Advanced dynamical sytems

1. | Discrete maps

Maps in $\mathbb{T}^1$

Proposition 1. Let $\alpha = \frac{p}{q} \in \mathbb{Q}$ and let $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation of angle α. Then, all the points of $\mathbb{T}^1$ are periodic for R_α with period q.

Proof. We identify the elements of $\mathbb{T}^1$ as $\mathbb{R}/\mathbb{Z}$. Let $x \in \mathbb{T}^1$. Then, ${R_\alpha}^q x = x + \alpha q = x + p = x$. And q is the smallest integer such that ${R_\alpha}^q x = x$ because we assume that p and q are coprime. □

Proposition 2. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation of angle α. Then, all the orbits of R_α are dense in $\mathbb{T}^1$.

Proof. Let $\varepsilon > 0$, $x, y \in \mathbb{T}^1$. Discretize $\mathbb{T}^1$ in intervals of length at most $\frac{1}{\varepsilon}$. Then, $\exists m, n \in \mathbb{N}$ with $m < n \leq \frac{1}{\varepsilon} + 1$ such that ${R_\alpha}^m x$ and ${R_\alpha}^n x$ are in the same interval. Thus, $\left|{R_\alpha}^{n-m} x - x\right| < \varepsilon$. Now, concatenating ${R_\alpha}^{n-m} x$ repeatedly, we will eventually have $\left|{R_\alpha}^{k(n-m)} x - y\right| < \varepsilon$ for some $k \in \mathbb{N}$. □

Corollary 3. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \subset \mathbb{T}^1$ be a non-empty closed invariant set for R_α. Then, $A = \mathbb{T}^1$.

Proof. Let $x \in \mathbb{T}^1$ and $y \in A$. Then, $\forall k \in \mathbb{N}\ \exists n_k \in \mathbb{N}$ such that $R_\alpha^{n_k} y \in (x - \frac{1}{k}, x + \frac{1}{k})$. Thus, $R_\alpha^{n_k} y \overset{k\to\infty}{\longrightarrow} x$ and $x \in A$ because A is closed and $R_\alpha^{n_k} y \in A\ \forall k \in \mathbb{N}$ because A is invariant. □

Definition 4. Consider the set

$$\Sigma_m := \{(x_1, x_2, \ldots) : x_i \in \{0, 1, \ldots, m-1\}\}$$

We define the *shift map* as:

$$\begin{aligned} \sigma_m : \quad \Sigma_m &\longrightarrow \Sigma_m \\ (x_1, x_2, \ldots) &\longmapsto (x_2, x_3, \ldots) \end{aligned}$$

Remark. Note that some elements in $[0,1]$ have two different representations in base-m identified as elements of Σ_m. So we can think of Σ_m as the quotient space $\Sigma_m/\sim$, where $(x_1, x_2, \ldots) \sim (y_1, y_2, \ldots)$ if and only if $\sum_{i=1}^\infty \frac{x_i}{m^i} = \sum_{i=1}^\infty \frac{y_i}{m^i}$.

Proposition 5. Let $m \in \mathbb{N}$. Consider the *expansion map*

$$\begin{aligned} E_m : \mathbb{T}^1 &\longrightarrow \mathbb{T}^1 \\ x &\longmapsto mx \end{aligned}$$

Then, if $\phi : \Sigma_m \to \mathbb{T}^1$ is the map $\phi(x_1, x_2, \ldots) = \sum_{i=1}^\infty \frac{x_i}{m^i}$, we have that $E_m \circ \phi = \phi \circ \sigma_m$. In particular, ϕ is a bijection, and thus it is a conjugacy between E_m and σ_m.

Proof. Let $x = (x_1, x_2, \ldots) \in \Sigma_m$. Then, $\phi \circ \sigma_m(x) = \sum_{i=1}^\infty \frac{x_{i+1}}{m^i}$. Moreover:

$$E_m \circ \phi(x) = m \sum_{i=1}^\infty \frac{x_i}{m^i} = x_i + \sum_{i=1} \frac{x_{i+1}}{m^i} \equiv \sum_{i=1} \frac{x_{i+1}}{m^i}$$

□

Remark. Note that E preserves the Lebesgue measure *backwards*: $\left|{E_m}^{-1}(A)\right| = |A|$ for all $A \subseteq \mathbb{T}^1$, but $|E_m(A)| \neq |A|$ in general.

Definition 6. We define the following distance in Σ_m. For all $x, x' \in \Sigma_m$:

$$d(x, x') := \frac{1}{2^\ell} \quad \text{with } \ell := \min\{i : x_i \neq x_i'\}$$

Proposition 7. Periodic points of E_m are dense in $\mathbb{T}^1$.

Proof. By conjugacy it suffices to show that periodic points of σ_m are dense in Σ_m. Let $x \in \Sigma_m$ and $\varepsilon > 0$. Then, $\varepsilon > \frac{1}{2^\ell}$ for some ℓ. And so the orbit of

$$y = (x_1, \ldots, x_\ell, x_1, \ldots, x_\ell, x_1, \ldots, x_\ell, \ldots)$$

is periodic and $d(x, y) < \varepsilon$. So periodic points of σ_m are dense in Σ_m. □

Proposition 8. There exists $x \in \mathbb{T}^1$ such that its orbit under E_m is dense in $\mathbb{T}^1$.

Proof. By conjugacy, we only prove it for σ_m. But this is clear by taking he sequence of *all sequences*:

$$\begin{aligned} x = (0, 1, \ldots, m-1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 2, 2, 0, 1, 2, 2, 1, \\ 2, 2, \ldots, (m-1), (m-1), 0, 0, 0, \ldots) \end{aligned}$$

□

A hyperbolic automorphism of $\mathbb{T}^2$

Proposition 9. Consider $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Then, $\mathbf{A}(\mathbb{Z}^2) = \mathbb{Z}^2$ and this induces an automorphism $\tilde{\mathbf{A}}$ of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Definition 10. We define the set of periodic points of $\tilde{\mathbf{A}}$ as $\operatorname{Per} \tilde{\mathbf{A}}$.

Lemma 11. $\operatorname{Per} \tilde{\mathbf{A}} = \mathbb{Q}^2/\mathbb{Z}^2$. Thus, $\operatorname{Per} \tilde{\mathbf{A}}$ is dense in $\mathbb{T}^2$.

Proof. Let $\mathbf{x} \in \operatorname{Per} \tilde{\mathbf{A}}$. Then, $\exists k \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{Z}^2$ such that $\mathbf{A}^k \mathbf{x} = \mathbf{x} + \mathbf{n}$. One can easily check that $\sigma(\tilde{\mathbf{A}}) = \left\{\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\} =: \{\lambda_\pm\}$ with $\lambda_- < 1 < \lambda_+$. Thus,

$$\det\left(\mathbf{A}^k - \mathbf{I}\right) = ({\lambda_+}^k - 1)({\lambda_-}^k - 1) \neq 0$$

and so the equation $\mathbf{A}^k \mathbf{x} = \mathbf{x} + \mathbf{n}$ has a unique solution. Now suppose the solution is $\mathbf{x} = (\alpha, \beta) \notin \mathbb{Q}^2/\mathbb{Z}^2$. We have a system of the form:

$$\begin{cases} a\alpha + b\beta = n_1 \\ c\alpha + d\beta = n_2 \end{cases}$$

An easy check shows that we must necessarily have both $\alpha, \beta \notin \mathbb{Q}/\mathbb{Z}$. Since $\mathbf{A}^k - \mathbf{I}$ is invertible, we may assume that $b \neq d$ (otherwise it's $a \neq c$). So, we can write $\beta = \frac{n_1 - n_2}{b-d} - \frac{a-c}{b-d}\alpha$. So:

$$n_1 = a\alpha + b\beta = b\frac{n_1 - n_2}{b-d} - \alpha\frac{ad-bc}{b-d} \implies \alpha \in \mathbb{Q}/_{\mathbb{Z}}$$

because $ad - bc \neq 0$. Now let $(\frac{p_1}{q_1}, \frac{p_2}{q_2}) \in \mathbb{Q}^2/\mathbb{Z}^2$ and $N \geq 1$ left to be chosen. We define the set $Q_N := \frac{\mathbb{Z}^2}{N} \mod \mathbb{Z}^2$, which is a subset finite set of $\mathbb{T}^2$. Observe that Q_N is invariant under $\tilde{\mathbf{A}}$, and thus, all of its points are periodic because the set is finite. For the above rational numbers, just choose $N = q_1 q_2$. □

Remark. The *hyperbolicity* comes from the fact that there is one eigenvector with eigenvalue greater than 1 and another with eigenvalue less than 1, both eigenvalues being positive.

Theorem 12. The iterates of $\tilde{\mathbf{A}}$ smear every domain $F \subseteq \mathbb{T}^2$ uniformly over $\mathbb{T}^2$, that is, for every domain $G \subseteq \mathbb{T}^2$, we have that the following limit exists:

$$\left|(\tilde{\mathbf{A}}^{-n} F) \cap G\right| \overset{n\to\infty}{\longrightarrow} |F||G|$$

This property of $\tilde{\mathbf{A}}$ is called *mixing*.

Proof. We can prove a more general property in terms of functions in the torus (and then apply it to $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$):

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} f(\tilde{\mathbf{A}}^n \mathbf{x}) g(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{\mathbb{T}^2} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \int_{\mathbb{T}^2} g(\mathbf{x})\, \mathrm{d}\mathbf{x}$$

We will prove this for the orthonormal basis of Fourier series $\{\mathrm{e}^{2\pi i \mathbf{p}\cdot\mathbf{x}}\}_{\mathbf{p}\in\mathbb{Z}^2}$. Note that:

$$\int_{\mathbb{T}^2} \mathrm{e}^{2\pi i((\tilde{\mathbf{A}}^n)^{\mathrm{T}}\mathbf{p})\cdot\mathbf{x}}\, \mathrm{d}\mathbf{x} = \begin{cases} 1 & \text{if } \mathbf{p} = \mathbf{0} \\ 0 & \text{if } \mathbf{p} \neq \mathbf{0} \end{cases}$$

Now for large n, the norm of the vector $(\tilde{\mathbf{A}}^n)^{\mathrm{T}}\mathbf{p}$ is large for $\mathbf{p} \neq \mathbf{0}$ as we have:

$$\tilde{\mathbf{A}}^n \mathbf{p} \simeq \lambda_+^n \langle \mathbf{p}, \mathbf{e}_+ \rangle \mathbf{e}_+$$

And so its transpose will eventually be different from $-\mathbf{q}$. Therefore, we have that if $g = \mathrm{e}^{2\pi i \mathbf{q}\cdot\mathbf{x}}$ then:

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} \mathrm{e}^{2\pi i((\tilde{\mathbf{A}}^n)^{\mathrm{T}}\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\, \mathrm{d}\mathbf{x} = 0$$

So for any $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$ we have the equality. Then, we use that any function nice enough can be approximated uniformly with the Féjer means of the Fourier series (see ?? ??). □

Theorem 13. On the torus $\mathbb{T}^2$ there exist two direction fields invariant with respect to the automorphism $\tilde{\mathbf{A}}$. The integral curves of each of these directions fields are everywhere dense on the torus. The automorphism $\tilde{\mathbf{A}}$ converts the integral curves of each field into integral curves of the same field, expanding by λ_+ for the first field and contracting by λ_- for the second.

