

Real and functional analysis

1. | Measure theory and Lebesgue integral

Measures

Definition 1 (σ -algebra). Let Ω be a set and $\Sigma \subseteq \mathcal{P}(\Omega)$. We say that Σ is a σ -algebra over Ω if:

1. $\Omega \in \Sigma$.
2. If $A \in \Sigma$, then $A^c \in \Sigma$.
3. If $(A_n) \in \Sigma$, then:

$$\bigcup_{n=1}^{\infty} A_n \in \Sigma$$

Proposition 2. Let Σ be a σ -algebra over a set Ω . Then:

1. $\emptyset \in \Sigma$.
2. If $A, B \in \Sigma$, then $A \setminus B \in \Sigma$.
3. If $(A_n) \in \Sigma$, then:

$$\bigcap_{n=1}^{\infty} A_n \in \Sigma$$

Sketch of the proof. Use ?? ?? to show that $A \cap B \in \Sigma$ if $A, B \in \Sigma$. $\square$

Definition 3 (Measure). Let Σ be a σ -algebra over a set Ω . A *measure* over Ω is any function

$$\mu : \Sigma \longrightarrow [0, \infty]$$

satisfying the following properties:

1. $\mu(\emptyset) = 0$.
2. σ -additivity: If $(A_n) \in \Sigma$ are pairwise disjoint, then:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 4. Let Σ be a set and $(A_n) \in \Sigma$ be subsets. We say that $A_n \nearrow A$ if $A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\infty} A_n$. Analogously, we say that $A_n \searrow A$ if $A_n \supseteq A_{n+1} \forall n \in \mathbb{N}$ and $A = \bigcap_{n=1}^{\infty} A_n$.

Proposition 5. Let Σ be a σ -algebra over a set Ω , $\mu : \Sigma \longrightarrow [0, \infty]$ be a measure over Ω and $A_n, A, B \in \Sigma$, $n \in \mathbb{N}$. Then:

1. If $A \subseteq B$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.
2. If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
3. If $A_n \nearrow A$, then $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.

¹Note that for each $m \in \mathbb{N}$ we can make a partition of $\mathbb{R}^n$ in dyadic cubes.

4. If $A_n \searrow A$ and $\mu(A_1) < \infty$, then $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.

Sketch of the proof.

- 1, 2.

$$\mu(B) = \mu(A \sqcup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$$

3. Let $E_1 = A_1$ and $E_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then, $A = \bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} E_n$ and:

$$\begin{aligned} \mu(A) &= \sum_{n=1}^{\infty} \mu(E_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^N (\mu(A_n) - \mu(A_{n-1})) \\ &= \lim_{N \rightarrow \infty} \mu(A_N) \end{aligned}$$

4. Let $E_n = A_n \setminus A_{n+1}$ for $n \in \mathbb{N}$. Then, $A_1 = A \sqcup \bigsqcup_{n=1}^{\infty} E_n$ and:

$$\begin{aligned} \mu(A_1) &= \mu(A) + \sum_{n=1}^{\infty} \mu(E_n) \\ &= \mu(A) + \lim_{N \rightarrow \infty} \sum_{n=1}^N (\mu(A_n) - \mu(A_{n+1})) \\ &= \mu(A) + \mu(A_1) - \lim_{N \rightarrow \infty} \mu(A_N) \end{aligned}$$

And since $\mu(A_1) < \infty$, we get the desired result. $\square$

Definition 6. An *interval* $I \subseteq \mathbb{R}^n$ is a set of the form:

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

where $a_i, b_i \in \mathbb{R}_{\infty}$ and the notation $|a, b|$ represents either (a, b) , $[a, b)$, $(a, b]$ or $[a, b]$.

Definition 7. Let $I = \prod_{i=1}^n |a_i, b_i| \subseteq \mathbb{R}^n$ be an interval. We define its *volume* as:

$$\text{vol}(I) := \prod_{i=1}^n (b_i - a_i)$$

Definition 8. Let $m \in \mathbb{N} \cup \{0\}$. We define the m -th *dyadic cube* as the set:

$$[a_1, a_1 + 2^{-m}) \times \cdots \times [a_n, a_n + 2^{-m})$$

where $a_i \in 2^{-m}\mathbb{Z}^1$.

Lemma 9. Let $m \in \mathbb{N} \cup \{0\}$. Then the sidelength of the m -th dyadic cube is 2^{-m} , its volume 2^{-mn} and its diameter is $2^{-m}\sqrt{n}$.

Proposition 10. Any nonempty open set $U \subseteq \mathbb{R}^n$ can be written as a countable union of disjoint dyadic cubes whose closure is in U .

Proof. Let $\mathcal{D}_0$ be a union of disjoint 0-th dyadic cubes whose closure is contained in U . Now let $\mathcal{D}_1$ be a family of disjoint 1-th dyadic cubes contained in $U \setminus \mathcal{D}_0$ whose closure is in $U \setminus \overline{\mathcal{D}_0}$. In general, let $\mathcal{D}_n$ be a family of disjoint n -th dyadic cubes contained in $U \setminus \bigcup_{k=0}^{n-1} \mathcal{D}_k$ whose closure is in $U \setminus \bigcup_{k=0}^{n-1} \overline{\mathcal{D}_k}$. By construction, we have that $\bigcup_{n=0}^{\infty} \overline{\mathcal{D}_0} \subseteq U$ and the $\mathcal{D}_n$ are clearly pairwise disjoint. Moreover, if $x \in U$, $\delta := d(x, U^c) > 0$ and so it will be contained (at least) in an m -th dyadic cube, with $2^{-m}\sqrt{n} < \delta$. Hence, $\bigcup_{n=0}^{\infty} \mathcal{D}_0 = U$ $\square$

Definition 11. Let $A \subseteq \mathbb{R}^n$ be a set. We denote by $\mathcal{I}(A)$ the set of sequences of intervals that cover A . Analogously, we denote by $\mathcal{I}_0(A)$ the set of sequences of open intervals that cover A .

Definition 12 (Outer measure). Let $A \subseteq \mathbb{R}^n$ be a set. We define its *outer measure* as the function $|\cdot|^*$ defined by:

$$|A|^* := \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(I_k) : \{I_k : k \geq 1\} \in \mathcal{I}(A) \right\}$$

Proposition 13. Let $A \subseteq \mathbb{R}^n$ be a set. Then:

$$|A|^* = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(I_k) : \{I_k : k \geq 1\} \in \mathcal{I}_0(A) \right\}$$

Proof. Let

$$|A|_0^* = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(I_k) : \{I_k : k \geq 1\} \in \mathcal{I}_0(A) \right\}$$

We shall see $|A|^* = |A|_0^*$. As $\mathcal{I}_0(A) \subset \mathcal{I}(A)$, we have $|A|^* \leq |A|_0^*$.

Now let $\varepsilon > 0$ and by the definition of infimum we can take $(I_k) \in \mathcal{I}(A)$ of the form

$$I_k = [a_1^k, b_1^k] \times \cdots \times [a_n^k, b_n^k]$$

such that $\sum_{k=1}^{\infty} \text{vol}(I_k) \leq |A|^* + \frac{\varepsilon}{2}$. It is clear $\forall k \in \mathbb{N}$ $\exists \delta_k > 0$ such that:

$$J_k = [a_1^k - \delta_k, b_1^k + \delta_k] \times \cdots \times [a_n^k - \delta_k, b_n^k + \delta_k]$$

satisfy $\text{vol}(J_k) \leq \text{vol}(I_k) + \frac{\varepsilon}{2^{k+1}}$ and so:

$$|A|_0^* \leq \sum_{k=1}^{\infty} \text{vol}(J_k) \leq \sum_{k=1}^{\infty} \text{vol}(I_k) + \frac{\varepsilon}{2} \leq |A|^* + \varepsilon$$

This is true $\forall \varepsilon > 0$. Thus, $|A|_0^* \leq |A|^*$. $\square$

Lemma 14. Let $I, J_1, \dots, J_N \subseteq \mathbb{R}^n$ be intervals such that $I \subseteq \bigcup_{k=1}^N J_k$. Then, $\text{vol}(I) \leq \sum_{k=1}^N \text{vol}(J_k)$.

Sketch of the proof. Note that we can suppose $I = \bigcup_{k=1}^N J_k$ by intersecting (if necessary) with I . Suppose $I = I_1 \times \cdots \times I_n$ and let $\{R_k : k = 1, \dots, M\}$ be an appropriate partition of subrectangles of I . Note that $\text{vol}(I) = \sum_{k=1}^M \text{vol}(R_k)$. If (J_k) were pairwise disjoint we would have $\sum_{k=1}^M \text{vol}(R_k) = \sum_{k=1}^N \text{vol}(J_k)$. If not, some of these R_k are “repeated” and so $\text{vol}(I) = \sum_{k=1}^M \text{vol}(R_k) \leq \sum_{k=1}^N \text{vol}(J_k)$. $\square$

²Here $A + x := \{a + x : a \in A\}$ and $-A := \{-a : a \in A\}$

Theorem 15. The outer measure has the following properties:

1. $|\emptyset|^* = 0$.
2. If $A \subseteq B \subseteq \mathbb{R}^n$, then $|A|^* \leq |B|^*$.
3. If $(A_k) \subseteq \mathbb{R}^n$, then:
$$\left| \bigcup_{k=1}^{\infty} A_k \right|^* \leq \sum_{k=1}^{\infty} |A_k|^*$$
4. If $I \subseteq \mathbb{R}^n$ is an open interval and $I \subseteq A \subseteq \bar{I}$, then $|A|^* = \text{vol}(I)$.
5. If $I_1, \dots, I_N \subseteq \mathbb{R}^n$ are disjoint intervals, then:
$$\left| \bigcup_{k=1}^N I_k \right|^* = \sum_{k=1}^N \text{vol}(I_k)$$

6. If $A, B \subseteq \mathbb{R}^n$ and $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\} > 0$, then $|A \sqcup B|^* = |A|^* + |B|^*$.
7. If $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then $|A + x|^* = |-A|^* = |A|^*$ ².

Sketch of the proof.

1. Clear because $\emptyset \subset I$ for any interval $I \subset \mathbb{R}^n$.
2. Use the fact that $\mathcal{I}(B) \subseteq \mathcal{I}(A)$.
3. Let $\varepsilon > 0$. For each $k \in \mathbb{N}$ let $(I_n^k) \in \mathcal{I}(A_k)$ be such that:

$$\sum_{n=1}^{\infty} \text{vol}(I_n^k) \leq |A_k|^* + \frac{\varepsilon}{2^k}$$

Since, $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k,n=1}^{\infty} I_n^k$ we have that:

$$\begin{aligned} \left| \bigcup_{k=1}^{\infty} A_k \right|^* &\leq \sum_{k,n=1}^{\infty} \text{vol}(I_n^k) \leq \sum_{k=1}^{\infty} \left(|A_k|^* + \frac{\varepsilon}{2^k} \right) \\ &= \sum_{k=1}^{\infty} |A_k|^* + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$.

4. If follows from [Item 15-2](#) and the fact that $\text{vol}(I) = \text{vol}(\bar{I})$.
5. The inequality $\leq$ follows from [Item 15-3](#).

For the other one, let $\varepsilon > 0$, $I := \bigcup_{k=1}^N I_k$ and $K \subset I$ be a compact interval such that $\sum_{k=1}^N \text{vol}(I_k) \leq \text{vol}(K) + \varepsilon$. Now take $(J_k) \in \mathcal{I}_0(I)$ such that $\sum_{k=1}^{\infty} \text{vol}(J_k) \leq |I|^* + \varepsilon$. In particular $(J_k) \in \mathcal{I}_0(K)$. Since K is compact, there exists a finite covering of K which without loss of generality we may assume it is $K \subset \bigcup_{k=1}^M J_k$. Then

$$\begin{aligned} \sum_{k=1}^N \text{vol}(I_k) &\leq \text{vol}(K) + \varepsilon \leq \sum_{k=1}^M \text{vol}(J_k) + \varepsilon \\ &\leq \sum_{k=1}^{\infty} \text{vol}(J_k) + \varepsilon \leq |I|^* + 2\varepsilon \end{aligned}$$

by [Items 15-2](#) and [15-3](#). Since this is true $\forall \varepsilon > 0$, we get $\sum_{k=1}^N \text{vol}(I_k) \leq |I|^*$.

6. If $\delta = d(A, B) > 0$, it suffices to consider two sequences $(I_k) \in \mathcal{I}(A)$ and $(J_k) \in \mathcal{I}(B)$ such that $\text{diam}(I_k) < \delta/3$ and $\text{diam}(J_k) < \delta/3 \forall k \in \mathbb{N}$.
7. It follows from the property that the volume of intervals is invariant under translations and reflections.

□

Definition 16. A set $N \subset \mathbb{R}^n$ is called a *null set* if $|N|^* = 0$.

Definition 17. We say that a property holds *almost everywhere (a.e.)* if the set of points that doesn't hold it is null.

Lemma 18. The countable union of null sets is null.

Proof. Let $(N_n) \subset \mathbb{R}^n$ be a sequence of null sets. Then:

$$\left| \bigcup_{n=1}^{\infty} N_n \right|^* \leq \sum_{n=1}^{\infty} |N_n|^* = 0$$

□

Lemma 19. A point is null. Therefore, all countable sets are null.

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then, $\forall \varepsilon > 0$ we have:

$$|\{x\}|^* \leq \text{vol} \left(\prod_{i=1}^n \left(x_i - \frac{\varepsilon}{2}, x_i + \frac{\varepsilon}{2} \right) \right) = \varepsilon^n$$

And if $Q \subset \mathbb{R}^n$ is a countable subset, then using [Theorem 18](#) we deduce that $|Q|^* = 0$ from the fact that $Q = \bigcup_{q \in Q} \{q\}$.

□

Lebesgue measure

Definition 20 (Lebesgue measure). We say that $A \subseteq \mathbb{R}^n$ is *Lebesgue measurable* (or simply *measurable*) if $\forall \varepsilon > 0$, there exists an open set $U \supseteq A$ such that $|U \setminus A|^* < \varepsilon$. We denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all Lebesgue measurable sets of $\mathbb{R}^n$ and by $|\cdot|$ the restriction of $|\cdot|^*$ to $\mathcal{M}(\mathbb{R}^n)$.

Theorem 21. $\mathcal{M}(\mathbb{R}^n)$ is a σ -algebra and $|\cdot| : \mathcal{M}(\mathbb{R}^n) \rightarrow [0, \infty]$ is a measure (called *Lebesgue measure*) that satisfies:

1. The open sets, closed sets and null sets are in $\mathcal{M}(\mathbb{R}^n)$.
2. Each interval $I \subseteq \mathbb{R}^n$ is measurable and $|I| = \text{vol}(I)$.
3. If $A \in \mathcal{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then $A+x, -A \in \mathcal{M}(\mathbb{R}^n)$ and $|A+x| = |-A| = |A|$.
4. If $A \in \mathcal{M}(\mathbb{R}^n)$:

$$\begin{aligned} |A| &= \inf\{|U| : A \subseteq U \subseteq \mathbb{R}^n, U \text{ open}\} \\ &= \sup\{|C| : C \subseteq A \subseteq \mathbb{R}^n, C \text{ closed}\} \\ &= \sup\{|K| : K \subseteq A \subseteq \mathbb{R}^n, K \text{ compact}\} \end{aligned}$$

Sketch of the proof. Let's prove first that $\mathcal{M}(\mathbb{R}^n)$ is a σ -algebra.

- $\mathbb{R}^n \in \mathcal{M}(\mathbb{R}^n)$ because $\mathbb{R}^n$ is open and $0 = |\mathcal{M}(\mathbb{R}^n) \setminus \mathbb{R}^n|^* < \varepsilon \forall \varepsilon > 0$
- Let $(A_n) \in \mathcal{M}(\mathbb{R}^n)$. Then, $\forall \varepsilon > 0$ for each $n \in \mathbb{N}$ there exists an open set $U_n \supseteq A_n$ such that $|U_n \setminus A_n|^* < \frac{\varepsilon}{2^n}$. Now, $U := \bigcup_{n=1}^{\infty} U_n$ is open and satisfy:

$$\begin{aligned} \left| \bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n \right|^* &\leq \left| \bigcup_{n=1}^{\infty} (U_n \setminus A_n) \right|^* \\ &\leq \sum_{n=1}^{\infty} |U_n \setminus A_n|^* \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \end{aligned}$$

Let's see now the first property.

