Advanced topics in functional analysis and PDEs

1. | L^p spaces

Topologies of L^p spaces

Definition 1. Let E be a Banach space and $(x_n)_{n\in\mathbb{N}} \in E$. We say that $(x_n)_{n\in\mathbb{N}}$ *converges weakly* to $x \in E$ if $\forall f \in E^*$ we have:

$$\lim_{n\to\infty} f(x_n) = f(x)$$

We denote this by $x_n \rightharpoonup x$.

Definition 2. Let E be a Banach space and $(x_n)_{n\in\mathbb{N}} \in E$. We say that $(x_n)_{n\in\mathbb{N}}$ *converges strongly* to $x \in E$ if we have:

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

We denote this by $x_n \to x$.

Definition 3. Let E be a Banach space and $(L_n)_{n\in\mathbb{N}} \in E$. Assume $E = F^*$, where F is a Banach space. Then, we say that $(L_n)_{n\in\mathbb{N}}$ *converges weakly-** to $L \in E$ if $\forall x \in F$ we have:

$$\lim_{n\to\infty} L_n(x) = L(x)$$

We denote this by $L_n \overset{*}{\rightharpoonup} L$.

Theorem 4. Let E be a Banach space and $(x_n)_{n\in\mathbb{N}} \in E$. Then:

1. If $x_n \to x$, then $x_n \rightharpoonup x$.
2. If E is reflexive then weak convergence is equivalent to weak-* convergence.
3. If $x_n \to x$, $x_n \rightharpoonup x$ or $x_n \overset{*}{\rightharpoonup} x$, then (x_n) is bounded.
4. Let $(L_n)_{n\in\mathbb{N}} \in E^*$. If $x_n \to x$ in E and $L_n \overset{*}{\rightharpoonup} L$ in E^*, then $L_n(x_n) \to L(x)$ in $\mathbb{C}$.

Proof. We only prove the first and last points.

1. Let $L \in E^*$. Then, $\|L\| < \infty$ and:

$$|L(x_n) - L(x)| \leq \|L\| \, \|x_n - x\| \to 0$$

4. Let $(L_n) \in E^*$ converge to $L \in E^*$ weakly-* and let $x_n \to x$ in E. Then:

$$\begin{aligned}|L_n(x_n) - L(x)| &\leq |(L_n - L)(x_n)| + |L(x_n) - L(x)| \\ &\leq \|L_n - L\| \, \|x_n\| + \|L\| \, \|x_n - x\|\end{aligned}$$

And the result follows.

□

Theorem 5 (Banach-Alaoglu theorem). Let $\Omega \subseteq \mathbb{R}^d$ be a set and $1 < p < \infty$. If (f_n) is a bounded sequence in $L^p(\Omega)$, then there is a subsequence (f_{n_k}) and $f \in L^p(\Omega)$ so that $f_{n_k} \rightharpoonup f$ in $L^p(\Omega)$. If $p = \infty$, then there is a subsequence (f_{n_k}) and $f \in L^\infty(\Omega)$ so that $f_{n_k} \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$.

Lower-semicontinuous functions and convexity

Definition 6. Let E be a Banach space, $(x_n) \in E$ and $f : E \to \mathbb{R}$. We say that f is *strongly lower-semicontinuous* if:

$$x_n \to x \implies f(x) \leq \liminf_{n\to\infty} f(x_n)$$

Remark. Analogously, we can define *weakly lower-semicontinuity* and *weak-* lower-semicontinuity* by replacing $x_n \to x$ by $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$ respectively.

Theorem 7. Let E be a Banach space and $f : E \to \mathbb{R}$ be convex. Then, f is strongly lower-semicontinuous if and only if f is weakly lower-semicontinuous. In particular, the map $\|\cdot\|_E$ is weakly lower-semicontinuous.