Proof. Let $\mathbf{e}_+$ and $\mathbf{e}_-$ be the eigenvectors of $\mathbf{A}$ with eigenvalues λ_+ and λ_- respectively. Let $\mathbf{x} \in \mathbb{T}^2$ and

$$\begin{aligned} \gamma_+ : \mathbb{R} &\longrightarrow \mathbb{T}^2 \\ t &\longmapsto \mathbf{x} + t\mathbf{e}_+ \end{aligned} \qquad \begin{aligned} \gamma_- : \mathbb{R} &\longrightarrow \mathbb{T}^2 \\ t &\longmapsto \mathbf{x} + t\mathbf{e}_- \end{aligned}$$

be the expanding and contracting curves and let $\boldsymbol{\xi}_{\mathbf{x}} = \operatorname{im}(\gamma_+)$, $\boldsymbol{\eta}_{\mathbf{x}} = \operatorname{im}(\gamma_-)$ be the corresponding direction fields. The density of the curves is a consequence of the density of orbits in rotation maps in the circle with irrational angle. □

Definition 14. Let $\mathbf{A}, \mathbf{B} : \mathbb{T}^2 \to \mathbb{T}^2$ be $\mathcal{C}^1$ functions and $\varepsilon > 0$. We say that B is $\mathcal{C}^0$-ε-*close* to $\mathbf{A}$ if:

$$\sup_{\mathbf{x}\in\mathbb{T}^2} \|\mathbf{B}(\mathbf{x}) - \mathbf{A}(\mathbf{x})\| < \varepsilon$$

We say that $\mathbf{B}$ is $\mathcal{C}^1$-ε-*close* to $\mathbf{A}$ if they are $\mathcal{C}^0$-ε-close and:

$$\sup_{\mathbf{x}\in\mathbb{T}^2} \|\mathbf{DB}(\mathbf{x}) - \mathbf{DA}(\mathbf{x})\| < \varepsilon$$

Theorem 15 (Structal stability). Let $\mathbf{B}$ be a diffeomorphism on $\mathbb{T}^2$ which is $\mathcal{C}^1$-ε-close to $\tilde{\mathbf{A}}$. Then, $\mathbf{B}$ is $\mathcal{C}^0$-conjugate to $\tilde{\mathbf{A}}$.

Proof. We need to find a $\mathcal{C}^0$-conjugacy $\mathbf{H}$ between $\mathbf{B}$ and $\tilde{\mathbf{A}}$. Since, $\mathbf{B}$ is $\mathcal{C}^1$-close to $\tilde{\mathbf{A}}$, we may expect that both $\mathbf{H}$ and $\mathbf{B}$ are small perturbations of the identity and $\tilde{\mathbf{A}}$ respectively. So set $\mathbf{H} = \mathbf{I} + \mathbf{h}$ and $\mathbf{B} = \tilde{\mathbf{A}} + \mathbf{b}$. Then, we want to find $\mathbf{h}$ and $\mathbf{b}$ such that:

$$\mathbf{H} \circ \tilde{\mathbf{A}} = \mathbf{B} \circ \mathbf{H} \iff \mathbf{h}(\tilde{\mathbf{A}}\mathbf{x}) - \tilde{\mathbf{A}}\mathbf{h}(\mathbf{x}) = \mathbf{b}(\mathbf{x} + \mathbf{h}(\mathbf{x}))$$

This equation is called *conjugacy equation*. Consider the operators

$$\begin{aligned} \mathbf{S}_{\tilde{\mathbf{A}}} : \mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^2) &\longrightarrow \mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^2) \\ \mathbf{h} &\longmapsto \mathbf{h}(\tilde{\mathbf{A}}(\mathbf{x})) \\ \mathbf{L}_{\tilde{\mathbf{A}}} : \mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^2) &\longrightarrow \mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^2) \\ \mathbf{h} &\longmapsto \mathbf{S}_{\tilde{\mathbf{A}}}\mathbf{h} - \tilde{\mathbf{A}}\mathbf{h} \end{aligned}$$

where we consider the diffeomorphisms $\tilde{\mathbf{A}}$ and $\mathbf{B}$ as operators *lifted* to $\mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. Observe that:

$$\sup_{\mathbf{x}\in\mathbb{R}^2} \|\mathbf{S}_{\tilde{\mathbf{A}}}\mathbf{h}(\mathbf{x})\| = \sup_{\mathbf{x}\in\mathbb{R}^2} \left\|\mathbf{S}_{\tilde{\mathbf{A}}}\mathbf{h}(\tilde{\mathbf{A}}^{-1}\mathbf{x})\right\| = \sup_{\mathbf{x}\in\mathbb{R}^2} \|\mathbf{h}(\mathbf{x})\|$$

Hence, $\|\mathbf{S}_{\tilde{\mathbf{A}}}\| = 1$ and similarly $\left\|\mathbf{S}_{\tilde{\mathbf{A}}}^{-1}\right\| = 1$, where $\mathbf{S}_{\tilde{\mathbf{A}}}^{-1} : \mathbf{h} \mapsto \mathbf{h}(\tilde{\mathbf{A}}^{-1}(\mathbf{x}))$. We'll now prove that $\mathbf{L}_{\tilde{\mathbf{A}}}$ is invertible. Note that $\mathbb{R}^2 = \langle \mathbf{e}_+ \rangle \oplus \langle \mathbf{e}_- \rangle$ because $\tilde{\mathbf{A}}$ is invertible. Thus:

$$\mathbf{L}_{\tilde{\mathbf{A}}}\mathbf{h} = \mathbf{c} \iff \begin{cases} \mathbf{L}_{\tilde{\mathbf{A}}}\mathbf{h}_+ = \mathbf{S}_{\tilde{\mathbf{A}}}\mathbf{h}_+ - \lambda_+\mathbf{h}_+ = \mathbf{c}_+ \\ \mathbf{L}_{\tilde{\mathbf{A}}}\mathbf{h}_- = \mathbf{S}_{\tilde{\mathbf{A}}}\mathbf{h}_- - \lambda_-\mathbf{h}_- = \mathbf{c}_- \end{cases}$$

where $\mathbf{h} = \mathbf{h}_+ + \mathbf{h}_-$, $\mathbf{c} = \mathbf{c}_+ + \mathbf{c}_-$ and $\mathbf{h}_\pm, \mathbf{c}_\pm \in \langle \mathbf{e}_\pm \rangle$. Now, note that $\left\|\frac{\mathbf{S}_{\tilde{\mathbf{A}}}}{\lambda_+}\right\| < 1$ and so

$$(\mathbf{S}_{\tilde{\mathbf{A}}} - \lambda_+\mathbf{I}) = \lambda_+\left(\frac{\mathbf{S}_{\tilde{\mathbf{A}}}}{\lambda_+} - \mathbf{I}\right)$$

is invertible. Similarly, we have $\left\|\mathbf{S}_{\tilde{\mathbf{A}}}^{-1}\lambda_-\right\| < 1$ and so

$$(\mathbf{S}_{\tilde{\mathbf{A}}}^{-1} - \lambda_-\mathbf{I}) = \mathbf{S}_{\tilde{\mathbf{A}}}^{-1}\left(\mathbf{I} - \lambda_-\mathbf{S}_{\tilde{\mathbf{A}}}^{-1}\right)$$

is invertible because it is a product of invertible operators. Thus, $\mathbf{L}_{\tilde{\mathbf{A}}}$ is invertible and its inverse is linear because $\mathbf{L}_{\tilde{\mathbf{A}}}$ is linear. Now, we return to our initial problem. Find $\mathbf{h}$ such that $\mathbf{h} = {\mathbf{L}_{\tilde{\mathbf{A}}}}^{-1}(\mathbf{b}(\mathbf{x}+\mathbf{h}(\mathbf{x}))) =: \mathbf{\Psi}(\mathbf{h})$, which is a fixed-point problem. Note that $\mathbf{\Psi}$ is a contraction. Indeed:

$$\begin{aligned}\left\|\mathbf{\Psi}(\mathbf{h}) - \mathbf{\Psi}(\mathbf{h}')\right\| &\leq \left\|{\mathbf{L}_{\tilde{\mathbf{A}}}}^{-1}\right\|\left\|\mathbf{b}(\mathbf{x}+\mathbf{h}(\mathbf{x})) - \mathbf{b}(\mathbf{x}+\mathbf{h}'(\mathbf{x}))\right\| \\ &\leq \left\|{\mathbf{L}_{\tilde{\mathbf{A}}}}^{-1}\right\| \left\|\mathbf{D}\mathbf{b}\right\| \left\|\mathbf{h}-\mathbf{h}'\right\|\end{aligned}$$

because of ?? ??. This last term is arbitrarily small ($\|\mathbf{D}\mathbf{b}\|$ is arbitrarily small) because $\mathbf{B}$ is $\mathcal{C}^1$-close to $\tilde{\mathbf{A}}$. Thus, $\mathbf{h}$ exists and it's unique. □

Definition 16. A dynamical system $f : X \to X$ has *sensitive dependence on initial conditions* on X if $\exists \varepsilon > 0$ such that for each $x \in X$ and any neighborhood N_x of x, exists $y \in N_x$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \varepsilon$.