1. Clearly open sets are measurable and so are the null sets since $|U \setminus N|^* \leq |U|^*$ for all open sets U and null sets N . To show that the closed sets are measurable, let's prove first that the compact sets are measurable. Given a compact set K and $\varepsilon > 0$, there exists an open set U such that $K \subset U$, $|U| \leq |K|^* + \varepsilon$ and $U \setminus K = \bigsqcup_{k=1}^{\infty} D_k$, where (D_k) are dyadic cubes. Thus:

$$|K|^* + \varepsilon \geq |U| \geq \left| K \sqcup \bigcup_{k=1}^{\infty} D_k \right|^* = |K|^* + \sum_{k=1}^{\infty} \text{vol } D_k$$

by [Item 15-5](#). Thus, $|U \setminus K|^* \leq \sum_{k=1}^{\infty} \text{vol } D_k \leq \varepsilon$. Finally, as any closed set can be written a countable union of compact sets, since $\mathcal{M}(\mathbb{R}^n)$ is a σ -algebra, we get the desired result.

Finally:

- Let $A \in \mathcal{M}(\mathbb{R}^n)$. Then, $\forall k \in \mathbb{N}$ there exists an open set $U_k \supseteq A$ such that $|U_k \setminus A|^* < \frac{1}{k}$. Let $U := \bigcup_{k=1}^{\infty} U_k^c$. Then, $U \subseteq A^c$ and if $N = A^c \setminus U$ we have that $N \subseteq U_k \setminus A \forall k \in \mathbb{N}$. So $|N|^* \leq \frac{1}{k} \forall k \in \mathbb{N}$ and therefore N is null. So $A^c = U \cup N$ is union of two measurable sets (U is a countable union of closed sets) and thereby measurable.

So $\mathcal{M}(\mathbb{R}^n)$ is a σ -algebra. Now let's see that $|\cdot|$ is a measure.

- Consequence of $\emptyset \in \mathcal{M}(\mathbb{R}^n)$ and $|\emptyset|^* = 0$.
- Let $(A_n) \in \mathcal{M}(\mathbb{R}^n)$ be pairwise disjoint. We need to prove that:

$$\left| \bigcup_{n=1}^{\infty} A_n \right| = \sum_{n=1}^{\infty} |A_n|$$

One inequality has already been given in [Item 15-3](#). For the other one, first suppose that the (A_n) are bounded. Then, $\forall \varepsilon > 0$ there exist compact sets $K_n \subset A_n$ such that $|A_n| \leq |K_n|^* + \frac{\varepsilon}{2^k}$. Then:

$$\sum_{n=1}^{\infty} |A_n| \leq \sum_{n=1}^{\infty} \left(|K_n|^* + \frac{\varepsilon}{2^k} \right) \leq |A| + \varepsilon$$

If the (A_n) aren't bounded, then take (B_m^n) measurable and bounded such that $A_n = \bigsqcup_{m=1}^{\infty} B_m^n$ and so:

$$\sum_{n=1}^{\infty} |A_n| = \sum_{n,m=1}^{\infty} |B_m^n| = \left| \bigcup_{n,m=1}^{\infty} B_m^n \right| = |A|$$

So $|\cdot|$ is a measure. Now let's continue with the remaining properties:

2. $I = \partial I \cup \text{Int}(I)$ which is a union of a closed set and an open set. Hence, $I \in \mathcal{M}(\mathbb{R}^n)$ and $|I| = \text{vol}(I)$ (by [Item 15-4](#)).
3. It follows from [Item 15-7](#).
4. Note that if $A \in \mathcal{M}(\mathbb{R}^n)$, then there exists a closed set $C \subseteq A$ such that $|A \setminus C|^* < \varepsilon \forall \varepsilon > 0$. Indeed, there exists an open set $U \supseteq A^c$ such that $|U \setminus A^c|^* < \varepsilon$. Now take $C = U^c$. From here we deduce the first two equalities. For the third one, it suffices to bound to closed sets by intersecting them with balls to become compact sets.

□

Definition 22. A *real function* is a function $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$. We will say that f is *finite* if $\pm\infty \notin \text{im } f$.

Definition 23. Let f be a real function. We say that f is *Lebesgue measurable* (or simply *measurable*) if $\{f(x) > r\} \in \mathcal{M}(\mathbb{R}^n) \forall r \in \mathbb{R}$.

Lemma 24. Let $a, b \in [-\infty, +\infty]$ and f be a real function. The sets:

- $\{a < f(x) < b\}$
- $\{a \leq f(x) < b\}$
- $\{a < f(x) \leq b\}$
- $\{a \leq f(x) \leq b\}$

are all measurable.

Sketch of the proof. Rewrite the sets as union of antiimages of f . For example, for the second one:

$$\begin{aligned} \{a \leq f(x) < b\} &= \{f(x) \geq a\} \cup \{f(x) \geq b\}^c \\ &= \bigcap_{n=1}^{\infty} \left\{ f(x) > a - \frac{1}{n} \right\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ f(x) > b - \frac{1}{n} \right\} \right)^c \end{aligned}$$

Since all the sets in the unions are measurable, so is the initial set. □

Proposition 25. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable if and only if for all open set $U \subseteq \mathbb{R}$, $f^{-1}(U) \in \mathcal{M}(\mathbb{R}^n)$.

Sketch of the proof.

$\Rightarrow$) Let $U \subseteq \mathbb{R}$ be an open set. We can write U as a countable union of intervals I_n . Then:

$$f^{-1}(U) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$$

And $f^{-1}(I_n) \in \mathcal{M}(\mathbb{R}^n)$ by [Theorem 24](#).

$\Leftarrow$) Take the open set $(r, \infty) \subset \mathbb{R}$ and note that:

$$\{f > r\} = f^{-1}((r, \infty)) \in \mathcal{M}(\mathbb{R}^n)$$

□

Proposition 26. Let f be a finite measurable real function, $U \subseteq \mathbb{R}$ be an open set such that $\text{im } f \subseteq U$ and $\varphi : U \rightarrow \mathbb{R}$ be a continuous function. Then, $\varphi \circ f$ is also measurable.

Proof. For any open set $V \subseteq \mathbb{R}$ we have that:

$$(\varphi \circ f)^{-1}(V) = f^{-1}(\varphi^{-1}(V)) \in \mathcal{M}(\mathbb{R}^n)$$

because $\varphi^{-1}(V)$ is open as φ is continuous. □

Proposition 27. Let u, v be two finite measurable real functions, $U \subseteq \mathbb{R}^2$ be an open set such that $(u(x), v(x)) \in U \forall x \in \mathbb{R}^n$ and $\varphi : U \rightarrow \mathbb{R}$ be a continuous function. Then, $\varphi(u(x), v(x))$ is also measurable.

Sketch of the proof. For any open set $V \subseteq \mathbb{R}$, $\varphi^{-1}(V)$ is open, and therefore it is a countable union of intervals of the form $I \times J$. Now, use the fact that $\{(u, v) \in I \times J\} = \{u \in I\} \cap \{v \in J\}$. □

Proposition 28. Let f, g be two measurable real functions. Then, so are $f \pm g$, fg and f/g if $g(x) \neq 0 \forall x \in \mathbb{R}^n$.

Proof. Use [Theorem 27](#) with $\varphi(x, y) = x|y|^*$, $\varphi(x, y) = xy$ and $\varphi(x, y) = x/y$, respectively. □

Proposition 29. Let f, g be two real functions such that f is measurable and $f \stackrel{\text{a.e.}}{=} g$. Then, g is also measurable.

Sketch of the proof. Let $N := \{f \neq g\}$ which is null. Moreover, note that $\forall r \in \mathbb{R}$:

$$\{g > r\} = \{f > r\} \cup (\{g > r\} \cap N)$$

which is measurable. □

Proposition 30. Let (f_m) be a sequence of measurable real functions. Then, the following functions are measurable:

- $\sup\{f_m : m \in \mathbb{N}\}$
- $\inf\{f_m : m \in \mathbb{N}\}$
- $\limsup_{m \rightarrow \infty} f_m$
- $\liminf_{m \rightarrow \infty} f_m$

Furthermore, any function being pointwise limit a.e. of a sequence of measurable functions is measurable.

Sketch of the proof. Use the following identities for each case:

- $\{\sup\{f_m : m \in \mathbb{N}\} > r\} = \bigcup_{n=1}^{\infty} \{f_n > r\}$
- $\inf\{f_m : m \in \mathbb{N}\} = -\sup\{-f_m : m \in \mathbb{N}\}$
- $\limsup_{m \rightarrow \infty} f_m = \inf\{\sup\{f_k : k \geq n\} : n \in \mathbb{N}\}$
- $\liminf_{m \rightarrow \infty} f_m = \sup\{\inf\{f_k : k \geq n\} : n \in \mathbb{N}\}$

For the last property, if $f(x) = \lim_{m \rightarrow \infty} f_m(x) \forall x \in \mathbb{R}^n \setminus N$ with f_m measurable functions and $|N| = 0$, then $f_m \mathbf{1}_{\mathbb{R}^n \setminus N}$ are measurable, $f \mathbf{1}_{\mathbb{R}^n \setminus N} = \lim_{m \rightarrow \infty} f_m \mathbf{1}_{\mathbb{R}^n \setminus N}$ and $f \stackrel{\text{a.e.}}{=} f \mathbf{1}_{\mathbb{R}^n \setminus N}$. □

Definition 31. The Borel σ -algebra over $\mathbb{R}^n$, $\mathcal{B}(\mathbb{R}^n)$, is the smallest σ -algebra that contains the open sets of $\mathbb{R}^n$.

Lemma 32. We have that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mathbb{R}^n)$.

Definition 33. A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable if $\{x \in \mathbb{R} : g(x) > r\} \in \mathcal{B}(\mathbb{R}^n) \forall r \in \mathbb{R}$.

Proposition 34. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lebesgue measurable function and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. Then, $g \circ f$ is Lebesgue measurable.

Sketch of the proof. The set

$$\Sigma_f = \{A \in \mathcal{M}(\mathbb{R}^n) : f^{-1}(A) \text{ is measurable}\}$$

is a σ -algebra that contain the open sets. Keeping this in mind, for any open set $U \subseteq \mathbb{R}$ we have that:

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{M}(\mathbb{R}^n)$$

because $g^{-1}(U) \in \mathcal{B}(\mathbb{R}) \subseteq \Sigma_f$. $\square$

Definition 35. Let f be a measurable function. We define the following measurable functions:

$$f^+ := \sup\{f, 0\} \quad f^- := \sup\{-f, 0\}$$

Note that then, $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Definition 36. A simple function is a linear combination

$$s := \sum_{k=1}^N \alpha_k \mathbf{1}_{A_k}$$

where $\alpha_k \in \mathbb{R}$ and $A_k \in \mathcal{M}(\mathbb{R}^n)$ for $k = 1, \dots, N$ ³.

Theorem 37. Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be a measurable function and $\forall k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ let:

$$E(k, m) := \left\{ \frac{k-1}{2^m} \leq f < \frac{k}{2^m} \right\} \quad \text{and} \quad F(m) := \{f \geq m\}$$

Then, the sequence of positive simple functions

$$s_m = m \mathbf{1}_{F(m)} + \sum_{k=1}^{m2^m} \frac{k-1}{2^m} \mathbf{1}_{E(k, m)}$$

is increasing and $\lim_{m \rightarrow \infty} s_m(x) = f(x) \forall x \in \mathbb{R}^n$.

Sketch of the proof. The sets $E(k, m)$ and $F(m)$ are measurable and satisfy:

$$E(k, m) = E(2k-1, m+1) \sqcup E(2k, m+1)$$

and:

$$F(m) = \left(\bigsqcup_{h=m2^{m+1}+1}^{(m+1)2^{m+1}} E(h, m+1) \right) \sqcup F(m+1)$$

The proof of $s_m(x) \leq s_{m+1}(x)$ follows from distinguish the following three cases: $x \in F(m+1)$, $x \in F(m) \setminus F(m+1)$ and $x \in E(k, m)$ for some $k \in \mathbb{N}$.

Now given $x \in \{f \neq \infty\}$, we have that $x \in E(k, m)$ for some $k \in \mathbb{N}$. Thus:

$$\frac{k-1}{2^m} \leq f(x) < \frac{k}{2^m} \implies 0 \leq f(x) - s_m(x) < \frac{1}{2^m}$$

If $x \in \{f = \infty\}$, then $\lim_{m \rightarrow \infty} s_m(x) = m = \infty = f(x)$. $\square$

³We may suppose that the sets A_k are pairwise disjoint, the quantities α_k are all different and that $A_k = s^{-\alpha_k}$.

Theorem 38. Let $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ be a measurable function. Then, there exists a sequence of simple functions (s_m) such that $\lim_{m \rightarrow \infty} s_m(x) = f(x) \forall x \in \mathbb{R}^n$ and $|s_m| \leq |s_{m+1}| \leq |f| \forall m \in \mathbb{N}$.

Sketch of the proof. Apply Theorem 37 to the functions f^+ and f^- and use that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. $\square$

Lebesgue integral

Definition 39. Let $N \in \mathbb{N}$, $E_1, \dots, E_N$ be disjoint measurable sets and $s = \sum_{k=1}^N \alpha_k \mathbf{1}_{E_k}$ be a positive simple function such that $0 \leq \alpha_1 < \dots < \alpha_N < \infty$. We define the integral of s over $\mathbb{R}^n$ as:

$$\int s := \sum_{k=1}^N \alpha_k |E_k|$$

We define the integral of s over a measurable set E as:

$$\int_E s := \int s \mathbf{1}_E = \sum_{k=1}^N \alpha_k |E_k \cap E|$$

Proposition 40. Let (E_n) be a sequence of measurable sets and s, t be simple functions. Then:

1. If $E = \bigsqcup_{n=1}^{\infty} E_n$, then $\int_E s = \sum_{n=1}^{\infty} \int_{E_n} s$.
2. $\int (s + t) = \int s + \int t$.
3. If $\lambda \in \mathbb{R}_{\geq 0}$, then $\int \lambda s = \lambda \int s$.
4. If $s \leq t$, then $\int s \leq \int t$.

Sketch of the proof. Note that we can suppose $s = \sum_{k=1}^N \alpha_k \mathbf{1}_{F_k}$ and $t = \sum_{k=1}^N \beta_k \mathbf{1}_{F_k}$ with $\alpha_k, \beta_k \geq 0$.

1.

$$\begin{aligned} \int_E s &= \sum_{k=1}^N \alpha_k \left| F_k \cap \bigsqcup_{n=1}^{\infty} E_n \right| = \sum_{k=1}^N \alpha_k \left| \bigsqcup_{n=1}^{\infty} (F_k \cap E_n) \right| \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^N \alpha_k |F_k \cap E_n| = \sum_{n=1}^{\infty} \int_{E_n} s \end{aligned}$$

The other proofs are straightforward by considering $t \pm s = \sum_{k=1}^N (\beta_k \pm \alpha_k) \mathbf{1}_{F_k}$ and $\lambda s = \sum_{k=1}^N \lambda \alpha_k \mathbf{1}_{F_k}$. $\square$

Proposition 41. Given a simple function s , the function

$$\begin{aligned} \mu_s : \mathcal{M}(\mathbb{R}^n) &\longrightarrow [0, \infty] \\ E &\longmapsto \int_E s \end{aligned}$$

is a measure.

