Partial differential equations

1. | PDEs in Physics

Wave and membrane dynamics

Proposition 1 (Wave equation). Consider a one-dimensional string of length L and constant $k(x)$, $\rho(x,t)$ be its linear density and $u(x,t)$ be the displacement of the point x at the time t from its equilibrium point. Then, the dynamics of the string are given by:

$$(\rho u_t)_t = (k u_x)_x$$

If both k and ρ are constant, this equation is sometimes written as:

$$u_{tt} = c^2 u_{xx} \tag{1}$$

These kinds of equations are called *hyperbolic equations*.

Proposition 2 (Navier-Cauchy equation). Consider a solid of mass density ρ and let μ and λ be the so-called *Lamé coefficients* that describe the material. If $\mathbf{u}(\mathbf{x},t)$ is the displacement vector at the point $\mathbf{x}$ and the instant t, the equation that describes the deformation of the solid (*elastodynamics*) is:

$$\rho \mathbf{u}_{tt} = \mu \Delta \mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u})$$

Fluid dynamics

Definition 3. Given a vector field $\mathbf{u}(\mathbf{x},t)$, we define the *material derivative operator* as:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} := \mathbf{u}_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$$

Definition 4. An *incompressible flow* is a flow in which the material density is constant.

Proposition 5 (Continuous equation). Consider a fluid of density ρ moving at a velocity $\mathbf{u}(\mathbf{x},t)$. The conservation of mass implies that the following equation (called *continuous equation*) must hold:

$$\rho_t + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0 \tag{2}$$

If the fluid is incompressible, the previous equation becomes:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

Proposition 6 (Cauchy momentum equation). Consider an inviscid fluid of density ρ moving at a velocity $\mathbf{u}(\mathbf{x},t)$ and undergoing a pressure of $p(\mathbf{x},t)$. The conservation of momentum implies that the following equation (called *Cauchy momentum equation*) must hold:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p = 0 \tag{3}$$

Theorem 7 (Inviscid flow). Consider an incompressible inviscid flow of density ρ moving at a velocity $\mathbf{u}(\mathbf{x},t)$ and undergoing a pressure of $p(\mathbf{x},t)$. The equations describing the dynamics of the flow are:

$$\begin{cases} \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{cases}$$

If however the flow is compressible, the equations become:

$$\begin{cases} \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p = 0 \\ \rho_t + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0 \end{cases}$$

Theorem 8 (Viscid flow). Consider an incompressible viscid fluid of density ρ, viscosity η, moving at a velocity $\mathbf{u}(\mathbf{x},t)$ and undergoing a pressure of $p(\mathbf{x},t)$. The equations describing the dynamics of the flow are:

$$\begin{cases} \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p = \eta \Delta \mathbf{u} \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{cases}$$

If however the flow is compressible, the equations become:

$$\begin{cases} \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla} p = \eta \left(\Delta \mathbf{u} + \dfrac{1}{3} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) \right) \\ \rho_t + \boldsymbol{\nabla}\cdot(\rho \mathbf{u}) = 0 \end{cases}$$

Potential theory

Proposition 9. Consider a body $\Omega \subset \mathbb{R}^3$ with a density of mass ρ. The gravitational force done by this body to a mass m located at the position $\mathbf{x} \in \mathbb{R}^3$ is given by:

$$\mathbf{F}(\mathbf{x}) = -Gm \int_\Omega \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^3} \rho(\mathbf{y}) \mathrm{d}^3\mathbf{y}$$

Proposition 10. Consider a body $\Omega \subset \mathbb{R}^3$ with a density of mass ρ. Then, $\mathbf{F}(\mathbf{x}) = m \boldsymbol{\nabla} u(\mathbf{x})$ where

$$u(\mathbf{x}) = G \int_\Omega \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \rho(\mathbf{y}) \mathrm{d}^3\mathbf{y}$$

is the *potential* created by the body Ω at the point $\mathbf{x} \in \mathbb{R}^3$. Furthermore, if ρ is regular enough, we have $\boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{x}) = -4\pi\rho(\mathbf{x})$[1]. Combining these two equation, we get:

$$\Delta u = -4\pi \rho$$

which is the *Poisson equation* (and also it is a *elliptic equation*).