Definition 17. Let $U \subseteq \mathbb{R}^n$, $\mathbf{f} : U \to U$ be a dynamical system and $\mathbf{x} \in U$ and $\mathbf{v} \in \mathbb{R}^n$. We define the *Lyapunov exponent* as:

$$\chi(x, \mathbf{v}) := \limsup_{n\to\infty} \frac{1}{n} \log \|\mathbf{D}(\mathbf{f}^n)(x)\mathbf{v}\|$$

Remark. The Lyapunov exponent measures the exponential growth rate of tangent vectors along orbits. It can rarely be computed explicitly, but if we can show that $\chi(x, \mathbf{v}) > 0$ for some $\mathbf{v}$, then we know that the system is *chaotic*.

Hamiltonian systems

Definition 18. Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be open and $H : U \to \mathbb{R}$ be a $\mathcal{C}^1$ function. We define the *Hamiltonian vector field* associated to H as:

$$\begin{cases} \dot{\mathbf{x}} = \dfrac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\dfrac{\partial H}{\partial \mathbf{x}} \end{cases} \tag{1}$$

Remark. Recall that H is a first integral of the system Eq. (1).

Lemma 19. Let $H : U \to \mathbb{R}$ be a $\mathcal{C}^1$ function, $W \subseteq U$. Then, the volume of W under the field of Eq. (1) is preserved.

Proof. Let $W_t := \boldsymbol{\phi}_t(W)$, where ϕ_t is the flow of Eq. (1). Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{vol}(W_t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\phi_t(W)} \mathrm{d}\mathbf{x} = \int_W \frac{\mathrm{d}}{\mathrm{d}t}\det \mathbf{D}\boldsymbol{\phi}_t = \int_W \operatorname{\mathbf{div}} \mathbf{X}_H$$

where $\mathbf{X}_H$ is the vector field of Eq. (1). But an easy computation shows that $\operatorname{\mathbf{div}} \mathbf{X}_H = 0$. Let's make the last step of the computation of above more explicit. Given $\mathbf{A} \in \mathrm{GL}_n(\mathbb{R})$, we have that:

$$\begin{aligned}\det(\mathbf{A} + \varepsilon\mathbf{T}) &= \varepsilon^n \det(\mathbf{A}) \det\left(\frac{1}{\varepsilon}\mathbf{I} + \mathbf{T}\mathbf{A}^{-1}\right) \\ &= \det(\mathbf{A})\left(1 + \varepsilon \operatorname{tr}(\mathbf{T}\mathbf{A}^{-1}) + \mathrm{O}\left(\varepsilon^2\right)\right)\end{aligned}$$

And so, $\det'(\mathbf{A})\mathbf{T} = \operatorname{tr}(\mathbf{T}\mathbf{A}^{-1})$. Finally, taking $\mathbf{A} = \mathbf{D}\boldsymbol{\phi}_t$ and using that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\boldsymbol{\phi}_t = \mathbf{D}\mathbf{X}_H\mathbf{D}\boldsymbol{\phi}_t$ we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\det \mathbf{D}\boldsymbol{\phi}_t = {\det}'(\mathbf{D}\boldsymbol{\phi}_t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\boldsymbol{\phi}_t = \operatorname{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\boldsymbol{\phi}_t(\mathbf{D}\boldsymbol{\phi}_t)^{-1}\right) = \\ = \operatorname{tr}(\mathbf{D}\mathbf{X}_H) = \operatorname{\mathbf{div}} \mathbf{X}_H\end{aligned}$$

□

2. | Dynamics on the circle

Generalities

Definition 20. Let $x, x' \in \mathbb{R}$. We say that $x \sim x'$ if and only if $x - x' \in \mathbb{Z}$. We define the *circle* as $\mathbb{T}^1 := \mathbb{R}/\sim$. We define the following distance in $\mathbb{T}^1$:

$$d(\overline{x}, \overline{y}) = \min_{x' \in \overline{x}, y' \in \overline{y}} |x' - y'|$$

Proposition 21 (Existence of a lift).

1. For any continuous map $F : \mathbb{T}^1 \to \mathbb{T}^1$ there exists a *lift* f, i.e. a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $F \circ \pi = \pi \circ f$, where $\pi : \mathbb{R} \to \mathbb{T}^1$ is the canonical projection.

2. If g is another lift of F, then $g - f = k \in \mathbb{Z}$.

Proof. We only prove the second property. Since, f, g are both lifts of F, they belong to the same equivalence class. Thus, $f - g \in \mathbb{Z}$. And now use the continuity of $f - g$. □

Remark. Recall that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism if and only if f is strictly monotonous.

Definition 22. We say that a homeomorphism F *preserves orientation* if and only if f is strictly increasing. We define the set of $\mathrm{Homeo}_+(\mathbb{T}^1)$ as the set of homeomorphisms of $\mathbb{T}^1$ that preserve orientation.

Proposition 23. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then, F admits a lift f such that $f(x) = x + \varphi(x)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a 1-periodic function.

Proof. We already now that F admits a lift f. A straightforward calculation shows that $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = f(x + 1)$ is also a lift of F. Thus, $f_1 - f = k \in \mathbb{Z}$. Now, since f must be strictly increasing, we need $k \in \mathbb{N}$. Moreover, since F is injective, $f|_{[0,1)}$ is injective and its image cannot contain 2 points whose difference is an integer. Thus, $k = 1$. Now, define $\varphi(x) = f(x) - x$, which is 1-periodic:

$$\varphi(x + 1) = f(x + 1) - (x + 1) = f(x) - x = \varphi(x)$$

□

Definition 24. We define the set:

$$\begin{aligned}\mathcal{D}^0(\mathbb{T}^1) := \{f \in \mathrm{Homeo}(\mathbb{R}) : f \text{ increasing and} \\ f(x+1) = f(x) + 1\}\end{aligned}$$

Note that we have the projection:

$$\begin{array}{ccc} \mathcal{D}^0(\mathbb{T}^1) & \longrightarrow & \mathrm{Homeo}_+(\mathbb{T}^1) \\ f & \longmapsto & F \end{array}$$

We can define a distance in $\mathcal{D}^0(\mathbb{T}^1)$ as:

$$d(f,g) = \max\left\{\sup_{x\in\mathbb{R}} |f(x)-g(x)|, \sup_{x\in\mathbb{R}} \left|f^{-1}(x)-g^{-1}(x)\right|\right\}$$

Lemma 25. $\mathcal{D}^0(\mathbb{T}^1)$ is a complete metric space. Moreover, the functions:

$$\begin{array}{ccc} \mathcal{D}^0(\mathbb{T}^1) & \longrightarrow & \mathcal{D}^0(\mathbb{T}^1) \\ f & \longmapsto & f^{-1} \end{array} \qquad \begin{array}{ccc} \mathcal{D}^0(\mathbb{T}^1)\times\mathcal{D}^0(\mathbb{T}^1) & \longrightarrow & \mathcal{D}^0(\mathbb{T}^1) \\ (f,g) & \longmapsto & f\circ g \end{array}$$

are continuous. Thus, $\mathcal{D}^0(\mathbb{T}^1)$ is a topological group with the composition.

Definition 26. Let $\varepsilon \geq 0$ and $\alpha \in \mathbb{R}$. We define the *Arnold family* as:

$$\begin{array}{rcl} f_{\alpha,\varepsilon} : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x+\alpha+\varepsilon\sin(2\pi x) \end{array}$$

Lemma 27. If $0 \leq \varepsilon < \frac{1}{2\pi}$, then $f_{\alpha,\varepsilon} \in \mathcal{D}^0(\mathbb{T}^1)$.

Proof. Note that ${f_{\alpha,\varepsilon}}' > 0 \iff \varepsilon < \frac{1}{2\pi}$. Thus, $f_{\alpha,\varepsilon}$ is strictly increasing, and therefore it is a homeomorphism. Moreover, $f_{\alpha,\varepsilon}(x+1) = f_{\alpha,\varepsilon}(x)+1$. □

Rotation number

Lemma 28. Let $f = \mathrm{id}+\varphi \in \mathcal{D}^0(\mathbb{T}^1)$ with φ 1-periodic. Thus:

$$f^n = \mathrm{id} + \sum_{i=0}^{n-1} \varphi\circ f^i =: \mathrm{id}+\varphi_n$$

with φ_n 1-periodic.

Proof. Use induction on i to prove that all the terms of the sum $\varphi\circ f^i$ are 1-periodic. The case $i=0$ is clear. Now, for the inductive step:

$$\varphi\circ f^{i+1}(x+1) = \varphi\left(x+1+\sum_{k=0}^{i}\varphi\circ f^k(x+1)\right) = \\ = \varphi\left(x+\sum_{k=0}^{i}\varphi\circ f^k(x)\right) = \varphi\circ f^{i+1}(x)$$

□

Lemma 29. Let $f = \mathrm{id}+\varphi \in \mathcal{D}^0(\mathbb{T}^1)$ with φ 1-periodic. Let $m := \min_{x\in\mathbb{R}}\varphi$ and $M := \max_{x\in\mathbb{R}}\varphi$. Then, we have $m \leq M < m+1$.

Proof. By the periodicity and continuity of φ, we have that $\exists x_m, x_M \in \mathbb{R}$ such that $\varphi(x_m) = m$, $\varphi(x_M) = M$ and $0 \leq x_M - x_m < 1$. Since $f \in \mathcal{D}^0(\mathbb{T}^1)$, we must have $f(x_M)-f(x_m) < 1$. Thus:

$$M - m = f(x_M)-f(x_m)-(x_M-x_m) < 1$$

□

Definition 30. Let $(u_n) \in \mathbb{R}$ be a sequence. We say that (u_n) is *subadditive* if $u_{n+m} \leq u_n + u_m$ for all $n,m \in \mathbb{N}$. We say that (u_n) is *superadditive* if $u_{n+m} \geq u_n+u_m$ for all $n,m\in\mathbb{N}$, that is, if $(-u_n)$ is subadditive.

Lemma 31. Let $f \in \mathcal{D}^0(\mathbb{T}^1)$ be such that $f = \mathrm{id}+\varphi$. We can write $f^n = \mathrm{id}+\varphi_n$ and let $m_n := \min_{x\in\mathbb{R}}\varphi_n$ and $M_n := \max_{x\in\mathbb{R}}\varphi_n$. Then, (M_n) is subadditive and (m_n) is superadditive.