Sketch of the proof. Clearly $\mu_s(\emptyset) = 0$ and the property of σ -additivity is exactly [Item 40-1](#). $\square$

Definition 42. Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be a measurable function. We define:

$$\mathcal{S}(f) := \{s : s \text{ is a simple function such that } 0 \leq s \leq f\}$$

Definition 43. Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be a measurable function. We define the *integral of f over $\mathbb{R}^n$* as:

$$\int_{\mathbb{R}^n} f(x) dx := \sup_{s \in \mathcal{S}(f)} \int s$$

We define the *integral of f over a measurable set $E \subseteq \mathbb{R}^n$* as:

$$\int_E f(x) dx := \int_{\mathbb{R}^n} f(x) \mathbf{1}_E(x) dx = \sup_{s \in \mathcal{S}(f \mathbf{1}_E)} \int s$$

Proposition 44. Let $E \subseteq \mathbb{R}^n$ be a measurable set, s be a simple function and f, g be measurable functions such that $f(x) \leq g(x) \forall x \in E$. Then:

1. $\int_E s = \int_E s(x) dx$
2. $\int_E f(x) dx \leq \int_E g(x) dx$

Sketch of the proof. The first property is clear. Regarding the second one, just note that:

$$\mathcal{S}(f \mathbf{1}_E) \subseteq \mathcal{S}(g \mathbf{1}_E) \implies \sup_{s \in \mathcal{S}(f \mathbf{1}_E)} \int s \leq \sup_{s \in \mathcal{S}(g \mathbf{1}_E)} \int s$$

$\square$

Theorem 45 (Monotone convergence theorem). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $f \geq 0$ be a non-negative measurable function such that $\exists (f_m) \geq 0$ with f_m measurable $\forall m \in \mathbb{N}$ and $f_m \nearrow f$. Then:

$$\int_E f(x) dx = \lim_{m \rightarrow \infty} \int_E f_m(x) dx$$

Proof. The inequality $\int_E f_m(x) dx \leq \int_E f(x) dx$ is obvious. We need to prove the other one. To do so it suffices to show that $\forall \varepsilon > 0$ and $\forall s \in \mathcal{S}(f \mathbf{1}_E)$ we have $(1 - \varepsilon) \int_E s \leq \lim_{m \rightarrow \infty} \int_E f_m(x) dx$. Let $E_m := \{f_m \geq (1 - \varepsilon)s\}$. Note that $E_m \nearrow E$ and moreover:

$$\int_E f_m \geq \int_{E_m} f_m \geq (1 - \varepsilon) \int_{E_m} s$$

Since μ_s is a measure we can use [Item 5-3](#) to conclude that $\int_{E_m} s \nearrow \int_E s$. Therefore, $\forall \varepsilon > 0$ we have:

$$(1 - \varepsilon) \int_E s \leq \lim_{m \rightarrow \infty} \int_E f_m$$

$\square$

Proposition 46. Let $E \subseteq \mathbb{R}^n$ be a measurable set with $|E| > 0$, $f, g, (f_m) \geq 0$ be non-negative measurable functions. Then:

1. $\int_E (f + g)(x) dx = \int_E f(x) dx + \int_E g(x) dx$
2. $\int_E \sum_{m=1}^{\infty} f_m(x) dx = \sum_{m=1}^{\infty} \int_E f_m(x) dx$
3. If (E_k) is a sequence of measurable sets such that $E = \bigsqcup_{m=1}^{\infty} E_m$, then:

$$\int_E f(x) dx = \sum_{m=1}^{\infty} \int_{E_m} f(x) dx$$

4. If $\alpha \in [0, \infty)$, then $\int_E \alpha f(x) dx = \alpha \int_E f(x) dx$.
5. $\int_E f(x) dx = 0 \iff f \stackrel{\text{a.e.}}{=} 0$ on E .
6. If $N \subset E$ is a null set, then $\int_E f(x) dx = \int_{E \setminus N} f(x) dx$.
7. If $\int_E f(x) dx < \infty$, then $f \stackrel{\text{a.e.}}{<} \infty$ on E .
8. If $h \in \mathbb{R}^n$, then $\int_{E-h} f(x+h) dx = \int_{-E} f(-x) dx = \int_E f(x) dx$

Sketch of the proof.

1. Use the [45 Monotone convergence theorem](#) to the sequences of simple functions $s_m \nearrow f$ and $t_m \nearrow g$ (that exists by [Theorem 37](#)).
2. Use induction from [Item 46-1](#) and the [45 Monotone convergence theorem](#) to the sequence $F_N = \sum_{m=1}^N f_m(x)$.
3. Apply [Item 46-2](#) to $f_m = f \mathbf{1}_{E_m}$.
4. Use the [45 Monotone convergence theorem](#) and this property for simple functions (as in [Item 46-1](#)).

5. The implication to the left is clear. For the other one, define $A_m = \{f > \frac{1}{m}\}$. We would like to see that $|\{f > 0\}| = \lim_{m \rightarrow \infty} |A_m| = 0$. But $A_m \subseteq A_{m+1}$ and they are clearly measurable. So by [Item 5-3](#) we have:

$$\begin{aligned} |\{f > 0\}| &= \lim_{m \rightarrow \infty} |A_m| = \lim_{m \rightarrow \infty} \int_{A_m} dx \\ &\leq \lim_{m \rightarrow \infty} m \int_{A_m} f dx = 0 \end{aligned}$$

Hence, the set $\{f > 0\}$ is null, i.e. $f \stackrel{\text{a.e.}}{=} 0$ on E .

6. Just note that

$$\int_E f = \int_{E \setminus N} f + \int_N f$$

and $\int_N f = 0$ because $\int_N s = 0 \forall s \in \mathcal{S}(\mathbf{1}_N)$ by the construction of the Lebesgue integral.

7. Apply [Item 5-3](#) to the sets $A_m = \{f \geq m\}$.
8. For indicator functions the statement is clear since if $f(x) = \mathbf{1}_E(x)$, then $f(x+h) = \mathbf{1}_{E-h}(x)$ and $f(-x) = \mathbf{1}_{-E}(x)$. Now extend this to positive simple functions and then to positive measurable functions.

□

Corollary 47. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $f \geq 0$ be a non-negative measurable function such that $\exists (f_m) \geq 0$ with $f_m \nearrow^{\text{a.e.}} f$. Then:

$$\int_E f(x) dx = \lim_{m \rightarrow \infty} \int_E f_m(x) dx$$

Sketch of the proof. Consequence of [Item 46-6](#) with the null set $N = \{\lim_{m \rightarrow \infty} f_m \neq f\}$. □

Theorem 48 (Chebyshev's inequality). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $f : E \rightarrow \mathbb{R}$ be a measurable function and $\alpha \in \mathbb{R}_{>0}$. Then:

$$|\{f \geq \alpha\}| \leq \frac{1}{\alpha} \int_E |f(x)| dx$$

Proof.

$$\begin{aligned} |\{f \geq \alpha\}| &= \int_{\{f \geq \alpha\}} dx \leq \frac{1}{\alpha} \int_{\{f \geq \alpha\}} |f(x)| dx \\ &\leq \frac{1}{\alpha} \int_E |f(x)| dx \end{aligned}$$

Lemma 49 (Fatou's lemma). Let $E \subseteq \mathbb{R}^n$ be a measurable set and $(f_m) \geq 0$ be a sequence of non-negative measurable functions over E . Then:

$$\int_E \liminf_{m \rightarrow \infty} f_m(x) dx \leq \liminf_{m \rightarrow \infty} \int_E f_m(x) dx$$

Sketch of the proof. Remember that $\liminf_{m \rightarrow \infty} f_m = \lim_{k \rightarrow \infty} \inf_{m \geq k} f_m$. Now apply [45 Monotone convergence theorem](#) to the increasing sequence of functions (F_k) defined by $F_k = \inf_{m \geq k} f_m$. Bear in mind that $F_k \leq f_k$. □

Definition 50. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $f : E \rightarrow [-\infty, +\infty]$ be a measurable function such that either $\int_E f^+(x) dx < \infty$ or $\int_E f^-(x) dx < \infty$. Then, we define the *integral of f over E* as:

$$\int_E f(x) dx := \int_E f^+(x) dx - \int_E f^-(x) dx$$

We say that f is an *integrable function over E* if

$$\|f\|_1 := \int_E |f(x)| dx < \infty$$

The set of such functions is denoted by $\mathcal{L}^1(E)$.

Proposition 51. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $f : E \rightarrow [-\infty, +\infty]$ be a measurable function. Then, the function $g = f \mathbf{1}_{|f| < \infty}$ is finite and $f \stackrel{\text{a.e.}}{=} g$.

Proposition 52. Let $E \subseteq \mathbb{R}^n$ be a measurable set. Then:

1. $\mathcal{L}^1(E)$ is a vector space over $\mathbb{R}$.
2. The integral

$$\begin{aligned} \int_E : \mathcal{L}^1(E) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_E f \end{aligned}$$

is a linear form.

3. If $f, g \in \mathcal{L}^1(E)$ are such that $f \leq^{\text{a.e.}} g$ on E , then $\int_E f \leq \int_E g$. Moreover:

$$\left| \int_E f(x) dx \right| \leq \int_E |f(x)| dx$$

4. If $f \in \mathcal{L}^1(E)$ and $E = E_1 \sqcup E_2$ with E_1, E_2 measurable, then

$$\int_{E_1 \sqcup E_2} f(x) dx = \int_{E_1} f(x) dx + \int_{E_2} f(x) dx$$

5. If $h \in \mathbb{R}^n$, $f \in \mathcal{L}^1(E)$, then:

$$\int_{E-h} f(x+h) dx = \int_{-E} f(-x) dx = \int_E f(x) dx$$

Theorem 53 (Dominated convergence theorem).

□ Let $E \subseteq \mathbb{R}^n$ be a measurable set, f be a measurable function over E such that $\exists (f_m)$ measurable with $f_m \xrightarrow{\text{a.e.}} f$ and $|f_m(x)| \leq g(x)$ on E with $g \in \mathcal{L}^1(E) \forall m \in \mathbb{N}$. Then, $f, f_m \in \mathcal{L}^1(E) \forall m \in \mathbb{N}$ and:

$$\int_E f(x) dx = \lim_{m \rightarrow \infty} \int_E f_m(x) dx$$

Proposition 54. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $f, g \in \mathcal{L}^1(E)$ and $\lambda \in \mathbb{R}$. Then:

1. $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$
2. $\|\lambda f\|_1 = |\lambda| \|f\|_1$
3. $\|f\|_1 = 0 \iff f \stackrel{\text{a.e.}}{=} 0$.

Definition 55. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $(f_m), f \in \mathcal{L}^1(E)$. We say that (f_m) converge in mean to f if $\lim_{m \rightarrow \infty} \|f_m - f\|_1 = 0$, or equivalently $\lim_{m \rightarrow \infty} f_m = f$ on $\mathcal{L}^1(E)$.

Theorem 56. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $(f_m) \in \mathcal{L}^1(E)$.

1. If $\sum_{m=1}^{\infty} \|f_m\|_1 < \infty$, $\exists f \in \mathcal{L}^1(E)$ such that $\sum_{m=1}^{\infty} f_m = f$ on $\mathcal{L}^1(E)$ and $\sum_{m=1}^{\infty} f_m(x) = f(x)$ converges absolutely $\forall x \in E \setminus N$, where N is a null set.
2. If $\lim_{m \rightarrow \infty} f_m = f$ on $\mathcal{L}^1(E)$, then there exists a subsequence (f_{m_k}) such that $\lim_{k \rightarrow \infty} f_{m_k} = f$ on $\mathcal{L}^1(E)$ and $\lim_{k \rightarrow \infty} f_{m_k}(x) = f(x) \forall x \in E \setminus N$, where N is a null set.

Proposition 57. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $f \in \mathcal{L}^1(E)$. Then, there exists a sequence of integrable simple functions (s_m) such that $\lim_{m \rightarrow \infty} s_m = f$ on $\mathcal{L}^1(E)$, $\lim_{m \rightarrow \infty} s_m(x) = f(x) \forall x \in E$ and $|s_m| \leq |s_{m+1}| \leq |f| \forall m \in \mathbb{N}$.

Integral calculus in one variable and Riemann integral

Definition 58. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, we say that $\int_a^b f(x) dx$ exists and it is finite if f is integrable on $(\min\{a, b\}, \max\{a, b\})$ ⁴.

Theorem 59 (Mean value theorem for integrals).

Let $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a positive integrable function over (a, b) and $g : (a, b) \rightarrow \mathbb{R}$ be a measurable and bounded function such that $\alpha \leq g(x) \leq \beta$ almost everywhere on (a, b) . Then, $\exists \gamma \in [\alpha, \beta]$ such that:

$$\int_a^b g(x)f(x) dx = \gamma \int_a^b f(x) dx$$

Moreover if g is continuous, $\exists \xi \in (a, b)$ such that:

$$\int_a^b g(x)f(x) dx = g(\xi) \int_a^b f(x) dx$$

In particular, taking $f = 1$, we get:

$$\int_a^b g(x) dx = g(\xi)(b - a)$$

⁴Note that if f is measurable, the integral always exists, but it may be $\pm\infty$.

Theorem 60 (Barrow's law). If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and derivable on (a, b) with bounded derivative, then $f' \in \mathcal{L}^1((a, b))$ and

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Theorem 61 (Fundamental theorem of calculus).

Let $f : (a, b) \rightarrow \mathbb{R}$ be an integrable function and continuous at $y \in (a, b)$. Then, $F(x) = \int_a^x f(s) ds$ is derivable at y and $F'(y) = f(y)$.

Theorem 62. Let $\varphi : [\alpha, \beta] \rightarrow [c, d]$ be a continuous function and derivable with bounded derivative on (α, β) . Let $a = \varphi(\alpha)$ and $b = \varphi(\beta)$. If $f : [c, d] \rightarrow \mathbb{R}$ is a continuous function, then $(f \circ \varphi)\varphi'$ is integrable on (α, β) and:

$$\int_c^d f(x) dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t) dt$$

Theorem 63 (Integration by parts). Let $F, G : [a, b] \rightarrow \mathbb{R}$ be the primitives of the two bounded functions $f, g : (a, b) \rightarrow \mathbb{R}$. Then, $Fg, fG \in \mathcal{L}^1(a, b)$ and:

$$\int_a^b F(x)g(x) dx = F(a)G(a) - F(b)G(b) - \int_a^b f(x)G(x) dx$$

Theorem 64. Let $f : [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Then, f is also Lebesgue integrable and both integrals coincide.

Functions defined by integrals

Theorem 65. Let $E \subseteq \mathbb{R}^n$ be a measurable set, $I \subseteq \mathbb{R}$ be an interval, $g \in \mathcal{L}^1(E)$ be such that $g \geq 0$ and $f(\cdot, t)$ be an integrable function $\forall t \in I$. We denote:

$$\Phi(t) := \int_E f(x, t) dx$$

1. If $f(x, \cdot)$ is continuous on t_0 almost everywhere on E and $|f(x, t)| \stackrel{\text{a.e.}}{\leq} g(x) \forall t \in I$, then Φ is continuous at t_0 .
2. If $f(x, \cdot)$ is derivable on t_0 almost everywhere on E and

$$\left| \frac{\partial f}{\partial t}(x, t) \right| \stackrel{\text{a.e.}}{\leq} g(x) \quad \forall t \in I$$

then the function $\frac{\partial f}{\partial t}(x, \cdot)$ is integrable on E and

$$\Phi'(t_0) = \int_E \frac{\partial f}{\partial t}(x, t_0) dx$$

Definition 66 (Hardy-Littlewood maximal function). Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and $B \subseteq \mathbb{R}^n$ be a ball. We define the Hardy-Littlewood maximal function as:

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

Theorem 67. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and $B \subseteq \mathbb{R}^n$ be a ball. Then:

1. Mf is measurable.
2. $Mf \stackrel{\text{a.e.}}{<} \infty$.
3. $|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq \frac{A}{\alpha} \int_{\mathbb{R}^n} |f(x)| dx$.

Theorem 68 (Lebesgue differentiation theorem). Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and $B \subseteq \mathbb{R}^n$ be a ball. Then:

$$\lim_{|B| \rightarrow 0} \frac{1}{|B|} \int_B f(y) dy \stackrel{\text{a.e.}}{=} f(x) \quad x \in B$$

Proposition 69. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and suppose $Mf \in \mathcal{L}^1(\mathbb{R}^n)$. Then, $f = 0$.