[1] That is, $\boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \mathbb{R}^3 \setminus \Omega$.

Diffusion and heat equations

Proposition 11 (Fick's law of diffusion). Consider a material with *diffusivity* (or *diffusion coefficient*) D, *diffusion flux* $\boldsymbol{\phi}$ and concentration u. Then, *Fick's law* states that:

$$\boldsymbol{\phi} = -D\boldsymbol{\nabla} u$$

Proposition 12 (Diffusion equation). Consider a material with diffusivity D, the diffusion flux $\boldsymbol{\phi}$ and concentration u. Then, the concentration of the material satisfies:

$$\frac{\partial u}{\partial t} = \boldsymbol{\nabla} \cdot (D\boldsymbol{\nabla} u)$$

In particular, if $D = \text{const.}$, then we get $\frac{\partial u}{\partial t} = D\Delta u$.

Proposition 13 (Fourier's law). Consider a material with *thermal conductivity* k, $\mathbf{q}$ be the *heat flux* and $u(x,t)$ its temperature. Then, *Fourier's law* states that:

$$\mathbf{q} = -k\boldsymbol{\nabla} u$$

Proposition 14 (Heat equation). Consider a material with thermal conductivity k and u be its temperature. Then, the temperature of the material satisfies:

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\boldsymbol{\nabla} \cdot (k\boldsymbol{\nabla} u)$$

where c is the *specific heat capacity* and ρ is the *density*. In particular, if $k = \text{const.}$, then we get $\frac{\partial u}{\partial t} = \alpha\Delta u$, where $\alpha := \frac{k}{c\rho}$ is the *thermal diffusivity*.

Maxwell equations

Proposition 15 (Gauß' law). *Gauß' law* states that a static electric field points away from positive charges and towards negative charges, and the net outflow of the electric field through a closed surface $\partial\Omega$ is proportional to the enclosed charge.

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Differential form)}$$

$$\oiint_{\partial\Omega} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_{\Omega} \rho \,\mathrm{d}V \qquad \text{(Integral form)}$$

Proposition 16 (Gauß' law for magnetism). *Gauß' law for magnetism* states that for each volume element Ω in space, there are exactly the same number of magnetic field lines entering and exiting the volume. No total magnetic charge can build up in any point in space.

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\oiint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

Proposition 17 (Maxwell-Faraday equation). *Maxwell-Faraday equation* states that a time-varying magnetic field always accompanies a spatially varying (also possibly time-varying), non-conservative electric field, and vice versa

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

Proposition 18 (Ampère-Maxwell circuital law). The original *Ampère's law* ($\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$) stats a relation between the total amount of magnetic field around some closed path $\partial\Sigma$ due to the current that passes through that enclosed path Σ. The second term on the right-hand-side (added later by Maxwell) is the *displacement current* associated with the polarization of the individual molecules of the dielectric material.

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\oint_{\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \left(\int_{\Sigma} \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{S}\right)$$

Mechanics and optics

Definition 19. We define the *refractive index* is defined as:

$$n(\mathbf{x}) = \frac{c}{v(\mathbf{x})}$$

where c is the speed of the light in the vacuum and $v(\mathbf{x})$ the speed of the light at the position $\mathbf{x}$ (located in some medium).

Proposition 20 (Fermat's principle). *Fermat's principle* states that the path taken by a ray between two given points a and b is the path that can be traveled in the least time. Mathematically, we want to minimize the functional:

$$\mathcal{T}(\mathbf{x}) = \int_a^b \frac{|\mathrm{d}\mathbf{x}|}{v(\mathbf{x})}$$

So we shall solve the equation $\delta\mathcal{T} = 0$, which is equivalent to solve:

$$\delta \int_a^b n(\mathbf{x})\,\mathrm{d}s = 0$$

where s is the arc-length parameter. From the Euler-Lagrange equations, we get the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right) = \boldsymbol{\nabla} n$$

Proposition 21 (Eikonal equation). The time $T(x)$ taken by the light to travel from a fixed point x_0 to x in a medium of refractive index n is given by:

$$\|\boldsymbol{\nabla} T\|^2 = n^2$$

Definition 22. The *action* $\mathcal{S}$ of a physical system is defined as the integral of the Lagrangian $L := T - V$ between two instants of time t_1 and t_2. That is:

$$\mathcal{S}(\mathbf{x}, t) = \int_{t_1}^{t_2} L(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\,\mathrm{d}t = \int_{t_1}^{t_2} \left(\frac{1}{2}m\|\dot{\mathbf{x}}\|^2 - V(\mathbf{x})\right)\mathrm{d}t$$

where m is the mass of the particle, T is the kinetic energy of the particle and V is its potential energy.