Proof. We only prove one implication, and also we admit that if f is convex and strongly lower-semicontinuous, then there is a continuous linear operator L_x (called *support plane*) so that $\forall y \in E$ we have:

$$f(y) \geq f(x) + L_x(y - x)$$

In particular, if $(x_n) \rightharpoonup x$, then:

$$f(x_n) \geq f(x) + L_x(x_n - x) \implies \liminf_{n\to\infty} f(x_n) \geq f(x)$$

□

Theorem 8. Let $\Omega \subseteq \mathbb{R}^d$ be a set and $1 < p < \infty$. If $f_n \rightharpoonup f$ in $L^p(\Omega)$, then:

$$\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p$$

In addition, if $\|f\|_p = \lim_{n\to\infty} \|f_n\|_p$, then $f_n \to f$ in $L^p(\Omega)$.

Proof. The first point is Theorem 7 in the case $E = L^p(\Omega)$. We only prove the second point for $p = 2$. If $\|f\|_2 = \lim_{n\to\infty} \|f_n\|_2$, then:

$$\begin{aligned}{\|f_n - f\|_2}^2 &= \|f_n\|_2^2 + \|f\|_2^2 - 2\operatorname{Re}\int_\Omega f_n \overline{f} \\ &\to {\|f\|_2}^2 + {\|f\|_2}^2 - 2\operatorname{Re}\int_\Omega f\overline{f} \\ &= 0\end{aligned}$$

where the convergence of the integral is due to the weakly convergence of f_n to f. □

2. | Sobolev spaces

Basic definitions

Definition 9 (Sobolev spaces). Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the *Sobolev spaces* $W^{m,p}$ as:

$$W^{m,p}(\Omega) := \{f \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m, \partial^\alpha f \in L^p(\Omega)\}$$

Moreover we define the associate norm $\|\cdot\|_{W^{m,p}(\Omega)}$ as:

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha|\leq m} \|\partial^\alpha f\|_p^{\ p}\right)^{1/p}$$

If $p = 2$, we denote $H^m(\Omega) := W^{m,2}(\Omega)$.

Theorem 10. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Then, for all $m \in \mathbb{N}$ and all $1 \leq p \leq \infty$, $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is Banach. Moreover, if $p < \infty$, it is separable and if $1 < p < \infty$, it is reflexive. Finally, $H^m(\Omega)$ is a separable Hilbert space.

Proof. Let (f_j) be a Cauchy sequence in $W^{m,p}(\Omega)$. Then, (f_j) is Cauchy in $L^p(\Omega)$, and for all $|\alpha| \leq m$, $(\partial^\alpha f_j)$ is Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there are $f \in L^p(\Omega)$ and $f_\alpha \in L^p(\Omega)$, so that:

$$f_j \xrightarrow{L^p} f \qquad \partial^\alpha f_j \xrightarrow{L^p} f_\alpha$$

It remains to prove that $f_\alpha = \partial^\alpha f$. Since we have convergence in L^p, we also have convergence in the distributional sense, that is $f_j \overset{\mathcal{D}^*}{\to} f$. In particular, we must have $\partial^\alpha f_j \overset{\mathcal{D}^*}{\to} \partial^\alpha f$. By uniqueness of the limit in $\mathcal{D}^*(\Omega)$, we indeed have $f_\alpha = \partial^\alpha f$. This proves that $W^{m,p}(\Omega)$ is complete. Now, the map

$$\begin{aligned} W^{m,p}(\Omega) &\longrightarrow (L^p(\Omega))^N \\ f &\longmapsto (\partial^\alpha f)_{|\alpha|\leq m} \end{aligned}$$

with $N := \left|\{\alpha \in \mathbb{N}^d : |\alpha| \leq m\}\right|$ is an isometry. So $W^{m,p}(\Omega)$ can be identified with a closed vector space of $(L^p(\Omega))^N$. In particular, $W^{m,p}(\Omega)$ is separable for $p < \infty$, and it is reflexive for $1 < p < \infty$. □

Definition 11. Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the space $W_0^{m,p}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}$, where the closure is taken with the norm of $W^{m,p}(\Omega)$. Similarly, we set $H_0^m(\Omega) := W_0^{m,2}(\Omega)$.

Remark. Note that $W_0^{m,p}(\Omega)$ is also Banach (with the same norm as $W^{m,p}(\Omega)$) because it is a closed subspace in a Banach space.