Fubini-Tonelli theorem

Definition 70. Let $E \subseteq \mathbb{R}^{p+q}$ and $y \in \mathbb{R}^q$. We define the *section* of E at y as:

$$E(y) := \{x \in \mathbb{R}^p : (x, y) \in E\}$$

Proposition 71. Let $E, F, E_k \subseteq \mathbb{R}^{p+q}$, $k \in \mathbb{N}$, and $y \in \mathbb{R}^q$. Then:

1. If $E = A \times B$, with $A \subseteq \mathbb{R}^p$ and $B \subseteq \mathbb{R}^q$, then $E(y) = A$ if $y \in B$ and $E(y) = \emptyset$ if $y \notin B$.
2. $E \cap F = \emptyset \implies E(y) \cap F(y) = \emptyset$.
- 3.

$$\text{i) } \left(\bigcap_{k=1}^{\infty} E_k \right) (y) = \bigcap_{k=1}^{\infty} E_k(y)$$

$$\text{ii) } \left(\bigcup_{k=1}^{\infty} E_k \right) (y) = \bigcup_{k=1}^{\infty} E_k(y)$$

$$\text{iii) } (E \setminus F)(y) = E(y) \setminus F(y)$$

4. If $E(y)$ is measurable, then:

$$|E(y)| = \int_{\mathbb{R}^p} \mathbf{1}_E(x, y) dx$$

In particular, if E is an interval $E = I_p \times I_q$, then:

$$|I(y)| = |I_p| \mathbf{1}_{I_q}(y)$$

Lemma 72. Let $E \subseteq \mathbb{R}^{p+q}$ be a measurable set. Then:

1. There exists a null set $N \subset \mathbb{R}^q$ such that $E(y)$ is measurable $\forall y \in \mathbb{R}^q \setminus N$ (that is $E(y)$ is measurable almost everywhere $\forall y \in \mathbb{R}^q$).
2. The function

$$\Phi(y) = \begin{cases} |E(y)| & \text{if } y \in \mathbb{R}^q \setminus N \\ 0 & \text{if } y \in N \end{cases}$$

is measurable and positive on $\mathbb{R}^q$.

$$3. |E| = \int_{\mathbb{R}^q} |E(y)| dy$$

Theorem 73 (Tonelli's theorem). Let $f : \mathbb{R}^{p+q} \rightarrow [0, \infty]$ be a non-negative measurable function. Then:

1. $f(\cdot, y)$ and $f(x, \cdot)$ are measurable almost everywhere $x \in \mathbb{R}^p, y \in \mathbb{R}^q$.
2. Let N_p and N_q be the respective null sets where the above functions aren't measurable. Then the functions

$$\Phi(y) = \begin{cases} \int_{\mathbb{R}^p} f(x, y) dx & \text{if } y \in \mathbb{R}^q \setminus N_q \\ 0 & \text{if } y \in N_q \end{cases}$$

$$\Psi(x) = \begin{cases} \int_{\mathbb{R}^q} f(x, y) dy & \text{if } x \in \mathbb{R}^p \setminus N_p \\ 0 & \text{if } x \in N_p \end{cases}$$

are measurable on $\mathbb{R}^q$ and $\mathbb{R}^p$, respectively.

- 3.

$$\int_{\mathbb{R}^q} \Phi(y) dy = \int_{\mathbb{R}^{p+q}} f(x, y) d(x, y) = \int_{\mathbb{R}^p} \Psi(x) dx$$

Corollary 74. Let $f : \mathbb{R}^{p+q} \rightarrow [0, \infty]$ be a non-negative measurable function. Then:

$$\begin{aligned} \int_{\mathbb{R}^{p+q}} f(x, y) d(x, y) &= \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^p} f(x, y) dx \right) dy \\ &= \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x, y) dy \right) dx \end{aligned}$$

These identities are sometimes written as:

$$\begin{aligned} \int_{\mathbb{R}^{p+q}} f(x, y) dx dy &= \int_{\mathbb{R}^q} dy \int_{\mathbb{R}^p} f(x, y) dx \\ &= \int_{\mathbb{R}^p} dx \int_{\mathbb{R}^q} f(x, y) dy \end{aligned}$$

Corollary 75. Let $f : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ be a measurable function. Then, f is integrable if and only if:

$$\int_{\mathbb{R}^q} dy \int_{\mathbb{R}^p} |f(x, y)| dx < \infty$$

Theorem 76 (Fubini's theorem). Let $f \in \mathcal{L}^1(\mathbb{R}^{p+q})$. Then:

1. $f(\cdot, y) \stackrel{\text{a.e.}}{\in} \mathcal{L}^1(\mathbb{R}^p)$ and $f(x, \cdot) \stackrel{\text{a.e.}}{\in} \mathcal{L}^1(\mathbb{R}^q)$, $x \in \mathbb{R}^p, y \in \mathbb{R}^q$.
2. Let N_p and N_q be the respective null sets where the above functions aren't integrable. Then the functions

$$\Phi(y) = \begin{cases} \int_{\mathbb{R}^p} f(x, y) dx & \text{if } y \in \mathbb{R}^q \setminus N_q \\ 0 & \text{if } y \in N_q \end{cases}$$

$$\Psi(x) = \begin{cases} \int_{\mathbb{R}^q} f(x, y) dy & \text{if } x \in \mathbb{R}^p \setminus N_p \\ 0 & \text{if } x \in N_p \end{cases}$$

are integrable on $\mathbb{R}^q$ and $\mathbb{R}^p$, respectively.

3.

$$\int_{\mathbb{R}^q} \Phi(y) dy = \int_{\mathbb{R}^{p+q}} f(x, y) d(x, y) = \int_{\mathbb{R}^p} \Psi(x) dx$$

Change of variables

Definition 77. Let $U, V \subseteq \mathbb{R}^n$ be open sets. A *change of variables* is a diffeomorphism $\varphi : U \rightarrow V$ of class $\mathcal{C}^1$.

Theorem 78 (Change of variables). Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\varphi : U \rightarrow V$ be a change of variables. If $f : \mathbb{R}^n \rightarrow [0, \infty]$ is measurable or integrable on V , then so is $(f \circ \varphi)|J\varphi|$ and:

$$\int_V f(x) dx = \int_U f(\varphi(t))|J\varphi(t)| dt$$

2. | Banach spaces

Normed vector spaces

Definition 79. Let M be a set. A *distance* in M is a function $d : M \times M \rightarrow \mathbb{R}$ such that $\forall x, y, z \in M$ the following properties are satisfied:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ (*triangular inequality*)

We define a *metric space* as a pair (M, d) that satisfy the previous properties.

Proposition 80. Let $(M_1, d_1), \dots, (M_n, d_n)$ be metric spaces. Then, $M_1 \times \dots \times M_n$ with the distance

$$d(x, y) = \max\{d_i(x_i, y_i) : i = 1, \dots, n\}$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$, is a metric space.

Definition 81. A metric space (M, d) is *complete* if every Cauchy sequence in M converges in M .

Definition 82. Let E be a real (or complex) vector space. A *norm* on E is a function $\|\cdot\| : E \rightarrow \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}, \mathbb{C}$) such that $\forall \mathbf{u}, \mathbf{v} \in E$ and $\forall \lambda \in \mathbb{K}$ the following properties are satisfied:

1. $\|\mathbf{u}\| = 0 \iff \mathbf{u} = 0$
2. $\|\lambda \mathbf{u}\| = |\lambda| \|\mathbf{u}\|$
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (*triangular inequality*)

We define a *normed vector space* as a pair $(E, \|\cdot\|)$ that satisfy the previous properties.

Proposition 83. Let $(E_1, \|\cdot\|_1), \dots, (E_n, \|\cdot\|_n)$ be normed vector spaces. Then, $E_1 \times \dots \times E_n$ with the norm

$$\|(x_1, \dots, x_n)\| = \max\{\|x_i\|_i : i = 1, \dots, n\}$$

is a normed vector space.

Proposition 84. Let $(E, \|\cdot\|)$ be a normed vector space and consider the following functions:

$$\begin{aligned} S : E \times E &\longrightarrow E & P : \mathbb{R} \times E &\longrightarrow E \\ (x, y) &\longmapsto x + y & (\lambda, x) &\longmapsto \lambda x \end{aligned}$$

Then:

1. S is uniformly continuous.
2. P is continuous.
3. $\|\cdot\| : E \rightarrow \mathbb{R}$ is Lipschitz continuous:

$$|\|x\| - \|y\|| \leq \|x - y\| \quad \forall x, y \in E$$

Sketch of the proof.

- 1, 2. It follows from the [Triangular inequality](#).

3. Consider the inequalities:

$$\|x\| \leq \|x - y\| + \|y\| \quad \|y\| \leq \|y - x\| + \|x\|$$

□

Definition 85. Let $(E, \|\cdot\|)$ be a normed vector space and $(x_n) \in E$ be a sequence. We say that $\sum_{n=1}^{\infty} x_n$ is a *convergent series* in E that converges to $x \in E$ if:

$$\lim_{N \rightarrow \infty} \left\| x - \sum_{n=1}^N x_n \right\| = 0$$

We say that $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent* if:

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

Proposition 86. Let $(E, \|\cdot\|)$ be a normed vector space and $\sum_{n=1}^{\infty} x_n = x$ be a convergent series. Then:

$$\|x\| \leq \sum_{n=1}^{\infty} \|x_n\|$$

Proof.

$$\|x\| = \lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N x_n \right\| \leq \lim_{N \rightarrow \infty} \sum_{n=1}^N \|x_n\| = \sum_{n=1}^{\infty} \|x_n\|$$

□

Definition 87 (Banach space). A *Banach space* is a normed vector space which is complete with the distance associated with the norm.

Theorem 88. Let $(E, \|\cdot\|)$ be a normed vector space. Then, $(E, \|\cdot\|)$ is a Banach space if and only if every series in E that converges absolutely, converges.

Proof.

$\implies$) Let $\sum_{n=1}^{\infty} x_n$ be a sequence in E that converges absolutely. Then, $\forall \varepsilon > 0$ and $\forall M > N > 0$ large enough we have:

$$\left\| \sum_{n=1}^M x_n - \sum_{n=1}^N x_n \right\| \leq \sum_{n=N+1}^M \|x_n\| \leq \sum_{n=N+1}^{\infty} \|x_n\| < \varepsilon$$

because it is the tail of a convergent series. Thus, $\sum_{n=1}^{\infty} x_n < \infty$ because it is Cauchy and E is Banach.

$\Leftarrow$) Let $(y_n) \in E$ be a Cauchy sequence. Then, $\forall k \in \mathbb{N}, \exists n_k$ such that $\forall p, q \geq n_k$ we have:

$$\|y_p - y_q\| < \frac{1}{2^k}$$

Now consider the sequence $(x_m) \in E$ defined as $x_1 = y_{n_1}$ and $x_m = y_{n_m} - y_{n_{m-1}} \forall m \geq 2$. Therefore, $\|x_m\| \leq \frac{1}{2^{m-1}} \forall m \geq 2$ and so the series $\sum_{m=1}^{\infty} \|x_m\|$ converges, and so does the series $\sum_{m=1}^{\infty} x_m =: x \in E$. Since, $\sum_{m=1}^M x_m = y_{n_M}$ we have that $\lim_{M \rightarrow \infty} y_{n_M} = x$. Finally, as (y_{n_M}) is a subsequence of (y_n) , which is Cauchy, we have that $\lim_{n \rightarrow \infty} y_n = x$.

□

Proposition 89. The space $\mathcal{B}([a, b])$ of continuous and bounded functions with the *uniform norm*

$$\|f\|_{\infty} := \sup\{|f(t)| : t \in [a, b]\}$$

is Banach.

Proof. Let $(f_n) \in \mathcal{B}([a, b])$ be Cauchy. Then, for each $x \in [a, b]$, the numeric sequence $(f_n(x))$ is Cauchy and so it converges pointwise to a function $f(x)$. Thus, we have:

$$\lim_{n, m \rightarrow \infty} |f_n(x) - f_m(x)| = \lim_{n \rightarrow \infty} |f_n(x) - f(x)|$$

which is valid $\forall x \in [a, b]$. Hence, $\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty}$ and clearly $f \in \mathcal{B}([a, b])$, as it is the uniform limit of continuous and bounded functions. □

Proposition 90. Let $(E, \|\cdot\|)$ be a normed vector space and $F \subseteq E$ be a vector subspace. Then, $\overline{F}$ is also a vector subspace of E .

Sketch of the proof. Let $x, y \in \overline{F}$ and $\lambda \in \mathbb{K}$. Then, $\exists (x_n), (y_n) \in F$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. The continuity of the functions defined in Theorem 84 implies $x + y \in \overline{F}$ and $\lambda x \in \overline{F}$. □

Proposition 91. Let $(E, \|\cdot\|)$ be a normed vector space and $F \subseteq E$ be a vector subspace.

1. If F is complete, it is closed.
2. If F is closed and E is Banach, then F is complete.

Proof.

1. Let $x \in \overline{F}$. Then, $\exists (x_n) \in F$ such that $\lim_{n \rightarrow \infty} x_n = x$. In particular (x_n) is Cauchy and since F is complete, we conclude $x \in F$.
2. Let $(x_n) \in F$ be a Cauchy sequence. In particular, it is a Cauchy sequence in E and so $\exists x \in E$ with $\lim_{n \rightarrow \infty} x_n = x$. Moreover, this x satisfies $x \in \overline{F}$ and since F is closed, $x \in F$.

□

Definition 92. Let $(E, \|\cdot\|)$ be a normed vector space and $A \subseteq E$ be a subset. We say that A is a *total subset* if $\langle A \rangle$ is dense in E .

Remark. The linear span $\langle A \rangle$ exclude the infinite linear combinations of elements in A , even if A is a subspace of infinite dimension.

Definition 93. A metric space is called *separable* if it contains a countable dense subset.

Proposition 94. A normed vector space E is separable if and only if it contains a total countable subset.

Proof. The implication to the right is evident. For the other one, let A be the total countable subset and consider the set $\langle A \rangle_{\mathbb{Q}}$ of rational linear combinations. This set is countable and dense in $\langle A \rangle$. Since the latter one is also dense in E , we conclude that $\langle A \rangle_{\mathbb{Q}}$ is dense in E . □

Proposition 95. Let M be a separable metric space and $S \subseteq M$ be a subspace. Then, S is separable.

Proof. Let A be the countable dense subset of M . The balls $B(y, 1/n)$ with $y \in A$, $n \in \mathbb{N}$ cover S . For each $y \in N$, let A_n be the set formed by one point of each nonempty $B(y, 1/n) \cap S$. The set $\bigcup_{n=1}^{\infty} A_n$ is dense in S . □

Proposition 96. Let M, N be separable metric spaces. Then, $M \times N$ is separable.

Sketch of the proof. The product of two dense subsets is dense with the product topology. □

Quotient space

Definition 97. Let E be a normed vector space and F be a closed subspace. We define the *quotient space* E/F as the respective quotient vector space with the associated norm:

$$\begin{aligned} \|\bar{x}\|_{E/F} &:= \inf\{\|z\|_E : z \in E \text{ with } \bar{z} = \bar{x}\} \\ &= \inf\{\|x + y\|_E : y \in F\} \end{aligned}$$

Proposition 98. Let E be a normed vector space and F be a closed subspace. The quotient space E/F is a normed vector space. Moreover, the map

$$\begin{aligned} \pi : E &\longrightarrow E/F \\ x &\longmapsto \bar{x} \end{aligned}$$

is continuous with $\|\bar{x}\|_{E/F} \leq \|x\|_E$.

Sketch of the proof. Note that $\bar{x} = x + F$ and so

$$\|\bar{x}\|_{E/F} = d(0, x + F) = 0 \iff 0 \in x + F \iff \bar{x} = \bar{0}$$

The inequality follows from the fact that $\|\bar{y}\|_{E/F} = 0 \forall y \in F$. From here, the continuity is clear. □

Lemma 99. Let E be a normed vector space and F be a closed subspace.

1. If E is Banach, so is E/F .
2. If E is separable, so is E/F .

Sketch of the proof.

1. Let $\sum_{n=1}^{\infty} \|\bar{x}_n\|_{E/F} < \infty$ and for each $n \in \mathbb{N}$ take $x_n \in \bar{x}_n$ with $\|x_n\| \leq 2\|\bar{x}_n\|_{E/F}$. Then, $\sum_{n=1}^{\infty} \|x_n\| < \infty$ and since E is Banach, $\sum_{n=1}^{\infty} x_n = x \in E$. The continuity and linearity of π implies $\sum_{n=1}^{\infty} \bar{x}_n = \bar{x}$. Hence, E/F is Banach.
2. Note that the density and countability of sets are invariant under π .

□

L^p spaces

Definition 100. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $1 \leq p < \infty$. We define:

$$\mathcal{L}^p(E) := \left\{ f : E \rightarrow \mathbb{R}^n \text{ measurable} : \int_E |f|^p < \infty \right\}$$

$$\mathcal{L}^\infty(E) := \{ f : E \rightarrow \mathbb{R}^n \text{ measurable} : \exists M > 0 \text{ with } |f(x)| \stackrel{\text{a.e.}}{\leq} M, x \in E \}$$

$$\mathcal{N}(E) := \{ f : E \rightarrow \mathbb{R}^n \text{ measurable} : f \stackrel{\text{a.e.}}{=} 0 \}$$

Definition 101. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $1 \leq p \leq \infty$. We define:

$$L^p(E) := \mathcal{L}^p(E) / \mathcal{N}(E)$$

Lemma 102 (Young's inequality for products). Let $a, b > 0$ and $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

And the equality holds if and only if $a^p = b^q$.