Proposition 23 (Principle of least action). The path taken by a physical system between times t_1 and t_2 and configurations $\mathbf{x}_1$ and $\mathbf{x}_2$ is the one for which the action is stationary (no change) to first order. Mathematically, $\delta\mathcal{S} = 0$, where δ means a *small change*. This value $S(\mathbf{x}, t)$ of the action satisfies the *Hamilton-Jacobi equation*:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\|\boldsymbol{\nabla} S\|^2 + V = 0$$

Proposition 24 (Schrödinger equation). The *Schrödinger equation* is a PDE that governs the *wave function* Ψ, which describes the quantum state of an isolated quantum system, of a quantum-mechanical system. This is given by:

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\Psi$$

where m is the mass of the particle and V is the potential in which the particle exists. Furthermore, $|\Psi|^2$ is the probability density function of the position of the particle.

Proposition 25. Substituting $\Psi = \sqrt{\rho}\mathrm{e}^{\mathrm{i}\frac{S}{\hbar}}$ into the Schrödinger equation and taking the limit $\hbar \to 0$ in the resulting equation yield the Hamilton-Jacobi equation. Moreover, if we define $\mathbf{v} = \frac{\boldsymbol{\nabla} S}{m}$, from one real equation (from the original one complex equation) we get the continuous equation (Eq. (2)) and from the imaginary equation taking the limit $\hbar \to 0$ we get the Cauchy momentum equation (Eq. (3)).

2. | First order partial differential equations

Vector calculus

Definition 26. Let $\Omega \subseteq \mathbb{R}^n$ be a set. We define the space $\mathcal{C}_0^\infty(\Omega)$ as the set of all compactly supported functions in $\mathcal{C}^\infty(\Omega)$.

Theorem 27 (Fundamental lemma of calculus of variations). Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}$ be a continuous function. If

$$\int_U f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

for any subset $U \subseteq \Omega$, then $f = 0$ in Ω.

Proof. If there were a point $\mathbf{x}_0 \in \Omega$ such that (without loss of generality) $f(\mathbf{x}_0) > 0$, the continuity would imply the existence of an open set U containing $\mathbf{x}_0$ and a $\varepsilon > 0$ such that $f(\mathbf{x}) > \varepsilon\ \forall \mathbf{x} \in U$. But then we would have:

$$0 = \int_U f(\mathbf{x})\,\mathrm{d}\mathbf{x} > \varepsilon|U| > 0$$

□

Corollary 28. Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}$ be a continuous function such that

$$\int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then, $f = 0$ in Ω.

Proof. If there were a point $\mathbf{x}_0 \in \Omega$ such that (without loss of generality) $f(\mathbf{x}_0) > 0$, the continuity would imply the existence of an open set U containing $\mathbf{x}_0$ and a $\varepsilon > 0$ such that $f(\mathbf{x}) > \varepsilon\ \forall \mathbf{x} \in U$. Now take $\varphi \in \mathcal{C}_0^\infty(\Omega)$ such that $\varphi \geq 0$, $\operatorname{supp}\varphi \subset U$ and $\varphi > 0$ in some open set $V \subseteq U$. And we would have:

$$0 = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_U f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} > 0$$

□

Proposition 29. Let $\Omega \subseteq \mathbb{R}^n$ be a compact set with a piecewise smooth boundary, $U \supset \Omega$ be an open neighborhood of Ω, $k \in \mathcal{C}^1(U)$ and $f, g \in \mathcal{C}^2(U)$. Then:

$$\int_\Omega f\,\mathbf{div}(k\boldsymbol{\nabla} g) = \int_{\partial\Omega} fk\boldsymbol{\nabla} g \cdot \mathrm{d}\mathbf{S} - \int_\Omega k\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$$

$$\int_\Omega f\,\mathbf{div}(k\boldsymbol{\nabla} g) - g\,\mathbf{div}(k\boldsymbol{\nabla} f) =$$

$$= \int_{\partial\Omega} k\left(f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f\right) \cdot \mathrm{d}\mathbf{S}$$

