is the modified Bessel function of the first kind of order 1.