Sketch of the proof. Let $f : [0, a] \rightarrow \mathbb{R}$ be continuous and invertible such that $f(0) = 0$. Then, we have

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(x) dx$$

with equality if and only if $f(a) = b$ (see Fig. 1 for a better understanding). Now take $f(x) = x^{p-1}$ and so $f^{-1}(x) = x^{q-1}$. The equality is held if:

$$a^{p-1} = b \text{ and } b^{q-1} = a \iff a^p = b^q$$

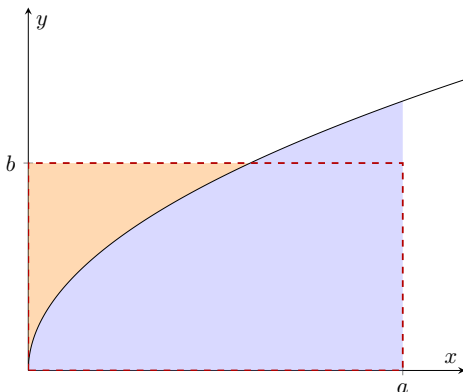


Figure 1

Definition 103. We say that $p, q \geq 1$ are *Hölder conjugates* if $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 104 (Hölder's inequality). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(E)$, $g \in L^q(E)$. Then:

$$\int_E |fg| \leq \left(\int_E |f|^p \right)^{1/p} \left(\int_E |g|^q \right)^{1/q}$$

Or equivalently (see Theorem 107):

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

And the equality holds if and only if $\exists \alpha, \beta \in \mathbb{R}_{\geq 0}$ such that $\alpha|f|^p \stackrel{\text{a.e.}}{=} \beta|g|^q$.

Proof. Dividing f by $\|f\|_p$ and g by $\|g\|_q$ we can assume that $\|f\|_p = \|g\|_q = 1$. Then, using 102 Young's inequality for products we have:

$$\int_E |f(x)g(x)| dx \leq \int_E \left(\frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \right) dx = \frac{1}{p} + \frac{1}{q} = 1$$

The equality follows from the equality in 102 Young's inequality for products. □

Corollary 105 (Hölder's inequality). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $f \in L^p(E)$, $g \in L^q(E)$. Then, $fg \in L^r(E)$ and:

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

Sketch of the proof. Use 104 Hölder's inequality with $F := |f|^r \in L^{\frac{p}{r}}(E)$ and $G := |g|^r \in L^{\frac{q}{r}}(E)$, noting that p/r and q/r are Hölder conjugates. □

Corollary 106 (Interpolation inequality). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p_1 \leq p_2 \leq \infty$ and $f \in L^{p_1}(E) \cap L^{p_2}(E)$. Then, $\forall p \in [p_1, p_2]$ we have $f \in L^p(E)$ and:

$$\|f\|_p \leq \|f\|_{p_1}^\alpha \|f\|_{p_2}^{1-\alpha}$$

with $\alpha \in [0, 1]$ such that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$.

Proposition 107. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $1 \leq p < \infty$. The set $L^p(E)$ is a normed vector space with the norm:

$$\|f\|_p := \left(\int_E |f|^p \right)^{1/p} \quad \forall f \in L^p(E)$$

And the set $L^\infty(E)$ is also a normed vector space with the norm:

$$\|f\|_\infty = \inf \{ M : |f(x)| \stackrel{\text{a.e.}}{\leq} M, x \in E \} \quad \forall f \in L^\infty(E)$$

Sketch of the proof. The case of $L^\infty(E)$ is easy and the first two properties for $L^p(E)$, $p \geq 1$, too (remember [Item 46-5](#)). It's missing to prove the [Triangular inequality](#) (also called *Minkowski inequality* in this case):

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

We have that:

$$\begin{aligned} \|f + g\|_p^p &= \int_E |f + g|^p \\ &\leq \int_E |f| |f + g|^{p-1} + \int_E |g| |f + g|^{p-1} \\ &\leq \left[\left(\int_E |f|^p \right)^{1/p} + \left(\int_E |g|^p \right)^{1/p} \right] \cdot \left(\int_E |f + g|^{(p-1) \frac{p}{p-1}} \right)^{1-1/p} \\ &= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p} \end{aligned}$$

Other important Banach spaces

Definition 108. Let I be an index set. We denote by $c_0(I)$ the space of all sequences convergent to 0; by $c(I)$, the space of all convergent sequences, and by $\ell^\infty(I)$, the space of all bounded sequences.

Proposition 109. Let I be an index set. The spaces $c_0(I)$, $c(I)$ and $\ell^\infty(I)$ with the *uniform norm*

$$\|(x_n)\|_\infty := \sup\{|x_n| : n \in \mathbb{N}\}$$

are Banach.

Definition 110. Let $1 \leq p < \infty$ and I be an index set. We define the space $\ell^p(I)$ as the space of the sequences $x = (x_n)$ such that:

$$\|x\|_p := \left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} < \infty$$

Proposition 111. Let I be an index set. The space $(\ell^p(I), \|\cdot\|_p)$ is Banach.

Space of continuous functions

Definition 112. Let $X \neq \emptyset$ be a set. We define the set $\mathcal{B}(X)$ as the vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of the functions $f : X \rightarrow \mathbb{K}$ that are bounded with the *uniform norm* (or *supremum norm*):

$$\|f\| := \|f\|_X := \sup\{|f(x)| : x \in X\}$$

Proposition 113. Let $X \neq \emptyset$ be a set and $(f_n), f \in \mathcal{B}(X)$ be functions. Then:

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0 \iff f_n \text{ converges uniformly to } f$$

⁵Note that it suffices for A to contain the constant functions so that it vanishes nowhere.

Sketch of the proof. Remember the characterization ??.

Definition 114. Let $K \subseteq \mathbb{K}^n$ be a compact set. We define $\mathcal{C}(K)$ as the closed subspace of $\mathcal{B}(K)$ containing the continuous functions.

Proposition 115. Let $K \subseteq \mathbb{K}^n$ be a compact set and $f, g \in \mathcal{C}(K)$. Then:

$$\|fg\|_K \leq \|f\|_K \|g\|_K$$

Definition 116. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subset. We say that A is a *subalgebra* if A is a vector subspace, and it is stable under the product, that is if $\forall f, g \in A$ we have $fg \in A$.

Proposition 117. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. Then, $\overline{A}$ is also a subalgebra.

Proof. In [Theorem 90](#) we saw that $\overline{A}$ is a vector subspace. To show that is stable under the product, consider $f, g \in \overline{A}$. Then, we can write $f = \lim_{n \rightarrow \infty} f_n$ and $g = \lim_{n \rightarrow \infty} g_n$ with $(f_n), (g_n) \in A$. Finally:

$$\begin{aligned} \|fg - f_n g_n\|_K &\leq \|fg - f_n g\|_K + \|f_n g - f_n g_n\|_K \\ &\leq \|g\|_K \|f - f_n\|_K + \|f_n\|_K \|g - g_n\|_K \end{aligned}$$

which has limit 0 when $n \rightarrow \infty$.

Definition 118. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. We say that A is a *separating set* (or *separate the points* of K) if $\forall x, y \in K \exists f \in A$ such that $f(x) \neq f(y)$.

Definition 119. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. We say that A *vanishes nowhere* if $\forall x \in K \exists f_x \in A$ such that $f_x(x) \neq 0$ ⁵.

Lemma 120. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a separating subalgebra that vanishes nowhere. Then, $\forall x, y \in K$ and $\forall \alpha, \beta \in \mathbb{K}$, $\exists f \in A$ such that $f(x) = \alpha$ and $f(y) = \beta$.

Proof. By hypothesis $\exists g, h_x, h_y \in A$ such that $g(x) \neq g(y)$, $h_x(x) \neq 0$ and $h_y(y) \neq 0$. Then consider:

$$f = \alpha \frac{h_x}{h_x(x)} \frac{g - g(y)}{g(x) - g(y)} + \beta \frac{h_y}{h_y(y)} \frac{g - g(x)}{g(y) - g(x)}$$

Definition 121. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. We say that A is *self-conjugate* if $\overline{f} \in A$ whenever $f \in A$.

Lemma 122. Let $K \subseteq \mathbb{R}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. If $f \in A$, then $|f| \in \overline{A}$.

Proof. First note that since $f \in \mathcal{C}(K)$, f is bounded and so $\exists a, b \in \mathbb{R}$ such that $f(K) \subseteq [a, b]$. Consider $v(x) = |x|$, $x \in [a, b]$. Then, by ?? $\exists (q_n) \in \mathbb{R}[x]$ such that $\lim_{n \rightarrow \infty} \|q_n - v\|_{[a, b]} = 0$. Now construct $p_n(x) = q_n(x) - q_n(0)$ which also satisfies $\lim_{n \rightarrow \infty} \|p_n - v\|_{[a, b]} = 0$ because $\lim_{n \rightarrow \infty} q_n(0) = v(0) = 0$. Since we can write $p_n(x) = \sum_{k=1}^n a_k x^k$, we have that $p_n(f) = \sum_{k=1}^n a_k f^k \in A$ and so:

$$\lim_{n \rightarrow \infty} \|p_n(f) - |f|\|_K \leq \lim_{n \rightarrow \infty} \|p_n - v\|_{[a, b]} = 0$$

□

Corollary 123. Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a subalgebra. If $f, g \in A$, then $\sup\{f, g\}, \inf\{f, g\} \in \bar{A}$.

Sketch of the proof.

$$\sup\{f, g\} = \frac{f + g + |f - g|}{2} \quad \inf\{f, g\} = \frac{f + g - |f - g|}{2}$$

□

Lemma 124. Let $K \subseteq \mathbb{K}^n$ be a compact set, $A \subseteq \mathcal{C}(K)$ be a separating subalgebra that vanishes nowhere, $x \in K$ and $f \in \mathcal{C}(K)$. Then, $\forall \varepsilon > 0 \exists g_x \in \bar{A}$ such that $g_x(x) = f(x)$ and $g_x < f + \varepsilon$ in K .

Proof. By Theorem 120, $\forall y \in K \exists h_y \in \mathcal{C}(A)$ such that $h_y(y) = f(y)$ and $h_y(x) = f(x)$. By continuity there is a neighbourhood N_y of y such that $h_y < f + \varepsilon$. Now note that $K \subset \bigcup_{y \in K} N_y$ and the compactness implies that $K \subset \bigcup_{i=1}^m N_{y_i}$ for certain $y_i \in K$, $i = 1, \dots, m$. Finally, take $g_x := \inf\{h_{y_i} : i = 1, \dots, m\} \in \bar{A}$. □

Theorem 125 (Stone-Weierstraß theorem). Let $K \subseteq \mathbb{K}^n$ be a compact set and $A \subseteq \mathcal{C}(K)$ be a separating self-conjugate subalgebra that vanishes nowhere. Then, A is dense in $\mathcal{C}(K)$.

Proof. We distinguish between $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

$\mathbb{K} = \mathbb{R}$: Let $f \in \mathcal{C}(K)$. We should find $g \in A$ such that $\forall \varepsilon > 0, \|f - g\|_K < \varepsilon$. For each $x \in K$, let g_x be the function of Theorem 124 that satisfies $g_x < f + \varepsilon$. By continuity there is a neighbourhood N_x of x such that $g_x > f - \varepsilon$. The compactness of K implies $K \subset \bigcup_{i=1}^m N_{x_i}$ for certain $x_i \in K$, $i = 1, \dots, m$. Finally, take $g := \sup\{g_{x_i} : i = 1, \dots, m\} \in \bar{A}$ that satisfies $\|f - g\|_K < \varepsilon$.

$\mathbb{K} = \mathbb{C}$: Note that $A_0 := \{\operatorname{Re} f : f \in A\} = \{\operatorname{Im} f : f \in A\}$ because A is self-conjugate. Moreover, A_0 is a separating subalgebra that vanishes nowhere. By the case $\mathbb{K} = \mathbb{R}$ we know that exists sequences $(u_n), (v_n) \in A_0$ such that $\lim_{n \rightarrow \infty} \|\operatorname{Re} f - u_n\|_K = \lim_{n \rightarrow \infty} \|\operatorname{Im} f - v_n\|_K = 0$. And it suffices to consider $g_n := u_n + iv_n$ that converges uniformly to f . □

Definition 126. Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $F \subset \mathcal{C}(X, Y)$ be a subset. We say that F is *pointwise bounded* if $\forall x \in X \exists M_x > 0$ such that $|f(x)| \leq M_x \forall f \in F$. We say that F is *locally bounded* if $\forall x \in X$ there exist a neighborhood N_x of x and a constant $M_x > 0$ such that $|f(N_x)| \leq M_x \forall f \in F$. We say that F is *uniformly bounded* if $\exists M > 0$ such that $|f(x)| \leq M \forall x \in X$ and $\forall f \in F$.

Definition 127. Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $F \subset \mathcal{C}(X, Y)$ be a subset. We say that F is *equicontinuous at a point* $x_0 \in X$ if $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x \in X$ with $d_X(x, x_0) < \delta$ we have:

$$d_Y(f(x), f(x_0)) < \varepsilon \quad \forall f \in F$$

We say that F is *pointwise equicontinuous* if it is equicontinuous at each point of X . Finally, we say that F is *uniformly equicontinuous* if $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x, y \in X$ with $d(x, y) < \delta$ we have:

$$d_Y(f(x), f(y)) < \varepsilon \quad \forall f \in F$$

Definition 128. Let (X, d) be a metric space and $F \subseteq X$. We say that F is *relatively compact* on X if $\bar{F}$ is compact on X .

Theorem 129 (Arzelà-Ascoli theorem). Let (X, d) be a metric space, $K \subset X$ be a compact set and $F \subset \mathcal{C}(K)$ be a subset. Then, F is relatively compact in $\mathcal{C}(K)$ if and only if F is pointwise equicontinuous and pointwise bounded.

Definition 130. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $U \subseteq \mathbb{R}^n$ be an open set. We define:

$$\mathcal{S}(E) = \{f : E \rightarrow \mathbb{R} : f \text{ is simple}\}$$

$$\mathcal{C}_0(U) = \{f \in \mathcal{C}(U) : \operatorname{supp} f \text{ is compact}\}$$

Theorem 131. Let $E \subseteq \mathbb{R}^n$ be a measurable space, $(f_k) \in L^p(E)$ be a sequence of functions and $1 \leq p < \infty$. Then:

1. If $\lim_{k \rightarrow \infty} f_k(x) \stackrel{\text{a.e.}}{=} f(x)$ with $|f_k| \leq g \in L^p(E)$, then $f \in L^p(E)$ and $\lim_{k \rightarrow \infty} \|f_k - f\|_p = 0$, and we will write $f_k \xrightarrow{L^p} f$.
2. If $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$, then $\sum_{k=1}^{\infty} |f_k(x)| \stackrel{\text{a.e.}}{<} \infty$ and $\exists f \in L^p(E)$ such that $\sum_{k=1}^{\infty} f_k(x) \stackrel{\text{a.e.}}{=} f(x)$ and $\sum_{k=1}^N f_k \xrightarrow{L^p} f$. In particular, $(L^p, \|\cdot\|_p)$ is a Banach space.
3. If $f_k \xrightarrow{L^p} f$, then $\exists (f_{k_j})$ such that $\lim_{j \rightarrow \infty} f_{k_j}(x) \stackrel{\text{a.e.}}{=} f(x)$.

Sketch of the proof.

1. A direct application of the 53 Dominated convergence theorem shows that $f \in L^p(E)$. Moreover, $|f_k - f|^p \leq 2g^p \in L^1(E)$. So again the 53 Dominated convergence theorem allows us to conclude $f_k \xrightarrow{L^p} f$.

2. Let $g_N(x) := \sum_{k=1}^N |f_k(x)| \nearrow g(x)$ (possibly infinity). Then:

$$\begin{aligned} \|g_N\|_p &= \left(\int_E |g_N|^p \right)^{1/p} \leq \sum_{k=1}^N \|f_k\|_p \\ &\leq \sum_{k=1}^{\infty} \|f_k\|_p < \infty \end{aligned}$$

where we have used the [Triangular inequality](#). Thus, $\|g_N\|_p^p < \infty$ and by the [45 Monotone convergence theorem](#) we have $\|g\|_p^p < \infty$ which implies:

$$\sum_{k=1}^{\infty} |f_k(x)| \stackrel{\text{a.e.}}{<} \infty$$

Now use [Item 131-1](#) to show that $F_N \xrightarrow{L^p} f$, where $F_N = \sum_{k=1}^N f_k$.

3. The Cauchy condition for (f_k) implies that $\forall m \in \mathbb{N}$ $\exists k_m$ such that if $p, q > k_m$ then $\|f_p - f_q\|_p < \frac{1}{2^m}$. Now consider the partial sequence defined by the series of partial sums:

$$f_{k_1} + (f_{k_2} - f_{k_1}) + \cdots + (f_{k_{m+1}} - f_{k_m}) + \cdots$$

Now use [Item 131-2](#). □

Theorem 132. Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$ be a measurable set and $U \subseteq \mathbb{R}^n$ be an open set. Then:

1. $\mathcal{S}(E)$ is dense in $L^p(E)$.
2. $\mathcal{C}_0(U)$ is dense in $L^p(U)$.