Sketch of the proof. For the first one apply the **?? ??** with the vector field $kf\boldsymbol{\nabla} g$ and for the second one, use the previous formula and the symmetry of f and g. □

Corollary 30 (Green identities). Let $\Omega \subseteq \mathbb{R}^n$ be a compact set with a piecewise smooth boundary, $U \supset \Omega$ be an open neighborhood of Ω and $f, g \in \mathcal{C}^2(U)$.

$$\int_\Omega f\Delta g = \int_{\partial\Omega} f\boldsymbol{\nabla} g \cdot \mathrm{d}\mathbf{S} - \int_\Omega \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$$

$$\int_\Omega f\Delta g - g\Delta f = \int_{\partial\Omega} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) \cdot \mathrm{d}\mathbf{S}$$

Method of characteristics

Proposition 31 (Method of characteristics). Let a, b, c, x_0, t_0 and u_0 be of class $\mathcal{C}^1$. Consider the following quasilinear partial differential equation

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = c(x,t,u) \tag{4}$$

with initial condition $u(x_0(s), t_0(s)) = u_0(s)$, $s \in I$ where $I \subseteq \mathbb{R}$ is an interval. The solutions curves of the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = a(x,t,u) \\ \dfrac{\mathrm{d}t}{\mathrm{d}\tau} = b(x,t,u) \\ \dfrac{\mathrm{d}u}{\mathrm{d}\tau} = c(x,t,u) \end{cases}$$

with initial conditions $x(0,s) = x_0(s)$, $t(0,s) = t_0(s)$ and $u(0,s) = u_0(s)$, form the surface of the graph $u(x,t)$. Such curves are called *characteristic curves*.

Sketch of the proof. Note that we can rewrite Eq. (4) as:

$$\begin{pmatrix} a(x,t,u) & b(x,t,u) & c(x,t,u) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \\ -1 \end{pmatrix} = 0$$

And $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, -1)^{\mathrm{T}}$ is perpendicular to the surface of $u(x,t)$. Now for each $s \in I$ it suffices to find a curve $C \subset \mathbb{R}^3$ parametrized by τ whose tangent vector is $(a(x,t,u), b(x,t,u), c(x,t,u))^{\mathrm{T}}$. □

Traffic flow equation

Proposition 32 (Traffic flow equation). Consider a one lane motorway with one entry an one exit. Let $\rho(x,t)$ be the density of cars per unit of length, $u(\rho)$ the average speed of the cars and $q = \rho u$ be the flux of cars. Then, we can model the traffic in the motorway with the equation:

$$\rho_t + (\rho u)_x = \rho_t + q'(\rho)\rho_x = 0$$

The integral form of the latter equation is:

$$\frac{\partial}{\partial t}\int_a^b \rho(x,t)\,\mathrm{d}x = q(a,t) - q(b,t) \tag{5}$$

Sketch of the proof. The integral form is due to the conservation of "mass". Thus, using the regularity of the functions:

$$\int_a^b \rho_t\,\mathrm{d}x = \frac{\partial}{\partial t}\int_a^b \rho(x,t)\,\mathrm{d}x = q(a,t) - q(b,t) = -\int_a^b q_x\,\mathrm{d}x$$

Now use the 27 Fundamental lemma of calculus of variations. □

Proposition 33. In the hypothesis of the traffic equation, if $t_2 \geq t_1$, then:

$$\int_a^b [\rho(x,t_2) - \rho(x,t_1)]\,\mathrm{d}x = \int_{t_1}^{t_2} [q(a,t) - q(b,t)]\,\mathrm{d}t \tag{6}$$