Proof. We have that:

$$f^{n+m}(x)-x = (f^m - \mathrm{id})(f^n(x)) + f^n(x) - x \leq M_m + M_n$$

Now take the supremum in x. The other inequality is analogous. □

Lemma 32. Let $(u_n) \in \mathbb{R}$ be a subadditive sequence. Then, $\lim_{n\to\infty}\frac{u_n}{n}$ exists, and it is equal to $\inf_{n\in\mathbb{N}}\frac{u_n}{n}$. Analogously, if (u_n) is superadditive, then $\lim_{n\to\infty}\frac{u_n}{n}$ exists, and it is equal to $\sup_{n\in\mathbb{N}}\frac{u_n}{n}$.

Proof. Assume (u_n) is subadditive and fix $p\in\mathbb{N}$. Let $n \geq p$ be such that $n = k_n p + r_n$ with $r < p$. Then:

$$\frac{u_n}{n} \leq \frac{u_{k_n p}+u_{r_n}}{n} \leq \frac{k_n u_p}{n} + \frac{u_{r_n}}{n} = \frac{u_p}{p+\frac{r_n}{k_n}} + \frac{u_{r_n}}{n}$$

where in the first and second inequalities we used that (u_n) is subadditive. Now to show that the limit exists and that the value is the one of above, take first $\limsup$ in n and then $\inf$ in p:

$$\limsup_{n\to\infty}\frac{u_n}{n} \leq \inf_{p\in\mathbb{N}}\frac{u_p}{p} \leq \liminf_{p\to\infty}\frac{u_p}{p}$$

□

Theorem 33 (Existence of the rotation number). For all $f\in\mathcal{D}^0(\mathbb{T}^1)$, we have that the sequence of functions $\frac{1}{n}(f^n - \mathrm{id})$ convergence uniformly to constant function $\rho(f)\in\mathbb{R}$. This number is called the *rotation number* of f.

Proof. By Theorem 29 we have that $\frac{m_n}{n} \leq \frac{M_n}{n} < \frac{m_n}{n}+\frac{1}{n}$, where $m_n := \min_{x\in\mathbb{R}}\varphi_n$ and $M_n := \max_{x\in\mathbb{R}}\varphi_n$. By Theorems 31 and 32, we have that $\frac{m_n}{n}$ and $\frac{M_n}{n}$ have the same limit and moreover:

$$\frac{m_n}{n} \leq \frac{1}{n}(f^n(x)-x) \leq \frac{M_n}{n}$$

So we have the result, and in fact the convergence is uniform by domination. □

Proposition 34. The following properties are satisfied:

1. $\rho(R_\alpha) = \alpha\ \forall\alpha\in\mathbb{R}$.
2. $\rho(f^n) = n\rho(f)\ \forall f\in\mathcal{D}^0(\mathbb{T}^1)$, $n\in\mathbb{N}$.
3. $f \leq g \implies \rho(f)\leq\rho(g)\ \forall f,g\in\mathcal{D}^0(\mathbb{T}^1)$.
4. $\rho(f+k) = \rho(f)+k\ \forall f\in\mathcal{D}^0(\mathbb{T}^1)$, $k\in\mathbb{Z}$.
5. If $f,g\in\mathcal{D}^0(\mathbb{T}^1)$ commute, then $\rho(f\circ g) = \rho(f)+\rho(g)$.

Sketch of the proof. For the penultimate one, note that $f_k(x) := f(x+k) = f(x)+k$, since $f \in \mathcal{D}^0(\mathbb{T}^1)$. Thus:

$$f_k{}^n(x) = f(f(\cdots f(f(x+k)+k)+\cdots)+k) = f^n(x)+nk$$

□

Proposition 35. The function

$$\begin{aligned} R : \mathcal{D}^0(\mathbb{T}^1) &\longrightarrow \mathbb{R} \\ f &\longmapsto \rho(f) \end{aligned}$$

is continuous with respect to the $\mathcal{C}^0$-topology.

Proof. Let $\varepsilon > 0$ and $N > 0$ such that $\frac{1}{N} < \varepsilon$. Let $f, g \in \mathcal{D}^0(\mathbb{T}^1)$ be close enough such that $\left|f^N(x) - g^N(x)\right| < \varepsilon$ for all $x \in \mathbb{R}$. In particular, $f^N(0) < g^N(0) + \varepsilon$. A straightforward induction shows that in fact we have $f^{kN}(0) < g^{kN}(0) + k - 1 + \varepsilon$ for all $k \in \mathbb{N}$. Thus:

$$\rho(f) - \rho(g) = \lim_{k\to\infty} \frac{f^{kN}(0) - g^{kN}(0)}{kN} \leq \frac{1}{N} < \varepsilon$$

Exchanging the roles of f and g we get the other inequality. □

Definition 36. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ with lift f. We define the *rotation number* of F as $\rho(F) := [\rho(f)] \in \mathbb{T}^1$.

Definition 37. Let $F, G \in \mathrm{Homeo}_+(\mathbb{T}^1)$. We say that G is *semi-conjugate* to F if there exists a continuous surjective map $H : \mathbb{T}^1 \to \mathbb{T}^1$ such that $H \circ F = G \circ H$. We say that G is *conjugate* to F if H is a homeomorphism.

Lemma 38. Let $F, G \in \mathrm{Homeo}_+(\mathbb{T}^1)$ be such that G is semi-conjugate to F. Then, if F has a periodic point, then G has a periodic point.

Proof. Let $p \in \mathbb{T}^1$ be a periodic point of F with period n. Then, $H(p)$ is a periodic point of G with period at most n. Indeed:

$$G^n(H(p)) = H(F^n(p)) = H(p)$$

□

Remark. The converse is not true.

Theorem 39. Let $F, G \in \mathrm{Homeo}_+(\mathbb{T}^1)$ be conjugate by $H \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then, $\rho(F) = \rho(G)$.

Proof. Let h and f be lifts of H and F respectively. Then, an easy check shows that $g := h \circ f \circ h^{-1}$ is a lift of G. It suffices to prove that $\rho(g) = \rho(f)$. Note that, by induction we have $h \circ f^n = g^n \circ h$ for all $n \in \mathbb{N}$. Now write $h = \mathrm{id} + \varphi$ with $\varphi \in \mathcal{C}(\mathbb{T}^1)$. Then:

$$\frac{f^n(x) - x + \varphi(f^n(x))}{n} = \frac{g^n(h(x)) - h(x)}{n} + \frac{h(x) - x}{n}$$

Taking limits, we have that $\rho(f) = \rho(g)$, as φ is bounded. □

Remark. Note that the proof also works even if $\exists h = \mathrm{id} + \varphi \in \mathcal{C}(\mathbb{T}^1)$ such that $h \circ f = g \circ h$ (i.e. H is only a *special* semi-conjugacy).

Rotation number and invariant measure

Definition 40. We say that $\mu : \mathcal{C}(\mathbb{T}^1) \to \mathbb{R}$ is a *measure on* $\mathcal{C}(\mathbb{T}^1)$ if:

1. μ is linear.
2. μ is continuous.
3. $\mu(\varphi) \geq 0$, whenever $\varphi \geq 0$.

We say that μ is a *probability measure* if $\mu(1) = 1$. We denote by $\mathcal{M}(\mathbb{T}^1)$ the set of all probability measures on $\mathcal{C}(\mathbb{T}^1)$.

Remark. Usually we will denote $\mu(\varphi)$ by $\int_{\mathbb{T}^1} \varphi \, \mathrm{d}\mu$ or $\int_{\mathbb{T}^1} \varphi(x) \, \mathrm{d}\mu(x)$.

Remark. Note that we then have $\mu(\varphi) > 0$ whenever $\varphi > 0$, because φ attains its minimum at some point x_0 (by the compactness of $\mathbb{T}^1$). Similarly, $\mu(\varphi) \leq 0$ whenever $\varphi \leq 0$, and $\mu(\varphi) < 0$ whenever $\varphi < 0$.

Remark. Examples of such measures are the Dirac measures

$$\delta_x(\varphi) = \varphi(x) \qquad x \in \mathbb{T}^1$$

or the Lebesgue measure:

$$\mathrm{Leb}(\varphi) := \int_0^1 \varphi(x) \, \mathrm{d}x$$

Definition 41. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $\mu \in \mathcal{M}(\mathbb{T}^1)$. We define the *push-forward measure* of F as $F_*\mu(\varphi) := \mu(\varphi \circ F)$.

Definition 42. We say that a measure $\mu \in \mathcal{M}(\mathbb{T}^1)$ is *invariant* by $F \in \mathrm{Homeo}(\mathbb{T}^1)$ (or F*-invariant*) if $F_*\mu = \mu$. We will denote by $\mathcal{M}_F(\mathbb{T}^1)$ the set of F-invariant probability measures.

Proposition 43. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$, $x \in \mathbb{T}^1$ and $n \in \mathbb{N}$.

1. Leb is invariant under R_α $\forall \alpha \in \mathbb{R}$.
2. δ_x is F-invariant $\iff$ $F(x) = x$
3. $\dfrac{\delta_x + \cdots + \delta_{F^{n-1}(x)}}{n}$ is F-invariant $\iff$ $F^n(x) = x$

Proof. We prove the difficult implication in the second item. That is, suppose δ_x is F-invariant. We then have that $\varphi(F(x)) = \varphi(x)$ $\forall \varphi \in \mathcal{C}(\mathbb{T}^1)$. Now if $F(x) \neq x$ for some $x \in \mathbb{T}^1$, then we may assume $x < F(x)$ and consider a continuous function on $\mathbb{T}^1$ that equals one in a neighborhood of $F(x)$ not containing x and zero otherwise. □

Theorem 44. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then, $\mathcal{M}_F(\mathbb{T}^1) \neq \varnothing$.

Proposition 45. Let $F \in \mathrm{Homeo}(\mathbb{T}^1)$ and $f = \mathrm{id} + \varphi$ be a lift of F, with $\varphi \in \mathcal{C}(\mathbb{T}^1)$. Then, $\forall \mu \in \mathcal{M}_F(\mathbb{T}^1)$, $\rho(f) = \mu(\varphi)$. Moreover:

1. $\|f^n - \mathrm{id} - n\rho(f)\|_{\mathcal{C}(\mathbb{R})} < 1$ for all $n \in \mathbb{N}$.
2. $\forall n \in \mathbb{N}$, $\exists x_n \in \mathbb{R}$ such that $f^n(x_n) - x_n = n\rho(f)$.