Sketch of the proof.

1. Consequence of [Item 131-1](#) and [Theorem 37](#).
2. □

Theorem 133. Let $E \subseteq \mathbb{R}^n$ be a measurable space and $1 \leq p < \infty$. Then, $L^p(E)$ is separable.

Operators

Definition 134. Let E, F be normed vector spaces. An operator T is a linear function $T : E \rightarrow F$.

Definition 135. Let E, F be normed vector spaces and $T : E \rightarrow F$ be an operator. We define the *norm* of T as:

$$\|T\| := \sup\{\|Tx\|_F : \|x\|_E \leq 1\} \quad (1)$$

Lemma 136. Let E, F be normed vector spaces and $T : E \rightarrow F$ be an operator. Then:

$$\begin{aligned} \|T\| &= \sup\{\|Tx\|_F : \|x\|_E < 1\} \\ &= \sup\{\|Tx\|_F : \|x\|_E = 1\} \\ &= \sup\left\{\frac{\|Tx\|_F}{\|x\|_E} : x \neq 0\right\} \end{aligned}$$

Sketch of the proof. Note that due to the linearity of T we have

$$\|Tx\|_F = \lim_{\varepsilon \rightarrow 0} \|T((1-\varepsilon)x)\|_F \leq \sup_{\|x\|_E < 1} \|Tx\|_F$$

And this proves the first equality as the reverse inequality is clear. For the second one, use the latter one and the fact that:

$$\|Tx\|_F = \|x\|_E \left\| T\left(\frac{x}{\|x\|_E}\right) \right\|_F \leq \sup_{\|x\|_E = 1} \|Tx\|_F$$

The last one follows from applying the second equality to $\frac{x}{\|x\|_E}$. □

Theorem 137. Let E, F be normed vector spaces and $T : E \rightarrow F$ be an operator. The following are equivalent:

1. T is continuous at 0.
2. T is continuous.
3. $T(B_E)$ is bounded on F , where $B_E := \{x \in E : \|x\|_E \leq 1\}$.
4. $\|T\| < \infty$.
5. $\exists C \geq 0$ such that $\forall x \in E$ we have:

$$\|Tx\|_F \leq C \|x\|_E$$

If, moreover, T is continuous, $\|T\|$ is the least of such constants C .

Sketch of the proof.

- 1 $\implies$ 2: Let $x \in E$ and $(x_n) \in E$ such that $\lim_{n \rightarrow \infty} x_n = x$. Then $\lim_{n \rightarrow \infty} (x_n - x) = 0$ and the continuity and linearity imply $\lim_{n \rightarrow \infty} (Tx_n - Tx) = 0$.
- 2 $\implies$ 3: The continuity at the origin of T implies that given $\varepsilon = 1$, $\exists \delta > 0$ such that:

$$T(B_E(0, \delta)) \subseteq B_F(0, 1)$$

The linearity of T implies that $T(B_E(0, 1)) \subseteq B_F(0, 1/\delta)$.

- 3 $\implies$ 4: Consequence of [Theorem 136](#).
- 4 $\implies$ 5: By the definition of supremum we have:

$$\left\| T\left(\frac{x}{\|x\|_E}\right) \right\|_F \leq \|T\|$$

And so $\|Tx\|_F \leq \|T\| \|x\|_E$.

- 5 $\implies$ 1: Evident. □

Definition 138. Let T be an operator. We say that T is *sublinear* if:

$$\|T(f+g)\| \leq \|Tf\| + \|Tg\| \quad \forall f, g \in L^p$$

Theorem 139 (Marcinkiewicz interpolation theorem). Let T be a sublinear operator. Then:

1. $|\{x \in \mathbb{R}^n : |Tf(x)| > t\}| \leq \frac{A}{t} \|f\|_1$
2. $\|Tf\|_\infty \leq A_\infty \|f\|_\infty$

Corollary 140. Let T be a sublinear operator. Then, $\|Tf\|_p \leq A_p \|f\|_p \quad \forall 1 < p < \infty$ and $\forall f \in L^p$.

Finite dimensional normed vector spaces

Definition 141. A *topological homeomorphism* is a linear homeomorphism between any two normed vector spaces. If there exists such a homeomorphism we will say that the two normed vector spaces are *isomorphic*.

Definition 142. Let E be a normed vector space and $\|\cdot\|_1, \|\cdot\|_2$ be two norms on E . We say that $\|\cdot\|_1$ is *finer* than $\|\cdot\|_2$ if $\exists \alpha > 0$ such that:

$$\|x\|_1 \leq \alpha \|x\|_2 \quad \forall x \in E$$

Definition 143. Let E be a normed vector space and $\|\cdot\|_1, \|\cdot\|_2$ be two norms on E . We say that the norms $\|\cdot\|_1, \|\cdot\|_2$ are *equivalent* if $\exists \alpha, \beta > 0$ such that:

$$\alpha \|x\|_2 \leq \|x\|_1 \leq \beta \|x\|_2 \quad \forall x \in E$$

Theorem 144. Let E be a normed vector space over $\mathbb{K}$ of dimension $n \in \mathbb{N}$. Then, any algebraic isomorphism $T: \mathbb{K}^n \rightarrow E$ is a topological isomorphism.

Proof. We need to show that the Euclidean norm $\|\cdot\|$ and the norm $\|x\|' := \|Tx\|_E$ are equivalent. Let $(u_1, \dots, u_n)$ be a basis of $\mathbb{K}^n$ and suppose $x = \sum_{j=1}^n x_j u_j$. Then:

$$\|x\|' \leq \sum_{j=1}^n |x_j| \|u_j\| \leq \|x\| \sum_{j=1}^n \|u_j\| =: C \|x\|$$

To show the other inequality, consider the function $f(x) = \|x\|'$ defined on $B = \{x \in \mathbb{K}^n : \|x\| = 1\}$. Then, f is continuous and ?? ?? implies the existence of and absolute minimum $c \in B$ such that $f\left(\frac{x}{\|x\|}\right) \geq c$, i.e. $\|x\|' \geq c \|x\|$. $\square$

Corollary 145. In a finite-dimensional normed vector space any two norms are always equivalent.

Corollary 146. In a normed vector space any finite-dimensional subspace is complete and therefore closed.

Sketch of the proof. The topological isomorphisms preserve the completeness. $\square$

Corollary 147. Let E, F be normed vector spaces and $T: E \rightarrow F$ be an operator. If T is linear and $\dim E < \infty$, then T is continuous.

Sketch of the proof. Since T is linear, it is the composition of continuous functions $T: E \rightarrow \mathbb{K}^n \rightarrow \mathbb{K}^m \rightarrow F$. $\square$

Lemma 148 (Almost orthogonality lemma). Let E be a normed vector space and $F \subseteq E$ be a proper subspace of E . Then, $\forall \varepsilon > 0 \exists u \in E$ such that $\|u\| = 1$ and $d(u, F) \geq 1 - \varepsilon$.

Proof. We may suppose $\varepsilon < 1$. Let $v \in E$ such that $d(v, F) = \delta > 0$. Then, consider $u = \frac{v - x_0}{\|v - x_0\|}$, where $x_0 \in F$ satisfies $\delta \leq \|v - x_0\| \leq \frac{\delta}{1 - \varepsilon}$. Finally, $\forall x \in F$:

$$\|u - x\| = \frac{\|v - (x_0 + \|v - x_0\|x)\|}{\|v - x_0\|} \geq \frac{\delta}{\|v - x_0\|} \geq 1 - \varepsilon$$

Theorem 149 (Riesz's theorem). Let E be a normed vector space. If the unit closed sphere $\{x \in E : \|x\| = 1\}$ is compact, then $\dim E < \infty$.

Proof. Suppose $\dim E = \infty$. Then, there exists a strictly increasing sequence (E_n) of closed finite-dimensional subspaces of E . Using the 148 Almost orthogonality lemma to the spaces $E_n \subset E_{n+1}$ with $\varepsilon = \frac{1}{2}$, we can construct a sequence (x_n) such that $\|x_n\| = 1$, $x_n \in E_{n+1} \setminus E_n$ and $d(x_n, E_n) > \frac{1}{2}$. That is, $\|x_i - x_j\| > \frac{1}{2} \forall i, j \in \mathbb{N}, i \neq j$. Therefore, S is not compact because there is no convergent subsequence of (x_n) . $\square$

Space of bounded operators

Definition 150. Let E, F be normed vector spaces. We define the following set:

$$\begin{aligned} \mathcal{L}(E, F) &:= \{T: E \rightarrow F : T \text{ is a bounded operator}\} \\ &= \{T: E \rightarrow F : T \text{ is continuous}\} \end{aligned}$$

Theorem 151. Let E, F be normed vector spaces. Then, $\mathcal{L}(E, F)$ is a vector normed space with the norm of Eq. (1) and the usual operations. Moreover, if F is Banach, so is $\mathcal{L}(E, F)$.

Sketch of the proof. An easy check shows that $\mathcal{L}(E, F)$ is a vector space and that the associated norm is indeed a norm. It's missing to show that $\mathcal{L}(E, F)$ is Banach whenever F is Banach. Let $\sum_{n=1}^{\infty} \|T_n\| < \infty$. Then, $\forall x \in E$, $\sum_{n=1}^{\infty} \|T_n x\|_F \leq \sum_{n=1}^{\infty} \|T_n\| \|x\|_E < \infty$. Since F is Banach, $\exists y \in F$ with $y = \sum_{n=1}^{\infty} T_n x$. By the linearity and continuity of T_n , the operator

$$\begin{aligned} T: E &\longrightarrow F \\ x &\longmapsto \sum_{n=1}^{\infty} T_n x \end{aligned}$$

is linear and continuous. So $T = \sum_{n=1}^{\infty} T_n$ and finally $\forall \varepsilon > 0$ if $x \in E \exists N \in \mathbb{N}$ such that:

$$\left\| T - \sum_{n=1}^N T_n \right\| \leq \sum_{n=N+1}^{\infty} \|T_n x\|_F \leq \|x\|_E \sum_{n=N+1}^{\infty} \|T_n\| < \varepsilon$$

because is the tail of a convergent series. $\square$

Proposition 152. Let E, F, G be normed vector spaces and $T: E \rightarrow F, S: F \rightarrow G$ be operators. Then:

$$\|S \circ T\| \leq \|S\| \|T\|$$

Proof.

$$\|(S \circ T)x\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|$$

$\square$

Definition 153. We denote $\mathcal{L}(E) := \mathcal{L}(E, E)$ which, together with the composition, has a structure of *normed algebra*. That is, $\forall T, S \in \mathcal{L}(E)$, we have:

1. $T \circ S \in \mathcal{L}(E)$
2. $\|S \circ T\| \leq \|S\| \|T\|$

Definition 154. Let E be a normed vector space. The Banach space $E^* := \mathcal{L}(E, \mathbb{K})$ is called *dual space* of E . The *bidual space* of E is $E^{**} := (E^*)^*$.

Definition 155. Let E be a normed vector space. We say that E is *reflexive* if $E = E^{**}$. $\square$

Compact operators

Definition 156. Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be an operator. We say that T is a *compact operator* if $T(B_E)$ is relatively compact on F where:

$$B_E := \{x \in E : \|x\|_E \leq 1\}$$

□

Definition 157. Let $X, Y \subseteq \mathbb{R}^n$ be compact metric spaces and $K \in \mathcal{C}(X \times Y)$. We define the *Fredholm operator with kernel K* as the operator $T : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$ defined by:

$$Tf(x) = \int_Y K(x, y)f(y) dy$$

Definition 158. Let $K \in \mathcal{C}(\Delta)$, where $\Delta := \{(x, y) \in \mathbb{R}^2 : a \leq y \leq x \leq b\}$. We define the *Volterra operator with kernel K* as the operator $T : \mathcal{C}([a, b]) \rightarrow \mathcal{C}([a, b])$ defined by:

$$Tf(x) = \int_a^x K(x, y)f(y) dy$$

Definition 159. Let $X, Y \subseteq \mathbb{R}^n$ be measurable spaces and $K \in L^2(X \times Y)$. We define the *Hilbert-Schmidt operator with kernel K* as the operator $T : L^2(Y) \rightarrow L^2(X)$ defined by:

$$Tf(x) \stackrel{\text{a.e.}}{=} \int_Y K(x, y)f(y) dy$$

Proposition 160. Let X, Y be compact metric spaces and $K \in \mathcal{C}(X \times Y)$. The Fredholm operator T with kernel K is compact and satisfies $\|T\| \leq \|K\|_{X \times Y} |Y|$

Sketch of the proof. It is a direct application of [129 Arzelà-Ascoli theorem](#). The proof of the equicontinuity follows from the inequality

$$|Tf(a) - Tf(b)| \leq \|f\| \sup_{y \in Y} \{|K(a, y) - K(b, y)|\} |Y|$$

and the fact that $\|f\| \leq 1$ and that K is uniformly continuous. The pointwise boundedness follows from:

$$|Tf(a)| \leq \sup_{(x, y) \in X \times Y} \{|K(x, y)|\} |Y|$$

because $\|f\| \leq 1$. And from here the inequality of the norm is clear. □

Proposition 161. Let $K \in \mathcal{C}(\Delta)$. The Volterra operator T with kernel K is compact and $\|T\| \leq \|K\|_{\Delta} (b - a)$.

Sketch of the proof. Use [129 Arzelà-Ascoli theorem](#) and a similar scheme of the proof of [Theorem 160](#). □

Proposition 162. Let $X, Y \subseteq \mathbb{R}^n$ be measurable spaces and $K \in L^2(X \times Y)$. The Hilbert-Schmidt operator T with kernel K satisfies $\|T\| \leq \|K\|_{L^2(X \times Y)}$.

Proof. By [104 Hölder's inequality](#) we have:

$$\|Tf\|^2 = \int_X \left(\int_Y K(x, y)f(y) dy \right)^2$$

$$\begin{aligned} &\leq \int_X \left(\int_Y K(x, y)^2 dy \right) \left(\int_Y |f(y)|^2 dy \right) \\ &= \left(\|K\|_{L^2(X \times Y)} \|f\|_{L^2(Y)} \right)^2 \end{aligned}$$

□

Definition 163. Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be an operator. We say that T is a *finite-rank operator* if $\dim T(E) < \infty$.

Lemma 164. Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$ be a finite-rank bounded operator. Then, T is compact.

Sketch of the proof. Since $\dim T(E) < \infty$, the bounded spaces on $T(E)$ are the relatively compact sets and since $T(B_E)$ is bounded, it is relatively compact. □

Proposition 165. Let E, F be Banach spaces and $(T_n) \in \mathcal{L}(E, F)$ be a sequence of compact operators with limit $T \in \mathcal{L}(E, F)$. Then, T is compact. In particular, the limit of any sequence of finite-rank bounded operators is compact.

Sketch of the proof. Let $(x_k) \in B_E$ be a sequence. Since T_1 is compact, there exists a subsequence $(x_{k,1})$ of (x_k) such that $T_1 x_{k,1}$ converges. Similarly, there exists a subsequence $(x_{k,2})$ of $(T_1 x_{k,1})$ such that $T_2 x_{k,2}$ converges. In general for each $n \in \mathbb{N}$, there exists a subsequence $(x_{k,n})$ of $(T_{n-1} x_{k,n-1})$ such that $T_n x_{k,n}$ converges. Now consider the sequence $(x_{k,k})$ and show that $(T x_{k,k})$ is Cauchy. The particularity follows from [Theorem 164](#). □

Neumann series

Remark. In Theory of Differential Equations, many times we need to find the solution of $Tu - u = v$, with $T \in \mathcal{L}(E)$ and $E = \mathcal{C}([a, b])$. Here, we will describe a general approach. Let E be a normed vector space, $T \in \mathcal{L}(E)$ and $v \in E$. We would like to find the solution of the equation:

$$(T - \text{id})u = v$$

for some $\lambda \in \mathbb{K}^*$.