Sketch of the proof. Integrate Eq. (5) with respect to t between t_1 and t_2. □

Proposition 34. In the hypothesis of the traffic equation, ρ is constant in each line of the form $x(t) = x_0 + q'(\rho(x_0,0))t$. This determines $\rho(x,t)$ provided that we already know the initial condition $\rho_0(x) := \rho(x,0)$, $x \in \mathbb{R}$. In other words, $\rho(x,t)$ is the solution ξ of the density at the appropriate x-intercept of the line passing through (x,t):

$$\xi = \rho_0(x - q'(\xi)t)$$

Sketch of the proof. Apply the 31 Method of characteristics. On the other hand, if ξ is the density at (x,t), we have:

$$x = x_0 + q'(\xi)t$$

where x_0 is the x-intercept at $t = 0$ of the line passing through (x,t) with slope $q'(\xi)$. Rearranging the previous equation and applying ρ_0 we get the desired result:

$$x_0 = x - q'(\xi)t \implies \xi = \rho(x_0) = \rho_0(x - q'(\xi)t)$$

□

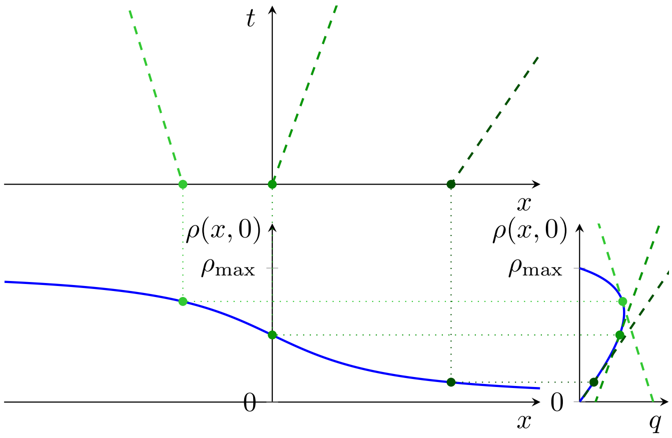

Figure 1: Characteristics of the traffic flow. In each line the density ρ is constant.

Proposition 35 (Rankine-Hugoniot equation). In the hypothesis of the traffic equation, let $x_\mathrm{s}(t)$ be the position at time t of a (jump) discontinuity in the function ρ. Then:

$$\frac{\mathrm{d}x_\mathrm{s}}{\mathrm{d}t} = \frac{[q]}{[\rho]} = \frac{(\rho u)_+ - (\rho u)_-}{\rho_+ - \rho_-}$$

where the notation $[x(t)]$ refers to:

$$[x(t_0)] := x_+(t_0) - x_-(t_0) := \lim_{t\to {t_0}^+} x(t) - \lim_{t \to {t_0}^-} x(t)$$

Sketch of the proof. Let $a(t) \leq x_\mathrm{s}(t) \leq b(t)$ and $t_2 \geq t_1$. Then, using Eq. (6) we have:

$$\begin{aligned}\int_{a(t_2)}^{b(t_2)} \rho(x,t_2)\,\mathrm{d}x - \int_{a(t_1)}^{b(t_1)} \rho(x,t_1)\,\mathrm{d}x &= \\ &= \int_{t_1}^{t_2} [q(a(t),t) - q(b(t),t)]\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} [\rho(a(t),t)(u - a') - \rho(b(t),t)(u - b')]\,\mathrm{d}t\end{aligned}$$

Letting $a(t) \nearrow x_\mathrm{s}(t) \swarrow b(t)$ and using the 27 Fundamental lemma of calculus of variations we get:

$$[\rho(x_\mathrm{s}(t),t)(u - {x_\mathrm{s}}')]_- - [\rho(x_\mathrm{s}(t),t)(u - {x_\mathrm{s}}')]_+ = 0$$