Proof. Let $\psi_n := f^n - \mathrm{id} - n\mu(\varphi)$ with $\mu \in \mathcal{M}_F(\mathbb{T}^1)$. We have that:

$$\mu(\psi_n) = \sum_{i=0}^{n-1} \mu(\varphi \circ f^i) - n\mu(\varphi) = \sum_{i=0}^{n-1} \mu(\varphi \circ F^i) - n\mu(\varphi) = 0$$

where we have used Theorem 28. Now we must have that ψ_n change their sign in $[0,1]$ because otherwise that would contradict $\mu(\psi_n) = 0$. So $\exists x_n \in [0,1]$ such that $\psi_n(x_n) = 0$. So:

$$f^n(x_n) - x_n = n\mu(\varphi)$$

Dividing by n and taking limits, we have that $\rho(f) = \mu(\varphi)$. This also shows the second point. To prove the first one, note that $\min \psi_n \leq 0$ and so by Theorem 29 we have $\max \psi_n < 1$. Moreover, $\min \psi_n = -\max(-\psi_n) > -1$ (using the same argument as before) and so $\|\psi_n\|_{\mathcal{C}(\mathbb{R})} < 1$ for all $n \in \mathbb{N}$. □

Rational rotation number

Proposition 46. Let $f \in \mathcal{D}^0(\mathbb{T}^1)$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ be such that the fraction $\frac{p}{q}$ is irreducible. Then:

$$\rho(f) = \frac{p}{q} \iff \exists x \in \mathbb{R} \text{ such that } f^q(x) = x + p$$
$$\rho(f) > \frac{p}{q} \iff \forall x \in \mathbb{R} \text{ we have } f^q(x) > x + p$$
$$\rho(f) < \frac{p}{q} \iff \forall x \in \mathbb{R} \text{ we have } f^q(x) < x + p$$

Proof. Since $\rho(f^q) = q\rho(f)$ and $\rho(f + p) = \rho(f) + p$, we have that if $g = f^q - p$, $\rho(g) = q\rho(f) - p$. Thus, an easy check shows that we can assume that $p = 0$ and $q = 1$. We will only prove the equivalences to the left, as it is sufficient.

1. Assume $f(x) = x$ for some $x \in \mathbb{R}$. Then, from the definition of $\rho(f)$ applied to the point x, we have that $\rho(f) = 0$.
2. Assume $f(x) > x$ and write $f = \mathrm{id} + \varphi$ with $\varphi \in \mathcal{C}(\mathbb{T}^1)$ and $\varphi > 0$. Since, $\mathbb{T}^1$ is compact, we have in fact that $\varphi \geq \min \varphi =: \varepsilon > 0$. Now:
$$f^n - \mathrm{id} = \sum_{i=0}^{n-1} \varphi \circ f^i \geq n\varepsilon$$
And so $\rho(f) \geq \varepsilon > 0$.
3. Proceed as in the previous case.

□

Definition 47. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $x \in \mathbb{T}^1$. We define the *orbit* of x as:

$$\mathcal{O}_F(x) := \{F^n(x) : n \in \mathbb{Z}\}$$

We also define the *positive orbit* of x and the *negative orbit* of x as:

$$\mathcal{O}_F^+(x) := \{F^n(x) : n \in \mathbb{Z}_{\geq 0}\}$$
$$\mathcal{O}_F^-(x) := \{F^n(x) : n \in \mathbb{Z}_{\leq 0}\}$$

If the homeomorphism is not specified, we will omit the subscript.

Definition 48. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $x \in \mathbb{T}^1$. We define the *omega limit* of x as the set of limit points of $\mathcal{O}_F^+(x)$, i.e.:

$$\omega(x) := \{y \in \mathbb{T}^1 : \exists (n_k) \nearrow +\infty \text{ such that } F^{n_k}(x) \to y\}$$

We define the *alpha limit* of x as the set of limit points of $\mathcal{O}_F^-(x)$, i.e.:

$$\alpha(x) := \{y \in \mathbb{T}^1 : \exists (n_k) \searrow -\infty \text{ such that } F^{n_k}(x) \to y\}$$

Definition 49. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $X \subset \mathbb{T}^1$. We say that X is *positively invariant* if $F(X) \subseteq X$ and *negatively invariant* if $F^{-1}(X) \subseteq X$. We say that X is *invariant* if $F(X) = X$.

Proposition 50. Let $X \subset \mathbb{T}^1$ and $x \in \mathbb{T}^1$. Then:

1. X is invariant $\iff \forall x \in X$, $\mathcal{O}(x) \subseteq X \iff X$ is a union of orbits.
2. $\mathcal{O}(x)$ is finite $\iff x$ is periodic.
3. The omega limit $\omega(x)$ and the alpha limit $\alpha(x)$ are non-empty compact invariant sets.

Definition 51. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. We define the *positively recurrent points* and *negatively recurrent points* as:

$$R^+(F) := \{x \in \mathbb{T}^1 : x \in \omega(x)\}$$
$$R^-(F) := \{x \in \mathbb{T}^1 : x \in \alpha(x)\}$$

Proposition 52. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then, $R^\pm(F)$ are invariant non-closed sets.

Definition 53. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $x \in \mathbb{T}^1$. We say that x is a *wandering point* if there exists a neighborhood U of x such that $\forall n \geq 1$ we have $F^n(U) \cap U = \varnothing$. The neighborhood U is called a *wandering domain*. We define the set:

$$\Omega(F) := \{x \in \mathbb{T}^1 : x \text{ is not wandering}\}$$

Remark. A point $x \in \mathbb{T}^1$ is *non-wandering* if it is not wandering, i.e. if $\forall U$ neighborhood of x $\exists n \geq 1$ such that $F^n(U) \cap U \neq \varnothing$.

Proposition 54. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then, $\Omega(F)$ is an invariant closed set.

Lemma 55. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$. Then:

$$\mathrm{Fix}(F) \subseteq \mathrm{Per}(F) \subseteq R^\pm(F) \subseteq \Omega(F) \subseteq \mathbb{T}^1$$

Proof. All the inclusions are clear except for maybe $R^\pm(F) \subseteq \Omega(F)$. Let $x \in R^\pm(F)$. Then, $\exists (n_k) \in \mathbb{N}$ with $n_k \nearrow \infty$ such that $F^{n_k}(x) \to x$. Now, let U be a neighborhood of x. Then, $x \in U$ and for k large enough, by the continuity of F, we must have $x \in F^{n_k}(U)$. So $F^{n_k}(U) \cap U \neq \varnothing$ and thus $x \in \Omega(F)$. □

Definition 56. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $X \subseteq \mathbb{T}^1$ be a non-empty closed invariant set. We say that X is *minimal* if $\forall x \in X$, $\overline{\mathcal{O}(x)} = X$. If $X = \mathbb{T}^1$, we say that F is *minimal*.

Proposition 57. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $X \subseteq \mathbb{T}^1$ be a closed and invariant. Then, X is minimal $\iff$ $\forall Y \subseteq X$ closed, invariant and non-empty, $Y = X$.

Proof.

$\Longrightarrow$) Let $Y \subseteq X$ be closed, invariant and non-empty and take $y \in Y$. We have:

$$Y \subseteq X = \overline{\mathcal{O}(y)} \subseteq \overline{Y} = Y$$

$\Longleftarrow$) Let $x \in X$. Since $\overline{\mathcal{O}(x)} \subseteq X$ is closed, invariant and non-empty, we have that $\overline{\mathcal{O}(x)} = X$.

□

Theorem 58. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ with $\rho(F) = \frac{p}{q} \in \mathbb{Q}/\mathbb{Z}$. Then:

1. F has periodic points of period q, and any periodic point of F has minimal period q.
2. For any $x \in \mathbb{T}^1$, $\omega(x)$ and $\alpha(x)$ are periodic orbits.

Proof. First we assume $q = 1$ and $p = 0$. Let $f \in \mathcal{D}^0(\mathbb{T}^1)$ be a lift of F. By Theorem 46, we have that $\exists x \in \mathbb{R}$ with $f(x) = x$. So $\mathrm{Fix}(f) \neq \varnothing$, and it is closed and invariant by integer translations. Now we write $\mathbb{R} \setminus \mathrm{Fix}(f)$ as union of open intervals. Let (a, b) be one of such connected components. Inside it, we must have either $f(x) > x$ or $f(x) < x$. In the first case we have that $(f^n(x))$ is strictly increasing $\forall x \in (a, b)$ and so $\omega(x) = \{b\} \in \mathrm{Fix}(f)$ and $\alpha(x) = \{a\} \in \mathrm{Fix}(f)$ $\forall x \in (a, b)$. The second case is exactly the opposite.
Now we do the general case. Assume $\rho(f) = \frac{p}{q}$. Then, again by Theorem 46, we have that $\exists x \in \mathbb{R}$ with $f^q(x) = x + p$. Assume we have $x' \in \mathbb{R}$ and $p', q' \in \mathbb{Z}$ with $q' \geq 1$ such that $f^{q'}(x') = x' + p'$. By Theorem 46, we have that $\frac{p}{q} = \frac{p'}{q'}$ and so $\exists k \in \mathbb{N}$ such that $q' = kq$ and $p' = kp'$ because $\frac{p}{q}$ is irreducible. Now let $g = f^q - p$. Then, an easy calculation shows that $g^k(x') = x'$. But $\rho(g) = 0$ and in the previous case we have seen that the periodic points are only fixed points, so $k = 1$. For the second part, we proceed as in the previous case with the function $g = f^q - p$. □

Irrational rotation number

Definition 59. Given $\mu \in \mathcal{M}(\mathbb{T}^1)$ and $U \subseteq \mathbb{T}^1$ open, we define the *measure of* U as:

$$\mu(U) := \sup\{\mu(\varphi) : \varphi \in \mathcal{C}(\mathbb{T}^1), \varphi \leq \mathbf{1}_U\}$$

Let $A \subset \mathcal{B}(\mathbb{T}^1)$ be a Borel measurable set. We define the *measure of* A as:

$$\mu(A) := \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$$

Remark. With this definition we have the usual properties of measure defined on subsets of $\mathbb{T}^1$. In particular, $\mathrm{Leb}([a, b]) = b - a$ and $\delta_x(A) = \mathbf{1}_{x \in A}$ $\forall x \in \mathbb{T}^1$ and $A \subseteq \mathbb{T}^1$.