Definition 166. Let E be a Banach space, $T \in \mathcal{L}(E)$ and $\lambda \in \mathbb{K}^*$. We define the *Neumann series* as the series:

$$-\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} T^n$$

Theorem 167. Let E be a Banach space, $T \in \mathcal{L}(E)$ and $\lambda \in \mathbb{K}^*$. If the Neumann series converges absolutely, then $(T - \lambda \text{id})^{-1} \in \mathcal{L}(E)$ and:

$$(T - \lambda \text{id})^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} T^n$$

Sketch of the proof. Remember that $\mathcal{L}(E)$ is Banach and note that:

$$\lim_{N \rightarrow \infty} -\frac{1}{\lambda} \sum_{n=0}^N \frac{1}{\lambda^n} T^n (T - \lambda \text{id}) = \text{id} - \lim_{N \rightarrow \infty} \frac{T^{N+1}}{\lambda^{N+1}}$$

□

Theorem 168. Let $T : \mathcal{C}([a, b]) \rightarrow \mathcal{C}([a, b])$ be the Volterra operator of kernel $K(x, y)$. Then, $\forall \lambda \neq 0$ the series

$$(T - \lambda \text{id})^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} T^n$$

is absolutely convergent on $\mathcal{L}(\mathcal{C}([a, b]))$.

Sketch of the proof. Use induction to prove:

$$\|T^n f(x)\| \leq \frac{M^n (x-a)^n}{n!} \|f\|$$

where $|K(x, y)| \leq M$. $\square$

Duality

Definition 169. Let E be a real vector space. A *convex functional* $p : E \rightarrow \mathbb{R}$ is a function that satisfies:

1. $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in E$
2. $p(\alpha x) = \alpha p(x) \quad \forall x \in E, \alpha \geq 0$

Theorem 170 (Hahn-Banach theorem). Let E be a real vector space, $F \subseteq E$ be a subspace, $p : E \rightarrow \mathbb{R}$ be a convex functional and $u \in F^*$. If $u(z) \leq p(z) \quad \forall z \in F$, then $\exists v \in E^*$ such that $v(z) = u(z) \quad \forall z \in F$ and $v(x) \leq p(x) \quad \forall x \in E$ (v is called an *extension* of u).

Definition 171 (Seminorm). Let E be a vector space over $\mathbb{K}$. A *seminorm* $p : E \rightarrow [0, \infty)$ is a functional that satisfies:

1. $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in E$
2. $p(\lambda x) = |\lambda| p(x) \quad \forall x \in E, \lambda \in \mathbb{K}$

Lemma 172. Let E be a vector space over $\mathbb{K}$. A norm defined on E is a seminorm.

Theorem 173 (Hahn-Banach theorem). Let E be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $F \subseteq E$ be a subspace, $p : E \rightarrow \mathbb{R}$ be a seminorm and $u \in F^*$. If $|u(z)| \leq p(z) \quad \forall z \in F$, then $\exists v \in E^*$ such that $v(z) = u(z) \quad \forall z \in F$ and $|v(x)| \leq p(x) \quad \forall x \in E$. That is, v extends u .

Theorem 174 (Hahn-Banach theorem). Let $E \neq \{0\}$ be a normed vector space.

1. If $F \subseteq E$ is a subspace and $u \in F^*$, then $\exists v \in E^*$ such that extends u and $\|v\| = \|u\|$.
2. For all $a \in E$, $\exists v \in E^*$ such that $v(a) = \|a\|$ and $\|v\| = 1$.
3. If $F \subseteq E$ is a closed subspace and $a \in E \setminus F$, then $\exists v \in E^*$ such that $v(a) = 1$ and $v(F) = \{0\}$.

Definition 175. Let E be a normed vector space, $x \in E$ and $v \in E^*$. We denote $\langle x, v \rangle := v(x)$ and $\hat{x} := \langle x, \cdot \rangle$. Thus, $\hat{x}(v) = \langle x, v \rangle$ and $\hat{x}$ is a bilinear form on E^* that satisfies:

$$|\hat{x}(v)| \leq \|v\|_{E^*} \|x\|_E$$

Thus, $\hat{x} \in E^{**}$ with $\|\hat{x}\|_{E^{**}} \leq \|x\|_E$.

Proposition 176. Let E be a normed vector space and $x \in E$ with $x \neq 0$. Then, $\|\hat{x}\|_{E^{**}} = \|x\|_E$.

Proposition 177. Let E be a normed vector space. The function

$$J : E \longrightarrow E^{**} \\ x \longmapsto \hat{x}$$

is linear, continuous, injective and isometric. Thus, $J(E) = E \subseteq E^{**}$. Moreover, if J is surjective, we have $E = E^{**}$. In this case, E is called *reflexive*.

Definition 178. Let E, F be normed vector spaces and $T \in \mathcal{L}(E, F)$. We define the *dual map*, $T^* \in \mathcal{L}(F^*, E^*)$, of T as $T^*(v) = v \circ T$.

Proposition 179. Let E, F be normed vector spaces and $T \in \mathcal{L}(E, F)$. Then, $\forall x \in E$ and $v \in F^*$ we have:

$$\langle T(x), v \rangle = \langle x, T^*(v) \rangle$$

Proposition 180. Let E, F be normed vector spaces. The function

$$\mathcal{L}(E, F) \longrightarrow \mathcal{L}(F^*, E^*) \\ T \longmapsto T^*$$

is linear, bijective and isometric. That is, $\|T\| = \|T^*\|$.

Theorem 181. Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p \leq \infty$ and q be the Hölder conjugate of p . Then:

- If $1 < p < \infty$, then $(L^p(\Omega))^* = L^q(\Omega)$.
- If $p = 1$, then $(L^1(\Omega))^* = L^\infty(\Omega)$.
- If $p = \infty$, then $(L^\infty(\Omega))^* \supsetneq L^1(\Omega)$.

In particular, for $1 < p < \infty$, $L^p(\Omega)$ is reflexive, while $L^1(\Omega)$ and $L^\infty(\Omega)$ are not.

Spectrum and eigenvalues

Proposition 182. Let E be a Banach space and $T \in \mathcal{L}(E)$. Then, $\forall \alpha \in \mathbb{K}$, $\text{im}(T - \alpha \text{id})$ and $\ker(T - \alpha \text{id})$ are invariant over T . Moreover, if $\alpha \neq 0$, the function

$$S : \ker(T - \alpha \text{id}) \longrightarrow \ker(T - \alpha \text{id}) \\ x \longmapsto \alpha x$$

is an isomorphism.

Proof. Let $y \in \text{im}(T - \alpha \text{id})$. Then, $y = Tx - \alpha x$ for some $x \in E$ and so:

$$Ty = T(Tx - \alpha x) = (T - \alpha \text{id})(Tx) \in \text{im}(T - \alpha \text{id})$$

Similarly if $x \in \ker(T - \alpha \text{id})$, then:

$$(T - \alpha \text{id})(Tx) = (T - \alpha \text{id})(\alpha x) = \alpha(Tx - \alpha x) = 0$$

The function $S = \alpha \text{id}$ is clearly an isomorphism because $\alpha \neq 0$. $\square$

Definition 183. Let E be a Banach space, $T \in \mathcal{L}(E)$ and $\alpha \in \mathbb{K}$. If $\ker(T - \alpha \text{id}) \neq \{0\}$, we say that $\ker(T - \alpha \text{id})$ is a *proper subspace* of T and that its non-zero elements are the *eigenvectors* of *eigenvalue* α of T .

Definition 184. Let E be a Banach space and $T \in \mathcal{L}(E)$. We define the *spectrum* of T as:

$$\sigma(T) := \{\alpha \in \mathbb{K} : T - \alpha \text{id is not bijective}\}$$

The elements of $\sigma(T)$ are called *spectral values* of T .

Proposition 185. Let E be a finite-dimensional Banach space and $T \in \mathcal{L}(E)$. Then:

$$\sigma(T) = \{\alpha \in \mathbb{K} : \alpha \text{ is eigenvalue of } T\} =: \Lambda(T)$$

Proof. The inclusion $\Lambda(T) \subseteq \sigma(T)$ is always true. For the other inclusion note that if $\alpha \notin \Lambda(T)$, then $Tx \neq \alpha x \forall x \in E, x \neq 0$, and so $\ker(T - \alpha \text{id}) = \{0\}$. Hence, $\alpha \notin \sigma(T)$ as in finite dimension injectivity in $\mathcal{L}(E)$ is equivalent to bijectivity in $\mathcal{L}(E)$. $\square$

Proposition 186. Let E be a Banach space and $T \in \mathcal{L}(E)$. Then, $\sigma(T)$ is compact and:

$$\sigma(T) \subseteq \{w \in \mathbb{K} : |w| \leq \|T\|\}$$

Proof. Let's first check that $\sigma(T) \subseteq \{w \in \mathbb{K} : |w| \leq \|T\|\}$. Let $\alpha \notin \{w \in \mathbb{K} : |w| \leq \|T\|\}$ (i.e. satisfies $|\alpha| > \|T\|$), $z \in E$ and define $f(x) = \frac{1}{\alpha}(Tx - z)$. Then, f is a contraction and by the ?? ?? $\exists! x \in E$ such that $Tx - \alpha x = z$, i.e. $T - \alpha \text{id}$ is bijective, and so $\alpha \notin \sigma(T)$.

Now let's see that $\sigma(T)^c$ is open. Let $\beta \notin \sigma(T)$ and $\alpha, z \in E$ be such that $|\alpha - \beta| < \frac{1}{\|(T - \beta \text{id})^{-1}\|}$. Now consider the function:

$$g(x) = (T - \beta \text{id})^{-1}(\alpha z - \beta x + z)$$

We have that g is a contraction and by the ?? ?? $\exists! x \in E$ such that $Tx - (\alpha - \beta)x = z$, i.e. $T - (\alpha - \beta)\text{id}$ is bijective. This is true $\forall \alpha \in E$ satisfying $|\alpha - \beta| < \frac{1}{\|(T - \beta \text{id})^{-1}\|}$.

Hence, $\sigma(T)^c$ is open.

Finally, since $\sigma(T)$ is a bounded closed subset of $\mathbb{C}$, it is compact. $\square$

3. | General theorems on linear maps

Open mapping and closed graph theorems

Theorem 187 (Baire's theorem). Let (U_n) be a sequence of dense open sets on a metric space X . Then, $\bigcap_{n=1}^{\infty} U_n$ is dense on X .

Corollary 188. Let X be a metric space such that $X = \bigcup_{n=1}^{\infty} C_n$, where $C_n \subseteq X$ are closed sets. Then, $\exists n_0$ such that $\text{Int } F_{n_0} \neq \emptyset$.

Sketch of the proof. If a closed set C has empty interior, then $X \setminus C$ is an open dense set. $\square$

Theorem 189 (Open mapping theorem). Let E, F be a Banach spaces and $T : E \rightarrow F$ be a surjective bounded operator. Then, T is open.

Theorem 190 (Closed graph theorem). Let E, F be a Banach spaces and $T : E \rightarrow F$ be an operator. Consider the graph of T :

$$\text{graph}(T) = \{(x, y) \in E \times F : y = Tx\}$$

Then, T is bounded if and only if $\text{graph}(T)$ is a closed set on $E \times F$.

Uniform boundedness principle

Theorem 191. Let E, F be a Banach spaces and $\{T_i : i \in I\} \subset \mathcal{L}(E, F)$ be a family of bounded operators. Then, one of the following statements holds:

1. $\sup\{\|T_i\| : i \in I\} < \infty$
2. $\exists A \subseteq E$ such that it is a countable intersection of dense open subsets (and therefore dense) such that:

$$\sup\{\|T_i x\|_F : i \in I\} = \infty \quad \forall x \in A$$

Corollary 192 (Banach-Steinhaus theorem). Let E, F be a Banach spaces and $(T_n) \in \mathcal{L}(E, F)$ be sequence of bounded operators such that:

- The limit $Tx := \lim_{n \rightarrow \infty} T_n x$ exists $\forall x \in D \subseteq E$, where D is a dense set in E .
- The sequence $(T_n x)$ is bounded $\forall x \in E$.

Then, T can be extended into a bounded operator such that:

$$\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$$

4. | Hilbert spaces

Inner products

Definition 193. Let E, F be vector spaces over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $u : E \rightarrow F$ be a function. We say that u is *semilinear* if $\forall x, y \in E$ and $\forall \lambda \in \mathbb{K}$ we have:

1. $u(x + y) = u(x) + u(y)$
2. $u(\lambda x) = \bar{\lambda}u(x)$

Definition 194. Let E be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. An *inner product* $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{K}$ is a function such that $\forall x, y \in E$ we have:

1. $\langle \cdot, y \rangle$ is linear and $\langle x, \cdot \rangle$ is semilinear.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle > 0 \iff x \neq 0$

We denote the norm associated with this inner product as $\|x\| := \sqrt{\langle x, x \rangle}$.

Definition 195. A *pre-Hilbert space* $(H, \langle \cdot, \cdot \rangle_H)$ is a vector space H over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ together with an inner product $\langle \cdot, \cdot \rangle_H$.⁶

Proposition 196. Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $x, y \in H$. Then:

1. $|\langle x, y \rangle| \leq \|x\| \|y\|$ (*Cauchy-Schwarz inequality*)
2. $\|x + y\| \leq \|x\| + \|y\|$ (*Minkowski inequality*)

Proof.

⁶In order to simplify the notation, if the context is clear, we will denote the inner product of H simply as $\langle \cdot, \cdot \rangle$.

1. Note that $\|x + \lambda y\|^2 \geq 0 \forall \lambda \in \mathbb{K}$. Taking $\lambda = -\frac{\langle x, y \rangle}{\|y\|^2}$ we have:

$$\begin{aligned} 0 &\leq \|x + \lambda y\|^2 \\ &= \|x\|^2 + \overline{\lambda} \langle x, y \rangle + \lambda \overline{\langle x, y \rangle} + |\lambda|^2 \|y\|^2 \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} - \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \end{aligned}$$

2. Using **Cauchy-Schwarz inequality** we have:

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\ &= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &\leq (\|x\| + \|y\|)^2 \end{aligned}$$

□

Lemma 197 (Polarization identity). Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $x, y \in H$. Then if $\mathbb{K} = \mathbb{C}$ we have:

$$\langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2}{4}$$

If $\mathbb{K} = \mathbb{R}$ we have:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

Sketch of the proof. Expand the right-hand side of the equalities using the definition of norm. □

Proposition 198. Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. Then:

- The inner product is a continuous function.
- The map

$$\begin{aligned} H &\longrightarrow H^* \\ x &\longmapsto \langle \cdot, x \rangle \end{aligned}$$

is semilinear, injective and isometric.

Definition 199. Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $x, y \in H$. We say that x and y are *orthogonal* if $\langle x, y \rangle = 0$. Moreover, we define the *orthogonal complement* of a subspace $A \subseteq H$ as:

$$A^\perp := \{x \in H : \langle x, a \rangle = 0 \forall a \in A\}$$

Lemma 200. Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $A \subseteq H$ be a subspace. Then:

$$A^\perp = \bigcap_{a \in A} \ker \langle \cdot, a \rangle$$

Hence, $A^\perp$ is a closed subspace of H .

Proof.

$$x \in A^\perp \iff \langle x, a \rangle = 0 \forall a \in A \iff x \in \ker \langle \cdot, a \rangle \forall a \in A$$

□

Proposition 201 (Pythagorean theorem). Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $x_1, \dots, x_n \in H$ be pairwise orthogonal elements of H . Then:

$$\|x_1\|^2 + \dots + \|x_n\|^2 = \|x_1 + \dots + x_n\|^2$$

Proof.

$$\begin{aligned} \|x_1 + \dots + x_n\|^2 &= \langle x_1 + \dots + x_n, x_1 + \dots + x_n \rangle \\ &= \|x_1\|^2 + \dots + \|x_n\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n \langle x_i, x_j \rangle \\ &= \|x_1\|^2 + \dots + \|x_n\|^2 \end{aligned}$$

□

Lemma 202 (Parallelogram law). Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and $x, y \in H$. Then:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Definition 203 (Hilbert space). A *Hilbert space* is a complete pre-Hilbert space.

Proposition 204. Let $X \subseteq \mathbb{R}$ and $A = \mathbb{N}, \mathbb{Z}, \{1, \dots, m\}$. Then:

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces with the inner product:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^n x_i \overline{y_i}$$

2. The $L^2(X)$ space with the inner product

$$\langle f, g \rangle_2 = \int_X f(x) \overline{g(x)} dx$$

is a Hilbert space.

3. The $\ell^2(A)$ space with the inner product

$$\langle (x_1, x_2, \dots), (y_1, y_2, \dots) \rangle = \sum_{a \in A} x_a \overline{y_a}$$

is a Hilbert space.

Projection theorem

Definition 205. Let X be a metric space and $A \subseteq X$ be a subset. We say that $y \in A$ is a *minimizer* of x in A if $d(x, A) = \|x - y\|$.

Proposition 206. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $C \subseteq H$ be a convex subset. Then, each $x \in H$ has at most one minimizer in C .