Lemma 12. Let $1 \leq p \leq \infty$ and (ϕ_ε) be an approximation of identity. For all $f \in L^p(\mathbb{R}^d)$, we set $f_\varepsilon := f * \phi_\varepsilon$. Then:

- f_ε is smooth.
- $f_\varepsilon \in L^p(\mathbb{R}^d)$ with $\|f_\varepsilon\|_p \leq \|f\|_p$.
- If $p < \infty$, then $\|f_\varepsilon - f\|_p \xrightarrow{\varepsilon\to 0} 0$.

Theorem 13. For $1 \leq p < \infty$ we have that $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$. In particular, $\mathcal{C}^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ are dense in $W^{m,p}(\mathbb{R}^d)$ for $1 \leq p < \infty$.

Proof. Let us first prove that $\mathcal{C}^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$. Let $f \in W^{m,p}(\mathbb{R}^d)$ and set $f_\varepsilon := f * j_\varepsilon$ for an approximation of identity j_ε. By Theorem 12, the functions f_ε are smooth. For all $|\alpha| \leq m$, the function $\partial^\alpha f$ is in $L^p(\mathbb{R}^d)$, and we have $\partial^\alpha(f_\varepsilon) = (\partial^\alpha f) * j_\varepsilon$. By ??, we deduce that $\forall |\alpha| \leq m$:

$$\|\partial^\alpha f * j_\varepsilon - \partial^\alpha f\|_p \to 0$$

This already proves that $\mathcal{C}^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$. For the second part, we take $f \in \mathcal{C}^\infty(\mathbb{R}^d) \cap W^{m,p}(\mathbb{R}^d)$ and set $f_n := \chi(x/n)f$, where χ is a smooth cut-off function satisfying $\chi(x) = 1$ for $|x| \leq 1$. Then, $f_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. By ?? ??, we have $\|f_n - f\|_p \to 0$. Moreover, we have:

$$\begin{aligned} \|\nabla f_n - \nabla f\|_p &= \|\nabla \chi f + \nabla f[\chi(x/n) - 1]\|_p \\ &\leq \|\nabla \chi\|_\infty \|f\|_p + \|\nabla f[\chi(x/n) - 1]\|_p \end{aligned}$$

and the last term goes to 0 again by ?? ??. So $\|\nabla f_n - \nabla f\|_p \to 0$. We go on with all derivatives, which proves that $\|\partial^\alpha f_n - \partial^\alpha f\|_p \to 0$ for all $|\alpha| \leq m$. This shows that $f_n \to f$ in $W^{m,p}(\mathbb{R}^d)$. □

Theorem 14 (Poincaré's inequality). Let Ω be a bounded open set in $\mathbb{R}^d$ and let $1 \leq p < \infty$. Then, there is a constant $C = C(\Omega, p)$ so that $\forall u \in W_0^{m,p}(\Omega)$ we have:

$$\|u\|_p \leq C \|\nabla u\|_p$$

Remark. 14 Poincaré's inequality is also valid when Ω is unbounded in one direction.

Corollary 15. If Ω is bounded, then the constant function $f(x) = C$ with $C \neq 0$ is not in $W_0^{1,p}$. Thus, we cannot approximate constant functions by $\mathcal{C}_0^\infty(\Omega)$ functions, with the $W^{1,p}(\Omega)$ norm.

Definition 16. Let Ω be a bounded set. We define the average of u in Ω as:

$$⨍_\Omega u := \frac{1}{|\Omega|}\int_\Omega u$$

Theorem 17 (Poincaré-Wirtinger's inequality). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded connected open set with $\mathcal{C}^1$ boundary, and let $1 \leq p < \infty$. Then, there is a constant $C = C(\Omega, p)$ so that $\forall u \in W^{1,p}(\Omega)$ we have:

$$\left\| u - ⨍_\Omega u \right\|_p \leq C \|\nabla u\|_p$$

Sobolev embeddings

Definition 18. Let E, F be Banach. We say that F is *embedded* in E if $F \subseteq E$ and the inclusion map $i : F \hookrightarrow E$ is continuous. We say that F is *compactly embedded* in E if $F \subseteq E$ and the inclusion map $i : F \hookrightarrow E$ is compact.