Definition 60. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$. We define the *support of* μ as:

$$\operatorname{supp} \mu := \{x \in \mathbb{T}^1 : \forall U \subseteq \mathbb{T}^1 \text{ open with } x \in U, \mu(U) > 0\}$$

Remark. Note that $\operatorname{supp} \mu$ is a closed set and $\mu(\mathbb{T}^1 \setminus \operatorname{supp} \mu) = 0$.

Remark. If $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $\mu \in \mathcal{M}_F(\mathbb{T}^1)$, then $\operatorname{supp} \mu$ is invariant.

Remark. $\operatorname{supp} \mathrm{Leb} = \mathbb{T}^1$ and $\operatorname{supp} \delta_x = \{x\}$.

Proposition 61. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$ and $F \in \mathrm{Homeo}(\mathbb{T}^1)$. μ is invariant by F if and only if $\forall A \subseteq \mathbb{T}^1$ Borel set, $\mu(A) = \mu(F^{-1}(A))$.

Proof.

$\Longrightarrow$) Let $A \subseteq \mathbb{T}^1$ be Borel. We have:

$$\begin{aligned}\mu(A) &= \inf_{\substack{U \text{ open} \\ A \subseteq U}} \sup_{\substack{\psi \text{ cont.} \\ \varphi \leq \mathbf{1}_U}} \mu(\varphi) = \inf_{\substack{U \text{ open} \\ A \subseteq U}} \sup_{\substack{\psi \text{ cont.} \\ \varphi \leq \mathbf{1}_U}} \mu(\varphi \circ F) = \\ &= \inf_{\substack{U \text{ open} \\ A \subseteq U}} \sup_{\substack{\psi \text{ cont.} \\ \psi \leq \mathbf{1}_{F^{-1}(U)}}} \mu(\psi) = \inf_{\substack{V \text{ open} \\ F^{-1}(A) \subseteq V}} \sup_{\substack{\psi \text{ cont.} \\ \psi \leq \mathbf{1}_V}} \mu(\psi) = \\ &= \mu(F^{-1}(A))\end{aligned}$$

□

Remark. If $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ and $\mu \in \mathcal{M}_F(\mathbb{T}^1)$, then $\mu(F^n(A)) = \mu(A)$ $\forall n \in \mathbb{Z}$ and $A \subseteq \mathbb{T}^1$ Borel.

Lemma 62. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$. We have a lift to a measure μ on $\mathbb{R}$ invariant by integer translations: $\mu(A + k) = \mu(A)$ $\forall k \in \mathbb{Z}$ and $A \subseteq \mathcal{B}(\mathbb{R})$.

Definition 63. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$. We define $h_\mu : [0, 1] \to [0, 1]$ as the function with $h_\mu(0) = 0$ and $h_\mu(x) = \mu([0, x))$ for $0 < x \leq 1$. This definition extends to a non-decreasing function $h_\mu : \mathbb{R} \to \mathbb{R}$ such that $h_\mu(x + k) = h_\mu(x) + k$ $\forall k \in \mathbb{Z}$.

Definition 64. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$. We say that μ has atoms if $\exists x \in \mathbb{T}^1$ such that $\mu(\{x\}) > 0$.

Lemma 65. Let $\mu \in \mathcal{M}(\mathbb{T}^1)$. h_μ is continuous if and only if $\forall x \in \mathbb{R}$, $\mu(\{x\}) = 0$, that is if μ has no atoms.

Proof.

$\Longrightarrow$) Let $x_n = x + \frac{1}{n} \to x$. Then, $[0, x_n) \supset [0, x_{n+1})$ and so by the continuity of h_μ and **??** we have:

$$\mu([0, x)) = \lim_{n \to \infty} \mu([0, x_n)) = \mu\left(\bigcap_{n \in \mathbb{N}} [0, x_n)\right) = \\ = \mu([0, x])$$

which implies that $\mu(\{x\}) = 0$.

$\Longleftarrow$) Let $x \in \mathbb{R}$ and $x_n \to x$. Since (x_n) is bounded, we can extract a monotone subsequence (x_{n_k}). Then:

$$\lim_{n \to \infty} h(x_n) = \lim_{k \to \infty} \mu([0, x_{n_k})) = \\ = \begin{cases} \mu\left(\bigcup_{k \in \mathbb{N}} [0, x_{n_k})\right) = \mu([0, x)) & \text{if } x_{n_k} \nearrow x \\ \mu\left(\bigcap_{k \in \mathbb{N}} [0, x_{n_k})\right) = \mu([0, x]) & \text{if } x_{n_k} \searrow x \end{cases}$$

The first case is fine, and for the second one, since $\mu(\{x\}) = 0$, we have that $\mu([0, x]) = \mu([0, x))$.

□

Definition 66. A subset $C \subseteq \mathbb{R}$ is a *Cantor set* if it is closed, it has no isolated points and it has empty interior.

Theorem 67. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ with $\rho(F) \notin \mathbb{Q}/\mathbb{Z}$. Then, there exists a surjective continuous map $H : \mathbb{T}^1 \to \mathbb{T}^1$ such that $H \circ F = R_{\rho(F)} \circ H$. Moreover, we have exactly one of the following two properties:

1. F is conjugated to $R_{\rho(F)}$ and in that case F is minimal.

2. $\exists X \subsetneq \mathbb{T}^1$ minimal which is a Cantor set and $X = \Omega(F)$.

Proof. Let $\mu \in \mathcal{M}_F(\mathbb{T}^1)$ and consider $h := h_\mu : \mathbb{R} \to \mathbb{R}$ as defined above. Now assume $x \in \mathbb{T}^1$ is such that $\mu(\{x\}) = c > 0$, then by invariance $\mu(A_n) = c > 0$, where $A_n := \{F^n(x)\}$. Note that since $\mu \leq 1$, (A_n) cannot be disjoint. So $\exists n, m \in \mathbb{N}$ with $n < m$ such that $F^n(x) = F^m(x)$. But then $F^{m-n}(x) = x$ and so x is periodic, which is not possible since $\rho(F) \in \mathbb{Q}/\mathbb{Z}$ by Theorem 46. Thus, μ has no atoms and so h is continuous by Theorem 65. Now, define $H : \mathbb{T}^1 \to \mathbb{T}^1$ as the projection of h to $\mathbb{T}^1$, which is continuous and surjective. Let $f \in \mathcal{D}^0(\mathbb{T}^1)$ be a lift of F. Then:

$$h(f(x)) - h(f(0)) = \mu([f(0), f(x))) = \mu([0, x)) = h(x)$$

where we have used the invariance of μ. Thus, $h \circ f = R_{h(f(0))} \circ h$ and necessarily we need $h(f(0)) = \rho(R_{h(f(0))}) = \rho(f)$ because of the invariance of the rotation number under semi-conjugacy[1]. This gives $H \circ F = R_{\rho(F)} \circ H$. Now, we can express the dichotomy as follows: either $\operatorname{supp} \mu = \mathbb{T}^1$ or $\operatorname{supp} \mu =: X \subsetneq \mathbb{T}^1$. The first case is equivalent to h being strictly increasing and so h is a homeomorphism. Then, H conjugates F and $R_{\rho(F)}$ and so F is minimal because $R_{\rho(F)}$ is minimal. In the second case, we have that X is a nonempty closed invariant set that has no isolated points because μ has no atoms. To show that X is minimal, let $\mathbb{T}^1 = X \sqcup U$ with U open, and so it can be written as a countable union of open intervals. Let $D \subseteq X$ be the set containing the endpoints of those intervals and let

$$Y := \{y \in \mathbb{T}^1 : H^{-1}(\{y\}) \text{ is a closed interval}\} \tag{2}$$

Y is countable ($H^{-1}(Y) = U \cup D$). Now take $M \subseteq X$ be nonempty, closed and invariant. We want to prove that $M = X$. We have that $H(M) \subseteq \mathbb{T}^1$ is nonempty, closed (in fact it's compact because it is the image of a compact set) and invariant by $R_{\rho(F)}$. So $H(M) = \mathbb{T}^1$ because $R_{\rho(F)}$ is minimal. Now, since H restricted to $X \setminus D$ is injective (h is strictly increasing in $X \setminus D$ thought inside $[0, 1]$), then $M \supseteq X \setminus D$ because if not there would exist $x \in X \setminus D$ such that $x \notin M$. We have $H(X \setminus D) = \mathbb{T}^1 \setminus Y$. Thus, $H^{-1}(H(X \setminus D)) = H^{-1}(\mathbb{T}^1 \setminus Y) = X \setminus D$. Then, $H(x) \in H(X \setminus D) \subseteq \mathbb{T}^1 = H(M)$. So $\exists y \in M$ such that $H(x) = H(y)$, and so $y \in H^{-1}(H(X \setminus X)) = X \setminus D$. But $H|_{X \setminus D}$ is injective, so $x = y \in M$, which is a contradiction because $x \notin M$. Thus, $M \supseteq X \setminus D$, which implies:

$$M = \overline{M} \supseteq \overline{X \setminus D} = \overline{X} = X$$

So $M = X$ and, thus, X is minimal. Moreover, X has empty interior. Indeed, if that wasn't the case, we would have $\partial X = \overline{X} \setminus \mathrm{Int}(X) = X \setminus \mathrm{Int}(X) \subsetneq X$ and so ∂X would be a nonempty closed invariant set, which is not possible because X is minimal. Finally, to prove $X = \Omega(F)$, by minimality it suffices to show that $\Omega(F) \subseteq X$. Let $x \in U$, where $\mathbb{T}^1 = X \sqcup U$, with $U = \bigsqcup_{i=1}^{\infty} I_i$ invariant and I_i intervals. Note that F maps connected components of U to connected components of U. We need to see that x is wandering. Let I be one of such intervals. We may have either $F^n(I) = I$ for some $n \geq 1$ or $F^n(I) \cap I = \varnothing$ for all $n \geq 1$. But in the first case, we would have that the extremities of I are periodic points, which is not possible because $\rho(F) \notin \mathbb{Q}/\mathbb{Z}$. So we must have the second case, which implies that I is a wandering domain, and thus so is U. □

Unique ergodicity

Definition 68. A homeomorphism $F : \mathbb{T}^1 \to \mathbb{T}^1$ is *uniquely ergodic* if it has a unique invariant probability measure.