Proof. Suppose there are two minimizers $y, z \in C$ and let $\delta := d(x, A)$. Then, using **202 Parallelogram law** we have:

$$4\delta^2 = \|2x - (y + z)\|^2 + \|y - z\|^2 \geq 4\delta^4 + \|y - z\|^2$$

because $\frac{y+z}{2} \in C$ as C is convex. Thus, $\|y - z\|$ and so $y = z$. □

Theorem 207. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $C \subseteq H$ be a nonempty convex complete subset. Then, $\forall x \in H$, there exists a unique minimizer of x in C , which is denoted by $P_C x \in C$.

Sketch of the proof. The uniqueness has been proved in Theorem 206. To show the existence, let $\delta := d(x, C)$ and let $(y_n) \in C$ such that $\|y_n - x\| \rightarrow \delta$. By the 202 Parallelogram law:

$$\|y_n - y_m\|^2 \leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2 \rightarrow 0$$

Hence (y_n) is Cauchy and so its limit $y \in C$ satisfies $\delta = d(x, y)$ by the continuity of the norm. $\square$

Theorem 208 (Projection theorem). Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $F \subseteq H$ be a closed subspace. Then:

1. $H = F \oplus F^\perp$ and $\forall x \in H$, we can write $x = P_F x + P_{F^\perp} x$.
2. If $x \in H$ and $y \in F$, then $y = P_F x \iff x - y \in F^\perp$.

Proof.

1. The equality $F \cap F^\perp = \{0\}$ follows from noting that $\langle u, u \rangle = 0 \ \forall u \in F \cap F^\perp$. Now let $x \in H$ and $y = P_F x$. We need to show that $z := x - y \in F^\perp$. Let $u \in F$. Then, $\exists \lambda \in \mathbb{K}$ such that $\|\lambda\| = 1$ and $\lambda \langle u, z \rangle = |\langle u, z \rangle|$. Now consider $f(t) = \|z - vt\|^2$, where $v = \lambda u \in F$. Note that f has a minimum at the origin because:

$$f(t) = \|x - y - vt\|^2 \geq \|x - y\|^2 = \|z\|^2 = f(0)$$

because $y + vt \in F$ and y is the minimizer of x in F . Thus, $0 = f'(0) = -2\langle v, z \rangle$ and so $\langle u, z \rangle = 0 \ \forall u \in F$. Moreover, z is the minimizer of x in $F^\perp$ because $\forall w \in F^\perp$ we have by the 201 Pythagorean theorem:

$$\|x - w\|^2 = \|x - z\|^2 + \|z - w\|^2 \geq \|z - w\|^2$$

2. We have just seen the implication to the right. For the other one note that by Item 208-1 we can write $x = P_F x + P_{F^\perp} x$. But, $x - y = P_F x + P_{F^\perp} x - y \in F^\perp$ and $F \cap F^\perp = \{0\}$, so $y = P_F x$.

$\square$

Corollary 209. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $F \subseteq H$ be a closed subspace. The function $P_F : H \rightarrow H$, called *orthogonal projection on F* , satisfy the following properties:

1. $P_F^2 = P_F$
2. $\langle P_F x_1, x_2 \rangle = \langle x_1, P_F x_2 \rangle \ \forall x_1, x_2 \in H$
3. P_F is linear, $\|P_F x\| \leq \|x\|$ and the equality holds if and only if $x \in F$.
4. If $F \neq \{0\}$, $\|P_F\| = 1$.
5. $\text{im } P_F = F = \{y \in H : P_F y = y\}$

⁷Here $\langle A \rangle$ denotes the space generated by A .

$$6. \ker P_F = F^\perp$$

$$7. \|x\|^2 = d(x, F)^2 + d(x, F^\perp)^2 \ \forall x \in H$$

Proof.

1. Note that $P_F|_F = \text{id}$ and $\text{im } P_F \subseteq F$.
- 2.

$$\begin{aligned} \langle P_F x_1, x_2 \rangle &= \langle P_F x_1, P_F x_2 + P_{F^\perp} x_2 \rangle \\ &= \langle P_F x_1, P_F x_2 \rangle \\ &= \langle P_F x_1 + P_{F^\perp} x_1, P_F x_2 \rangle \\ &= \langle x_1, P_F x_2 \rangle \end{aligned}$$

3. Let $x_1, x_2, y \in H$, $\lambda, \mu \in \mathbb{K}$. Then using Item 209-2:

$$\begin{aligned} \langle P_F(\lambda x_1 + \mu x_2), y \rangle &= \langle \lambda x_1 + \mu x_2, P_F y \rangle \\ &= \lambda \langle x_1, P_F y \rangle + \mu \langle x_2, P_F y \rangle \\ &= \lambda \langle P_F x_1, y \rangle + \mu \langle P_F x_2, y \rangle \\ &= \langle \lambda P_F x_1 + \mu P_F x_2, y \rangle \end{aligned}$$

for all $y \in H$. Thus, $P_F(\lambda x_1 + \mu x_2) = \lambda P_F x_1 + \mu P_F x_2$. Moreover, using Item 209-1 and the Cauchy-Schwarz inequality:

$$\|P_F x\|^2 = \langle P_F x, P_F x \rangle = \langle P_F x, x \rangle \leq \|P_F x\| \|x\|$$

and the inequality follows if and only if $P_F x = \lambda x$, for some $\lambda \in \mathbb{K}$, that is, $x \in F$.

4. Consequence of the equality case in Item 209-3.
5. Clearly $\text{im } P_F \subseteq F$. The other follows from the fact that if $u \in F$, then $P_F u = u$ by the 208 Projection theorem.
6. Clearly $\ker P_F \supseteq F^\perp$. Moreover, if $x \in \ker P_F$, the 208 Projection theorem implies $x \in F^\perp$.
7. By the 201 Pythagorean theorem we have:

$$\begin{aligned} \|x\|^2 &= \|P_F x\|^2 + \|P_{F^\perp} x\|^2 \\ &= \|x - P_{F^\perp} x\|^2 + \|x - P_F x\|^2 \\ &= d(x, F^\perp)^2 + d(x, F)^2 \end{aligned}$$

$\square$

Proposition 210. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $A \subseteq H$ be subset. Then:

1. $A^\perp = \overline{\langle A \rangle}^\perp$
2. $A^{\perp\perp} = \overline{\langle A \rangle}$

In particular if $F \subseteq H$ is a subspace, then $F^{\perp\perp} = \overline{F}$.

Proposition 211. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $F \subseteq H$ be a closed subspace and $x \notin F$. Then:

$$\begin{aligned} \max\{|\langle x, u \rangle| : u \in F^\perp, \|u\| \leq 1\} &= d(x, F) \\ &= \left\langle x, \frac{P_{F^\perp} x}{\|P_{F^\perp} x\|} \right\rangle \end{aligned}$$

Duality and adjoint operator

Theorem 212 (Riesz representation theorem). Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. The map

$$\begin{aligned} H &\longrightarrow H^* \\ x &\longmapsto \langle \cdot, x \rangle \end{aligned}$$

is semilinear, bijective and isometric.

Corollary 213. Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then, $\forall y \in H_2 \exists! T^*y \in H_1$ such that:

$$\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1} \quad (2)$$

Definition 214. Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. We define the *adjoint operator* of T as the unique map

$$\begin{aligned} T^* : H_2 &\longrightarrow H_1 \\ y &\longmapsto T^*y \end{aligned}$$

such that Eq. (2) is held.

Proposition 215. Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then:

1. $T^* \in \mathcal{L}(H_2, H_1)$, $\|T^*\| = \|T\|$ and $\|T^* \circ T\| = \|T \circ T^*\| = \|T\|^2$.
2. The map

$$\begin{aligned} \mathcal{L}(H_1, H_2) &\longrightarrow \mathcal{L}(H_2, H_1) \\ T &\longmapsto T^* \end{aligned}$$

is semilinear.

3. $(T^*)^* = T$.
4. If H_3 is Hilbert and $S \in \mathcal{L}(H_2, H_3)$, then $(S \circ T)^* = T^* \circ S^*$.

Proposition 216. Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. Then:

1. $(\text{im } T)^\perp = \ker T^*$
2. $\overline{\text{im } T} = (\ker T^*)^\perp$
3. $(\text{im } T^*)^\perp = \ker T$
4. $\overline{\text{im } T^*} = (\ker T)^\perp$

Definition 217. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. We say that T is *self-adjoint* if $T^* = T$.

Lemma 218. Let H be a Hilbert space. The orthogonal projections P_F on a closed subspace $F \subseteq H$ are self-adjoint.

Proposition 219. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be self-adjoint. Then:

$$\|T\| = \sup\{|\langle Tx, x \rangle| : \|x\| = 1\} = \max\{M(T), -m(T)\}$$

where $M(T) := \sup\{\langle Tx, x \rangle : \|x\| = 1\}$ and $m(T) := \inf\{\langle Tx, x \rangle : \|x\| = 1\}$

Orthonormal systems

Definition 220. Let H be a Hilbert space. An *orthogonal system* on H is a nonempty subset $E \subseteq H$ such that its vectors are pairwise orthogonal. If moreover $\|e\| = 1 \forall e \in E$, we will say that E is an *orthonormal system*.

Definition 221. Let H be a Hilbert space and E be an orthonormal system. We say that E is *complete* if $E^\perp = \{0\}$.

Lemma 222. Let H be a Hilbert space. E is complete if and only if there is no other orthogonal system $E' \subseteq H$ such that $E \subsetneq E'$.

Definition 223. Let H be a Hilbert space. A complete orthonormal system is called a *Hilbert basis*.

Lemma 224. Let H be a Hilbert space and $E = \{e_1, \dots, e_n\}$ be an orthonormal system and $F = \langle E \rangle$. Then, each $x \in H$ can be written as:

$$P_F x = \sum_{k=1}^n \langle x, e_k \rangle e_k$$

Definition 225. Let H be a Hilbert space and $\{e_i : i \in I\}$ be an orthonormal system. We define the *Fourier coefficients* of an element $x \in H$ as the numbers $\hat{x}(i) := \langle x, e_i \rangle \in \mathbb{K}$. We will denote $\hat{x} := \{\hat{x}(i) : i \in I\}$ and we will say that

$$\sum_{i \in I} \hat{x}(i) e_i$$

is the *Fourier series* of x .

Proposition 226. Let H be a Hilbert space, $E = \{e_i : i \in I\}$ be a finite orthonormal system, $x \in H$ and $s = \sum_{i \in I} d_i e_i \in \langle E \rangle$. Then, $\|x - s\|$ is minimum if and only if $d_i = \hat{x}(i) \forall i \in I$.

Proposition 227 (Gram-Schmidt process). Let H be a Hilbert space, $\mathcal{B} = \{x_1, x_2, \dots\} \subset H$ be a finite or countable sequence. The *orthonormalization* of $\mathcal{B}$ is the orthonormal sequence $U = \{u_1, u_2, \dots\} \subset H$ such that for each $n \in \mathbb{N}$, we have $\langle x_1, \dots, x_n \rangle = \langle u_1, \dots, u_n \rangle$. It is built as follows:

- For $n = 1$, define $y_1 := x_1$ and then $u_1 := \frac{y_1}{\|y_1\|}$
- For $n \geq 2$, define

$$y_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i$$

and then $u_n := \frac{y_n}{\|y_n\|}$.

Theorem 228 (Bessel's inequality). Let H be a Hilbert space and $\{e_i : i \in I\}$ be an orthonormal system. Then, $\forall x \in H$ we have:

$$\sum_{i \in I} |\hat{x}(i)|^2 \leq \|x\|^2$$

Lemma 229. Let H be a Hilbert space and $\{e_i : i \in I\}$ be an orthonormal system. The map

$$\begin{aligned} H &\longrightarrow \ell^2(I) \\ x &\longmapsto \hat{x} \end{aligned}$$

is linear, surjective and satisfies $\|\hat{x}\|_2 \leq \|x\|_H$. This map is called *Fourier transform*.

Definition 230. Let H be a Hilbert space, $\{e_k : k \in \mathbb{N}\}$ be an orthonormal system and $x \in H$. We define:

$$s_k(x) := \sum_{k=1}^n \hat{x}(k) e_k$$

Lemma 231. Let H be a Hilbert space, $\{e_k : k \in \mathbb{N}\}$ be an orthonormal system and $x \in H$. Then, the limit $z = \lim_{n \rightarrow \infty} s_n(x)$ exists and satisfies $\hat{z} = \hat{x}$, but x and z may be different.

Theorem 232 (Riesz-Fischer theorem). Let H be a Hilbert space and $E = \{e_i : i \in I\}$ be an orthonormal system. The following statements are equivalent:

1. E is complete.
2. For all $x \in H$, $x = \lim_{n \rightarrow \infty} s_n(x) = \sum_{i \in I} \hat{x}(i) e_i$ on H .
3. *Parseval identity*: For all $x \in H$, we have:

$$\|x\|^2 = \sum_{i \in I} |\hat{x}(i)|^2$$

4. $\langle x, y \rangle_H = \langle \hat{x}, \hat{y} \rangle_2 \quad \forall x, y \in H$

Proposition 233. Let $X, Y \subseteq \mathbb{R}^n$ be measurable spaces and $K \in L^2(X \times Y)$. The Hilbert-Schmidt operator T with kernel K is compact. Moreover, if $K(x, y) = K(y, x) \quad \forall (x, y) \in X \times Y$, then T is self-adjoint.

Spectral theorem

Proposition 234. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be self-adjoint. Then:

1. Any eigenvalue of T belongs to $[m(T), M(T)] \subset \mathbb{R}$.
2. Eigenvectors of distinct eigenvalues are orthogonal.
3. If $F \subseteq H$ is a closed subspace such that $T(F) \subseteq F$, then $T(F^\perp) \subseteq F^\perp$.

Theorem 235 (Spectral theorem). Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and self-adjoint. Then, $\exists \alpha \in \mathbb{K}$ such that α is eigenvalue of T with $|\alpha| = \|T\|$.

Corollary 236. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and self-adjoint. Suppose α_1 is an eigenvalue of T with $|\alpha_1| = \|T\|$. Now consider $T_1 := T|_{\ker(T - \alpha_1 \text{id})^\perp}$. By **Theorem 235**, we obtain an eigenvalue α_2 of T_1 (and therefore of T) such that $|\alpha_1| \geq |\alpha_2|$. Iterating the process, we get a sequence of eigenvalues (α_n) with the property that $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3| \geq \dots$.

Theorem 237. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and self-adjoint. The sequence (α_n) of eigenvalues of T (each repeated according its multiplicity) is a sequence of real numbers. If the sequence is countable (i.e. not finite), then $\lim_{n \rightarrow \infty} \alpha_n = 0$. Moreover, for each eigenvalue α_n , $\dim \ker(T - \alpha_n \text{id}) < \infty$.

Lemma 238. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and self-adjoint. Consider the sequence (v_n) of orthonormal eigenvectors associated with the eigenvalues (α_n) of T . Then:

$$H = \langle v_1, v_2, \dots \rangle \oplus \ker(T)$$

and $\langle v_1, v_2, \dots \rangle = \ker(T)^\perp$.

Theorem 239 (Hilbert-Schmidt spectral representation theorem). Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and self-adjoint. Consider the sequence (v_n) of orthonormal eigenvectors associated with the eigenvalues (α_n) of T . Then:

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, v_n \rangle v_n$$

assuming that $\alpha_n = 0$ eventually if the sequence (α_n) is finite.

Theorem 240 (Fredholm alternative). Let H be a Hilbert space, $T \in \mathcal{L}(H)$ be compact and self-adjoint and $\alpha \in \mathbb{K}^*$. Consider the sequence (v_n) of orthonormal eigenvectors associated with the eigenvalues (α_n) of T . Then:

1. If $\alpha \neq \alpha_n \quad \forall n \in \mathbb{N}$, then $\forall y \in H$ the unique solution to $Tx - \alpha x = y$ is:

$$x = \frac{1}{\alpha} \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n - \alpha} \langle y, v_n \rangle v_n - y \right)$$

2. If $\alpha = \alpha_{n_0}$ for some $n_0 \in \mathbb{N}$, then the equation $Tx - \alpha x = y$ has solution if and only if $y \in \ker(T - \alpha \text{id})^\perp$. In this case the solution is:

$$x = \frac{1}{\alpha} \left(\sum_{\substack{n \geq 1 \\ \alpha_n \neq \alpha}} \frac{\alpha_n}{\alpha_n - \alpha} \langle y, v_n \rangle v_n - y \right) + \sum_{n \geq 1} \beta_n u_n$$

where $\beta_n \in \mathbb{K}$ are arbitrary coefficients and the u_n are the eigenvectors associated with α_{n_0} .