Rearranging the terms we get the desired result. □

Lemma 36 (Entropy condition). In the hypothesis of the traffic equation, we will have existence and uniqueness of solutions for the traffic flow equation if:

$$q'(\rho_+) < \frac{[q]}{[\rho]} < q'(\rho_-)$$

3. | Wave equation

Proposition 37. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a two-times differentiable function such that:

$$\int_{x_1}^{x_2} (\rho u_t)(x,t_2)\,\mathrm{d}x - \int_{x_1}^{x_2} (\rho u_t)(x,t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} (k u_x)(x_2,t)\,\mathrm{d}t - \\ - \int_{t_1}^{t_2} (k u_x)(x_1,t)\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for certain smooth functions $\rho(x,t)$, $k(x)$, $f(x,t)$. Then, $u(x,t)$ is a solution to the wave equation with driven force f:

$$(\rho u_t)_t - (k u_x)_x = f(x,t)$$

If $f = 0$ and ρ and k are constant, the equation is sometimes rewritten as:

$$u_{tt} = c^2 u_{xx} \tag{7}$$

Sketch of the proof. Rewrite the equation as:

$$\int_{t_1}^{t_2} \frac{\partial}{\partial t} \int_{x_1}^{x_2} \rho u_t\,\mathrm{d}x\,\mathrm{d}t = \int_{x_1}^{x_2} \frac{\partial}{\partial x} \int_{t_1}^{t_2} k u_x\,\mathrm{d}t\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

Now use the regularity of the functions and the 27 Fundamental lemma of calculus of variations to get the result. □

Solution on $\mathbb{R}$

Proposition 38 (D'Alembert formula). Let $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ be functions. The solution $u(x,t)$ to the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases}$$

is:

$$u(x,t) = \frac{u_0(x+ct) + u_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,\mathrm{d}s \tag{8}$$

Sketch of the proof. Eq. (7) with the coordinates $(\xi,\eta) = (x+ct, x-ct)$ is simplified to $u_{\xi\eta} = 0$. Thus, $u(x,t) = \phi(x+ct) + \psi(x-ct)$ for certain smooth functions ϕ, ψ. Now use the initial conditions to conclude

$$\begin{cases} \phi(y) = \dfrac{1}{2}u_0(y) + \dfrac{1}{2c}\displaystyle\int_0^y v_0(s)\,\mathrm{d}s + C \\ \psi(y) = \dfrac{1}{2}u_0(y) - \dfrac{1}{2c}\displaystyle\int_0^y v_0(s)\,\mathrm{d}s - C \end{cases}$$

for certain constant $C \in \mathbb{R}$. □

Remark. 38 D'Alembert formula show us that the state at (x,t) depends entirely on the quantities $x+ct$ and $x-ct$ and the functions $\phi(x+ct)$, $\psi(x-ct)$ represent two waves traveling at velocities $-c$ and c respectively. Hence, a small perturbation far from (x,t) will not affect $u(x,t)$ in a neighborhood of (x,t) but it will do it eventually.

Theorem 39. Let $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be functions. The solution $u(x,t)$ to the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} + f \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases} \tag{9}$$

is:

$$u(x,t) = \frac{u_0(x-ct) + u_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,\mathrm{d}s + \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau$$

If we think $u(t) : x \to u(x,t)$, then we can write the expression above more compactly as:

$$u(t) = T'(t)u_0 + T(t)v_0 + \int_0^t T(t-\tau)f(\tau)\,\mathrm{d}\tau$$

where the operator $T(t)$ is defined as:

$$[T(t)\varphi](x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(s)\,\mathrm{d}s$$

Sketch of the proof. Eq. (9) with the coordinates $(\xi,\eta) = (x+ct, x-ct)$ is simplified to $u_{\xi\eta} = -\frac{f}{4c^2}$. Now integrate this equation using the **?? ??**. □

Theorem 40. Let $U \subseteq \mathbb{R}^2$ be an open set and $u : U \to \mathbb{R}$ be a function. Then, u satisfies the wave equation with density $\rho(x,t)$, constant $k(x)$ and driven force $f(x,t)$ if and only if:

$$\int_{\partial U} \rho u_t\,\mathrm{d}x + k u_x\,\mathrm{d}t = -\int_U f(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

Sketch of the proof. It is a consequence of the **?? ??** with the vector field $\mathbf{X} = (\rho u_t, k u_x)$ and the 27 Fundamental lemma of calculus of variations. □