Lemma 69. If $\alpha \notin \mathbb{Q}$, then R_α is uniquely ergodic and $\mathcal{M}_{R_\alpha}(\mathbb{T}^1) = \{\mathrm{Leb}\}$.

Proof. Let $\mu \in \mathcal{M}_{R_\alpha}$. We want to see that $\forall \varphi \in \mathcal{C}(\mathbb{T}^1)$:

$$\int_{\mathbb{T}^1} \varphi(x) \, \mathrm{d}\mu = \int_{\mathbb{T}^1} \varphi(x) \, \mathrm{d}x$$

An easy check shows that if $P_n = \sum_{k=-n}^{n} a_k \mathrm{e}^{2\pi \mathrm{i} k x}$ is a trigonometric polynomial, then $\int_{\mathbb{T}^1} P_n(x) \, \mathrm{d}x = a_0$. Moreover, if $k \neq 0$:

$$\int_{\mathbb{T}^1} \mathrm{e}^{2\pi \mathrm{i} k x} \, \mathrm{d}\mu = \mathrm{e}^{2\pi \mathrm{i} k \alpha} \int_{\mathbb{T}^1} \mathrm{e}^{2\pi \mathrm{i} k x} \, \mathrm{d}\mu \implies \int_{\mathbb{T}^1} \mathrm{e}^{2\pi \mathrm{i} k x} \, \mathrm{d}\mu = 0$$

where the equality is due to the invariance of μ. So, we also have $\int_{\mathbb{T}^1} P_n(x) \, \mathrm{d}\mu = a_0$. Now consider the Féjer means, which converge uniformly to φ (recall **??** **??**) and use **??**. □

Proposition 70. Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ with $\rho(F) \notin \mathbb{Q}/\mathbb{Z}$. Then, F is uniquely ergodic.

Proof. Let H be such that $H \circ F = R_{\rho(F)} \circ H$ (by Theorem 67) and so $F^{-1}(H^{-1}(A)) = H^{-1}({R_{\rho(F)}}^{-1}(A))$. Take $\mu \in \mathcal{M}_F(\mathbb{T}^1)$ and define $H_* \mu$ as:

$$H_* \mu(A) := \mu(H^{-1}(A)) \qquad \forall A \subseteq \mathbb{T}^1 \text{ Borel}$$

We have:

$$\begin{aligned} H_* \mu(A) = \mu(H^{-1}(A)) = \mu(F^{-1}(H^{-1}(A))) = \\ = \mu(H^{-1}({R_{\rho(F)}}^{-1}(A))) = H_* \mu({R_{\rho(F)}}^{-1}(A)) \end{aligned}$$

[1] Recall that since h is a lift, then $h(x+1)$ is also a lift and so $h(x+1) - h(x) = k \in \mathbb{Z}$ and this constant has to be 1 because $h(1) = 1$. By Theorem 23 we have an expression of $h = \mathrm{id} + \varphi$, and that gives us the invariance of the rotation number.

where the second equality is due to the invariance of μ. Hence, $H_*\mu$ is invariant by $R_{\rho(F)}$, and so $H_*\mu = \text{Leb}$. That, is $\mu(H^{-1}(A)) = \text{Leb}(A)$. Recall again the set Y of Eq. (2) and $\mathbb{T}^1 = X \sqcup U$, with $H^{-1}(Y) = \overline{U}$. Since Y is countable, $0 = \text{Leb}(Y) = \mu(H^{-1}(Y))$. Now since $H|_{X\setminus D} : X \setminus D \to \mathbb{T}^1 \setminus Y$ is a homeomorphism, we have that $\mu(B) = \text{Leb}(H(B))$, and so μ is uniquely determined. $\square$

Proposition 71. Let $F \in \text{Homeo}(\mathbb{T}^1)$. Then, F is uniquely ergodic if and only if $\forall \varphi \in \mathcal{C}(\mathbb{T}^1)\ \exists c_\varphi \in \mathbb{R}$ such that $\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ F^i$ converge uniformly to c_φ. In that case, $c_\varphi = \mu(\varphi)$, where $\mathcal{M}_F(\mathbb{T}^1) = \{\mu\}$.

Proof. Assume first that $\mathcal{M}_F(\mathbb{T}^1) = \{\mu\}$ and argue by contradiction. That, is $\exists \varepsilon > 0$, $(n_k) \in \mathbb{N}$ with $n_k \nearrow +\infty$ and $(x_k) \in \mathbb{T}^1$ such that $\forall k \geq 0$:

$$\left| \frac{1}{n_k}\sum_{i=0}^{n_k-1} \varphi \circ F^i(x_k) - \int_{\mathbb{T}^1} \varphi \, \mathrm{d}\mu \right| = \left| \int_{\mathbb{T}^1} \varphi \, \mathrm{d}\nu_k - \int_{\mathbb{T}^1} \varphi \, \mathrm{d}\mu \right| > \varepsilon \tag{3}$$

where $\nu_k = \frac{1}{n_k}\sum_{i=0}^{n_k-1} (F^i)_*\delta_{x_k}$. Note that $\nu_k \in \mathcal{M}(\mathbb{T}^1)$ and since $\mathcal{M}(\mathbb{T}^1)$ is compact with the weak*-topology[2], after extracting a subsequence, (ν_k) converges weakly to $\nu \in \mathcal{M}(\mathbb{T}^1)$. Now, ν is invariant. Indeed:

$$\|F_*\nu_k - \nu_k\| = \left\| \frac{1}{n_k}((F^{n_k})_*\delta_{x_k} - \delta_{x_k}) \right\| \leq \frac{2\|\varphi\|}{n_k} \overset{k\to\infty}{\longrightarrow} 0$$

So $\nu = \mu$, but this is a contradiction with Eq. (3). Now we prove the converse. Let $\mu \in \mathcal{M}_F(\mathbb{T}^1)$. Then:

$$c_\varphi = \int_{\mathbb{T}^1} c_\varphi \, \mathrm{d}\mu = \int_{\mathbb{T}^1} \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ F^i \, \mathrm{d}\mu =$$
$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_{\mathbb{T}^1} \varphi \circ F^i \, \mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_{\mathbb{T}^1} \varphi \, \mathrm{d}\mu = \int_{\mathbb{T}^1} \varphi \, \mathrm{d}\mu$$

where the third equality is due to the uniform convergence and the penultimate equality is due to the invariance of μ. This implies that μ is uniquely determined. $\square$

Remark. The unique ergodicity property is preserved under conjugation.

Definition 72. For $k \in \mathbb{N} \cup \{0\}$ we define the set $\mathcal{D}^k(\mathbb{T}^1)$ as:

$$\mathcal{D}^k(\mathbb{T}^1) := \{f : \mathbb{R} \to \mathbb{R} \text{ increasing } \mathcal{C}^k\text{-diffeomorphism such that } f(x+1) = f(x) + 1\}$$

Note that $f \in \mathcal{D}^k(\mathbb{T}^1)$ if and only if $f = \text{id} + \varphi$, with $\varphi \in \mathcal{C}^k(\mathbb{T}^1)$. We also define the set $\text{Diff}_+^k(\mathbb{T}^1)$ as:

$$\text{Diff}_+^k(\mathbb{T}^1) := \{F : \mathbb{T}^1 \to \mathbb{T}^1\ \mathcal{C}^k\text{-diffeomorphism with orientation preserving}\}$$

[2] $\mathcal{M}(\mathbb{T}^1)$ is closed in $E = (\mathcal{C}(\mathbb{T}^1))^*$ and it's contained in the unit ball of E, which is compact for the weak*-topology. Thus, $\mathcal{M}(\mathbb{T}^1)$ is compact.

Proposition 73. Let $F \in \text{Diff}_+^1(\mathbb{T}^1)$ with $\rho(F) \notin \mathbb{Q}/\mathbb{Z}$, μ be the unique invariant probability measure of F and $f \in \mathcal{D}^1(\mathbb{T}^1)$ be a lift of F. Then, $\displaystyle\lim_{n\to\infty} \frac{1}{n} \log Df^n(x) = \int_{\mathbb{T}^1} \log(Df) \, \mathrm{d}\mu = 0$.

Proof. An easy induction shows that $\forall n \in \mathbb{N}$ we have:

$$\log Df^n = \sum_{i=0}^{n-1} \log(Df \circ f^i)$$

So:

$$\frac{1}{n}\log Df^n = \frac{1}{n}\sum_{i=0}^{n-1} \log(Df \circ f^i) = \frac{1}{n}\sum_{i=0}^{n-1} \log(Df \circ F^i)$$

where in the last equality we have used the fact that $Df = 1 + D\varphi \in \mathcal{C}(\mathbb{T}^1)$. By Theorems 70 and 71, we have that $\frac{1}{n}\sum_{i=0}^{n-1} \log(Df \circ F^i)$ converges uniformly to $c := \int_{\mathbb{T}^1} \log(Df) \, \mathrm{d}\mu$. Moreover, since $Df^n = 1 + D\varphi_n \in \mathcal{C}(\mathbb{T}^1)$, we have that $\int_{\mathbb{T}^1} Df^n \, \mathrm{d}x = 1 + \int_{\mathbb{T}^1} D\varphi_n \, \mathrm{d}x = 1$. Now assume without loss of generality that $c > 0$. Then, for n large enough we must have $Df^n(x) \sim e^{nc}$ and so:

$$1 = \int_{\mathbb{T}^1} Df^n \, \mathrm{d}x \sim \int_{\mathbb{T}^1} e^{nc} \, \mathrm{d}x \overset{n\to\infty}{\longrightarrow} +\infty$$

If $c < 0$, we have a similar contradiction. Thus, $c = 0$. $\square$

Definition 74. Let $\varphi \in \mathcal{C}(\mathbb{T}^1)$. We say that φ has *bounded variation* if $\exists C \geq 0$ such that for all $0 = x_0 < x_1 < \cdots < x_n = 1$ we have:

$$\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| \leq C$$

The constant C is usually denoted as $\text{Var}(\varphi)$.

Remark. If $\varphi \in \text{Lip}(\mathbb{T}^1)$, then $\text{Var}(\varphi) = L$, where L is the Lipschitz constant of φ.