Theorem 19 (Gagliardo, Nirengerg and Sobolev's inequality). For all $1 \leq p \leq \frac{d}{m}$, there is a constant $C = C(p, m, d)$ so that $\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we have:

$$\|u\|_q \leq C \sum_{|\alpha|=m} \|\partial^\alpha u\|_p$$

where $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$. That is, we have the continuous embedding $W^{m,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$. In particular for $m = 1$, we have $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

Proof. By induction, it suffices to prove the result only for $m = 1$. We will prove only the case $d = 2$. We start with $p = 1$. Let $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. We have:

$$|u(x_1, x_2)| \leq \int_{-\infty}^{x_1} |\partial_{x_1} u(s, x_2)| \,\mathrm{d}s \leq$$

$$\leq \int_{\mathbb{R}} |\partial_{x_1} u(s, x_2)| \,\mathrm{d}s =: v_1(x_2)$$

Similarly, $|u(x_1, x_2)| \leq v_2(x_1)$. So:

$$\|u\|_2{}^2 \leq \int_{\mathbb{R}^2} |v_1(x_2)||v_2(x_1)| \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \|v_1\|_1 \|v_2\|_1 =$$

$$= \|\partial_{x_1} u\|_1 \|\partial_{x_2} u\|_1 \leq \|\nabla u\|_1{}^2$$

For the case $1 \leq p < 2$, we apply the result to the function $u_t := |u|^{t-1} u$. This function satisfies $|\nabla u_t| = t|u|^{t-1}|\nabla u|$ and so:

$$\|u\|_{2t}{}^t = \|u_t\|_2 \leq \|u_t\|_1 = t \left\| |u|^{t-1} \nabla u \right\| \leq$$

$$\leq t \left\| |u|^{t-1} \right\|_{p'} \|\nabla u\|_p = t \|u\|_{(t-1)p'}{}^{t-1} \|\nabla u\|$$

where p' is the Hölder conjugate of p. Now, we choose t so that $2t = (t-1)p'$, that is, $t = \frac{p}{2-p}$ and so:

$$\|u\|_{\frac{2p}{2-p}} \leq \frac{p}{2-p} \|\nabla u\|_p$$

□

Definition 20 (Hölder continuity). Let $k \in \mathbb{N}$ and $0 \leq \theta \leq 1$. We say that a map $u : \Omega \to \mathbb{C}$ is $\mathcal{C}^{k,\theta}$*-Hölder continuous* if there is $C \geq 0$ such that $\forall |\alpha| = k$ and all $x, y \in \Omega$ we have:

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C|x - y|^\theta$$

The set of such functions is denoted by $\mathcal{C}^{k,\theta}(\Omega)$.

Remark. Note that $\mathcal{C}^{k,0}(\Omega) = \mathcal{C}^k(\Omega)$ and that $\mathcal{C}^{0,1}(\Omega)$ is the set of Lipschitz continuous functions.

Remark. Note that $\mathcal{C}^{k,\theta}(\Omega)$ together with the norm

$$\|u\|_{\mathcal{C}^{k,\theta}(\Omega)} := \sup_{x \neq y} \sup_{|\alpha|=k} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^\theta}$$

is a Banach space.

Theorem 21 (Morrey's embedding). Let $m \geq 1$ and $p > \frac{d}{m}$. Then, $W^{m,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. In addition, let:

$$k := \left\lfloor m - \frac{d}{p} \right\rfloor \qquad \theta := m - \frac{d}{p} - k \in [0, 1)$$

If $\theta \neq 0$, then $W^{m,p}(\mathbb{R}^d) \subset \mathcal{C}^{k,\theta}(\mathbb{R}^d)$.

Theorem 22. For all $1 \leq p \leq \infty$, $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \cap \mathcal{C}^0(\mathbb{R})$.

Proof. The proof for $p > 1$ comes from 21 Morrey's embedding. For $p = 1$ we have that $\forall u \in \mathcal{C}_0^\infty(\mathbb{R})$ we have: $|u(x)| \leq \int_{-\infty}^x |u'| \leq \|u'\|_1$. So $\|u\|_\infty \leq \|u'\|_1$. By density, this proves that $\forall u \in W^{1,1}(\mathbb{R})$ we have $u \in L^\infty(\mathbb{R})$. Now, let $(u_n) \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $u_n \to u$ in $W^{1,1}(\mathbb{R})$. Then, $u_n \to u$ in $L^\infty(\mathbb{R})$ and so u is continuous because it is the uniform limit of continuous functions. □

Extension operators

Definition 23. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. An *extension* of $u \in W \in W^{m,p}(\Omega)$ is a function $\tilde{u} \in W^{m,p}(\mathbb{R}^d)$ so that $\tilde{u} \overset{\text{a.e.}}{=} u$ in Ω. An *extension operator* is a bounded linear operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$ so that Eu is an extension of u $\forall u \in W^{m,p}(\Omega)$.