Lemma 75. Let $\alpha \notin \mathbb{Q}$. Then, $\forall n \in \mathbb{N}$, $\exists \frac{p_n}{q_n} \in \mathbb{Q}$ such that:

1. $\displaystyle\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{{q_n}^2}$
2. $q_n \overset{n\to\infty}{\longrightarrow} +\infty$

Proof. Let $Q \in \mathbb{N}$. By the Pigeon-hole principle, there exist two elements among $0, \{\alpha\}, \ldots, \{Q\alpha\}$ (here $\{\cdot\}$ denotes the fractional part) such that they are in one of the intervals among $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \ldots, \left[\frac{Q-1}{Q}, 1\right]$. That is, $\exists q_1, q_2 \in \mathbb{Q}_{\geq 0}$ and $p \in \mathbb{Z}$ such that $|q\alpha - p| \leq \frac{1}{Q}$ with $q := q_2 - q_1 \leq Q$. Now apply this to $Q_n = n \geq 1$: $\exists \frac{p_n}{q_n} \in \mathbb{Q}$ with $1 \leq q_n \leq n$ such that:

$$|q_n\alpha - p_n| < \frac{1}{n} \leq \frac{1}{q_n}$$

To prove that $q_n \overset{n\to\infty}{\longrightarrow} +\infty$, we argue by contradiction. Assume that $\exists M \in \mathbb{N}$ such that $q_n \leq M$ for all $n \in \mathbb{N}$. Then, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have $q_n = q_{n_0}$. But then by the irrationality of α we have that $\exists c > 0$ such that:

$$c \leq |q_n\alpha - p_n| \leq \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0$$

which is a contradiction. □

Lemma 76. Let $\alpha \notin \mathbb{Q}$ and $\frac{p}{q} \in \mathbb{Q}$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$. Set $\alpha_i := \{i\alpha\}$ for $1 \leq i \leq q$, where $\{x\}$ denotes the fractional part of x. Then, each α_i belongs to a different interval of the form $\left(\frac{k_i}{q}, \frac{k_i+1}{q}\right)$ with $k_i \in \{0, \ldots, q-1\}$.

Proof. Assume without loss of generality that $0 < \alpha - \frac{p}{q} < \frac{1}{q^2}$. Then, for $1 \leq i \leq q$ we have:

$$0 \leq i\alpha - \frac{ip}{q} < \frac{i}{q^2} \leq \frac{1}{q} \tag{4}$$

We claim that the numbers $\{i\frac{p}{q}\}$ are all distinct for $1 \leq i \leq q$. Indeed, if $\exists i, j$ with $i\frac{p}{q} - j\frac{p}{q} = k \in \mathbb{Z}^*$, then $\frac{p}{q} = \frac{k}{i-j}$, which is not possible because $\frac{p}{q}$ is irreducible and $i - j \leq q - 1$. So we can write $\{i\frac{p}{q}\} = k_i\frac{p}{q}$ for some $k_i \in \{0, \ldots, q-1\}$. Finally, Eq. (4) implies that $\alpha_i \in \left(\frac{k_i}{q}, \frac{k_i+1}{q}\right)$. □

Proposition 77 (Denjoy-Koksma inequality). Let $F \in \mathrm{Homeo}_+(\mathbb{T}^1)$ with $\alpha := \rho(F) \notin \mathbb{Q}/\mathbb{Z}$, $\mu \in \mathcal{M}_F(\mathbb{T}^1)$ and $\frac{p}{q} \in \mathbb{Q}$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$. Then, $\forall \psi \in \mathcal{C}(\mathbb{T}^1)$ with $\mathrm{Var}(\psi) < \infty$ we have:

$$\left|\sum_{i=0}^{q-1} \psi(F^i(x)) - q\int_{\mathbb{T}^1} \psi \,\mathrm{d}\mu\right| \leq \mathrm{Var}(\psi) \qquad \forall x \in \mathbb{T}^1$$

Proof. We'll prove that $\forall x \in \mathbb{T}^1$:

$$\left|\sum_{i=1}^{q} \psi(F^i(x)) - q\int_{\mathbb{T}^1} \psi \,\mathrm{d}\mu\right| \leq \mathrm{Var}(\psi)$$

which is equivalent by replacing x by $F^{-1}(x)$. Let $x \in \mathbb{T}^1$ and choose $y_1, \ldots, y_{q-1} \in \mathbb{T}^1$ circularly ordered with $y_0 := x$ and such that $H(y_i) = \frac{i}{q} + H(x)$, where $H : \mathbb{T}^1 \to \mathbb{T}^1$ is the semi-conjugacy between F and R_α given by Theorem 67. By Theorem 76, we have that $\exists! k_i \in \{0, \ldots, q-1\}$ such that $H(x) + i\alpha \in \left(H(x) + \frac{k_i}{q}, H(x) + \frac{k_i+1}{q}\right)$. This implies that $F^i(x) \in [y_{k_i}, y_{k_i+1}] =: I_i$ because $H(x) + i\alpha = {R_\alpha}^i \circ H(x) = H \circ F^i(x)$ and H is increasing (thought in $[0,1]$). Now, we have:

$$\left|\sum_{i=1}^{q} \psi(F^i(x)) - q\int_{\mathbb{T}^1} \psi \,\mathrm{d}\mu\right| = \left|\sum_{i=1}^{q}\left(\psi(F^i(x)) - q\int_{I_i} \psi \,\mathrm{d}\mu\right)\right|$$

$$= \left|\sum_{i=1}^{q} q\left(\int_{I_i} \psi(F^i(x)) - \psi(t) \,\mathrm{d}\mu(t)\right)\right| \leq$$

$$\leq q\sum_{i=1}^{q} \sup_{t\in I_i} \left|\psi(F^i(x)) - \psi(t)\right| \mu(I_i) =$$

$$= \sum_{i=1}^{q} \left|\psi(F^i(x)) - \psi(t_i)\right| \leq \sum_{i=1}^{q} \mathrm{Var}(\psi|_{I_i}) = \mathrm{Var}(\psi)$$

where in the first equality we have used that:

$$\mu(I_i) = \mu(F^{-1}(I_i)) = \mu(F^{-1}(H^{-1}(J_i))) =$$
$$= \mu(H^{-1}({R_\alpha}^{-1}(J_i))) = \mu(H^{-1}(J_i)) = \mathrm{Leb}(J_i) = \frac{1}{q}$$

where $J_i = [H(y_{k_i}), H(y_{k_i+1})]$. Here we used first the invariance of μ, then the semi-conjugacy property of H, the fact that $H_*\mu$ is invariant under R_α and lastly $H_*\mu = \mathrm{Leb}$. The value $t_i \in I_i$ above is because the supremum is reached at some point, as the intervals are closed. □

Lemma 78. Let $f \in \mathcal{D}^1(\mathbb{T}^1)$. Df has bounded variation if and only if $\log Df$ has bounded variation.

Proof. Note that since $Df > 0$ (because f is an increasing homeomorphism) it attains a maximum $M > 0$ and a minimum $m > 0$ in $[0,1]$. Thus, for any $0 = x_0 < x_1 < \cdots < x_n = 1$ we have:

$$\sum_{i=1}^{n} \frac{1}{M}|Df(x_i) - Df(x_{i-1})| \leq$$
$$\leq \sum_{i=1}^{n} |\log Df(x_i) - \log Df(x_{i-1})| \leq$$
$$\leq \sum_{i=1}^{n} \frac{1}{m}|Df(x_i) - Df(x_{i-1})|$$

where we have used the **?? ??**. Hence, $\mathrm{Var}(Df) < \infty \iff \mathrm{Var}(\log Df) < \infty$. □

Theorem 79 (Denjoy theorem). Let $F \in \mathrm{Diff}_+^1(\mathbb{T}^1)$ with $\rho(F) \notin \mathbb{Q}/\mathbb{Z}$ and $f \in \mathcal{D}^1(\mathbb{T}^1)$ be a lift of F whose derivative Df has bounded variation. Then, F is topologically conjugated to $R_{\rho(F)}$.

Proof. By Theorem 67 it suffices to show that F has no wandering intervals. We argue by contraction. Assume that $J \subsetneq \mathbb{T}^1$ is a wandering interval, i.e. $\forall n \in \mathbb{Z}^*$, $F^n(J) \cap J = \varnothing$. This implies that $F^n(J) \cap F^m(J) = \varnothing$ if $n \neq m$ and since $\sum_{n\in\mathbb{Z}} \mathrm{Leb}(F^n(J)) \leq 1$, we must have $\mathrm{Leb}(F^n(J)) \overset{n\to\infty}{\longrightarrow} 0$. By assumption, $\mathrm{Var}(Df) < \infty$, so by Theorem 78, we have $\mathrm{Var}(\log Df) < \infty$. By Theorem 75, $\exists \frac{p_n}{q_n} \in \mathbb{Q}$ such that $\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{{q_n}^2}$ and $q_n \overset{n\to\infty}{\longrightarrow} +\infty$. Now use 77 Denjoy-Koksma inequality applied to $\psi = \log Df$ and the sequence $\frac{p_n}{q_n}$:

$$\left|\sum_{i=0}^{q_n-1} \log Df(F^i(x)) - q\int_{\mathbb{T}^1} \log Df \,\mathrm{d}\mu\right| =$$
$$= \left|\sum_{i=0}^{q_n-1} \log Df(F^i(x))\right| \leq \mathrm{Var}(\log Df) =: V$$

But

$$\sum_{i=0}^{q_n-1} \log Df(F^i(x)) = \sum_{i=0}^{q_n-1} \log Df(f^i(x)) = \log Df^{q_n}(x)$$

Thus, $-V \leq \log Df^{q_n} \leq V$, and so $\mathrm{e}^{-V} \leq Df^{q_n} \leq \mathrm{e}^{V}$. Hence, using the mean value theorem $\forall x, y \in \mathbb{R}$ we have:

$$\mathrm{e}^{-V}|x-y| \leq |f^{q_n}(x) - f^{q_n}(y)| \leq \mathrm{e}^{V}|x-y|$$

Applying this to the extremities of J, we have:

$$\mathrm{e}^{-V}\mathrm{Leb}(J) \leq \mathrm{Leb}(F^{q_n}(J)) \leq \mathrm{e}^{V}\mathrm{Leb}(J)$$

Since $q_n \overset{n\to\infty}{\longrightarrow} +\infty$, this contradicts the fact that $\mathrm{Leb}(F^n(J)) \overset{n\to\infty}{\longrightarrow} 0$. $\square$