Remark. From now on, we will denote $\mathbb{R}^d_\pm := \mathbb{R}^{d-1} \times \mathbb{R}_\pm$ and $\mathbb{R}^d_0 := \mathbb{R}^{d-1} \times \{0\}$.

Theorem 24. For all $m \in \mathbb{N}$ and all $1 \leq p < \infty$, $\mathcal{C}^\infty(\overline{\mathbb{R}^d_{\geq 0}})$ is dense in $W^{m,p}(\mathbb{R}^d_{\geq 0})$.

Proof. Let

$$\tau_h(u)(x_1, \ldots, x_d) := u(x_1, \ldots, x_{d-1}, x_d + h)$$

be the translation operator and set $u_\varepsilon := \tau_\varepsilon(u) * \phi_\varepsilon$, where $\varepsilon > 0$ and ϕ_ε is an approximation of identity. Then, $u_\varepsilon \in \mathcal{C}^\infty(\overline{\mathbb{R}^d_{\geq 0}})$ by the properties of the convolution. Moreover:

$$\|\partial^\alpha u_\varepsilon - \partial^\alpha u\|_p \leq \|\partial^\alpha u_\varepsilon - \partial^\alpha(\tau_\varepsilon u)\|_p + \|\partial^\alpha(\tau_\varepsilon u) - \partial^\alpha u\|_p$$
$$\leq \|(\partial^\alpha \tau_\varepsilon u) * \phi_\varepsilon - \partial^\alpha(\tau_\varepsilon u)\|_p + \|\tau_\varepsilon(\partial^\alpha u) - \partial^\alpha u\|_p$$

The first term goes to zero by the properties of smoothing sequences, and the second goes to zero since translations are continuous in L^p (check **??**). □

Remark. The same proof shows that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, if Ω is bounded with $\partial\Omega$ of class $\mathcal{C}^1$. This time, one needs to locally translate u along the normal direction.

Theorem 25. For all $m \in \mathbb{N}$ and all $1 \leq p < \infty$, there is an extension operator $E : W^{m,p}(\mathbb{R}^d_{\geq 0}) \to W^{m,p}(\mathbb{R}^d)$.

Proof. We only do the proof for $d = 1$ and $m = 1$ to highlight the main ideas. Let $u \in W^{1,p}(\mathbb{R}_{\geq 0})$. We define the *first order reflection*:

$$\bar{u} := \begin{cases} u(x) & \text{if } x \geq 0 \\ -3u(-x) + 4u(-x/2) & \text{if } x < 0 \end{cases}$$

By density, it is enough to prove the result for $u \in \mathcal{C}^1(\mathbb{R}_{\geq 0})$. An easy check shows that $\bar{u} \in \mathcal{C}^1(\mathbb{R})$. Moreover, we have:

$$\begin{aligned} \|\bar{u}\|_{W^{1,p}(\mathbb{R})}{}^p &= \int_{\mathbb{R}} |\bar{u}|^p + |\bar{u}'|^p \\ &= \int_{\mathbb{R}_{\geq 0}} |u|^p + |u'|^p + \int_{\mathbb{R}_{\leq 0}} [|-3u(-x) + 4u(-x/2)|^p + \\ &\qquad + |3u'(-x) - 2u'(-x/2)|^p] \\ &\leq C \|u\|_{W^{1,p}(\mathbb{R}_{\geq 0})}{}^p \end{aligned}$$

for some constant $C > 0$. Thus, E is a bounded extension operator. □

Remark. The reader can check that the same construction works on $\mathbb{R}^d$, by setting

$$\bar{u}(x_1, \ldots, x_d) := \begin{cases} u(x_1, \ldots, x_d) & \text{if } x_d \geq 0 \\ -3u(x_1, \ldots, x_{d-1} - x_d) + \\ +4u(x_1, \ldots, x_{d-1} - x_d/2) & \text{if } x_d < 0 \end{cases}$$

The proof for higher derivatives $m \geq 1$ needs to add more terms in order to make the junction smooth enough.

Definition 26. We say that a domain $\Omega \subseteq \mathbb{R}^d$ has boundary of class $\mathcal{C}^k$ if $\forall x \in \partial\Omega$ there is a neighborhood $\varepsilon, \delta > 0$ and a $\mathcal{C}^k$-diffeomorphism $\phi : B(x, \varepsilon) \to B(0, \delta)$ so that $\phi(x) = 0$ and $\phi(B(x, \varepsilon) \cap \Omega) = B(0, \delta) \cap \mathbb{R}^d_{\geq 0}$. Note that in particular this implies that $\phi(\partial\Omega \cap B(x, \varepsilon)) = B(0, \delta) \cap \mathbb{R}^d_0$.

Theorem 27. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^k$ boundary. Then, $\forall m \leq k$ and all $1 \leq p < \infty$, there is an extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$.

Theorem 28. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^k$ boundary. Then, $\forall m \leq k$, if $1 \leq p < \frac{d}{m}$, there is an embedding $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{d}$. If $p > \frac{d}{m}$, then $W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{\ell,\theta}(\overline{\Omega})$, where $\ell := \left\lfloor m - \frac{d}{p} \right\rfloor$ and $\theta := m - \frac{d}{p} - \ell \in [0, 1)$.

Theorem 29 (Reillich-Kondrachov's compactness theorem). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^k$ boundary. Then, $\forall m \leq k$ we have:

- If $1 \leq p < \frac{d}{m}$, $\forall r \in [p, q)$, where $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{d}$, the embedding $W^{m,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.
- If $p \geq \frac{d}{m}$, then $\forall r \in [p, \infty)$, the embedding $W^{m,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.
- If $p > \frac{d}{m}$, then the embedding $W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$ is compact.

Trace operators

Theorem 30. Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^d_{\geq 0})$. Then, the function $u|_{\mathbb{R}^d_0} : \mathbb{R}^{d-1} \to \mathbb{C}$ belongs to $L^p(\mathbb{R}^{d-1})$.

Definition 31. We define the *trace operator* as the map:

$$\begin{aligned} \operatorname{Tr} : W^{1,p}(\mathbb{R}^d_{\geq 0}) &\longrightarrow L^p(\mathbb{R}^{d-1}) \\ u &\longmapsto u|_{\mathbb{R}^d_0} \end{aligned}$$

Theorem 32. Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^d_{\geq 0})$. Then, $\operatorname{Tr} u = 0$ if and only if $u \in W^{1,p}_0(\mathbb{R}^d_{\geq 0})$.

Theorem 33. Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^1$ boundary. Then, the trace operator

$$\begin{aligned} \operatorname{Tr} : W^{1,p}(\Omega) &\longrightarrow L^p(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned}$$

is bounded. Here we are taking the norm of $L^p(\partial\Omega)$ as $\|u\|_{L^p(\partial\Omega)}{}^p := \int_{\partial\Omega} |u|^p$. In addition:

- $\forall u \in W^{1,p}(\Omega)$, $\operatorname{Tr} u = 0$ if and only if $u \in W^{1,p}_0(\Omega)$.
- For $p = 2$, Tr is surjective.

Lemma 34. Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then, $|u| \in W^{1,p}(\Omega)$ and $\|\boldsymbol{\nabla}|u|\| \overset{\text{a.e.}}{\leq} \|\boldsymbol{\nabla} u\|$.

Proof.

$$\|2|u|\boldsymbol{\nabla}|u|\| = \left\|\boldsymbol{\nabla}|u|^2\right\| = 2\,\|\mathrm{Re}(\overline{u}\boldsymbol{\nabla} u)\| \leq 2\,\|\boldsymbol{\nabla} u\|\,|u|$$

On the set $\{u \neq 0\}$, we can divide by $2|u|$, which gives the result. The proof on the set $\{u = 0\}$ is much difficult. □