Proposition 41. Let $U \subset \mathbb{R}^2$ be an open set, $u : U \to \mathbb{R}$ be a function. Then, u satisfies the wave equation with constant $c^2 = \frac{k}{\rho}$ and no driven force if and only if for any four points A, B, C and D delimiting the boundary of an open set $V \subseteq U$ (as in Fig. 2) we have:

$$u(A) - u(B) + u(C) - u(D) = 0 \tag{10}$$

Sketch of the proof. Prove

$$\int_{\partial V} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t = 2c(u(A) - u(B) + u(C) - u(D))$$

and then use Theorem 40. To show this latter equality note that $u_x = u_\xi + u_\eta$, $u_t = c(u_\xi - u_\eta)$ and use the fact that $\mathrm{d}\xi = \mathrm{d}x + c\,\mathrm{d}t = 0$ and $\mathrm{d}\eta = \mathrm{d}x - c\,\mathrm{d}t = 0$ in the respective characteristic lines of ∂V. □

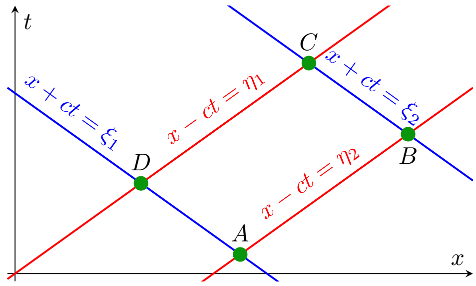

Figure 2: Characteristics of the waves equation.

Proposition 42 (Conservation of energy). Consider the wave equation $\rho u_{tt} - k u_{xx} = 0$ and assume the functions u_0, v_0 of the initial conditions have compact support. Then:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{-\infty}^{\infty} \left(\frac{1}{2}\rho {u_t}^2 + \frac{1}{2} k {u_x}^2 \right) \mathrm{d}x = 0$$

That is, the energy is conserved.

Sketch of the proof. Enter the derivate inside the integral and integrate by parts. □

Corollary 43. The problem of Eq. (9) in which the functions u_0 and v_0 have compact support has existence and uniqueness of solutions.

Sketch of the proof. The existence has already been proved for a sufficiently regular f. For the uniqueness, suppose u_1 and u_2 are two solutions. Then, $u = u_1 - u_2$ is a solution to $\rho u_{tt} - k u_{xx} = 0$ with initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 0$. Moreover:

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}\rho {u_t}^2 + \frac{1}{2} k {u_x}^2 \right) \mathrm{d}x = 0$$

because it is constant and attains the value of 0 at $t = 0$. This implies $u = 0$ using again the initial conditions. □

Solution with one fixed point

Proposition 44. Consider the problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \\ u(0,t) = \alpha(t) \end{cases}$$

where $u_0, v_0 : (0, \infty) \to \mathbb{R}$. Then, the d'Alembert solution is

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

where:

$$\phi(y) = \frac{1}{2} u_0(y) + \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s \quad \text{for } y \geq 0$$

$$\psi(y) = \begin{cases} \frac{1}{2} u_0(y) - \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s & \text{if } y \geq 0 \\ -\phi(-y) + \alpha(-y/c) & \text{if } y < 0 \end{cases}$$

In particular, if $\alpha(t) = 0$ and we make the odd extension of both u_0 and v_0, we have:

$$\psi(y) = \frac{1}{2} u_0(y) - \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s \quad \forall y \in \mathbb{R}$$

Sketch of the proof. We already saw the expressions of ϕ and ψ for $y \geq 0$ in 38 D'Alembert formula. For $y < 0$, note that we must have:

$$\alpha(t) = u(0,t) = \phi(ct) + \psi(-ct)$$

□

Proposition 45. Consider the problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \\ u_x(0,t) = \beta(t) \end{cases}$$

where $u_0, v_0 : (0, \infty) \to \mathbb{R}$. Then, the d'Alembert solution is

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

where:

$$\phi(y) = \frac{1}{2} u_0(y) + \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s \quad \text{for } y \geq 0$$

$$\psi(y) = \begin{cases} \frac{1}{2} u_0(y) - \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s & \text{if } y \geq 0 \\ \phi(-y) + \int_0^y \beta(-s/c)\, \mathrm{d}s & \text{if } y < 0 \end{cases}$$

In particular, if $\beta(t) = 0$ and we make the even extension of both u_0 and v_0, we have:

$$\psi(y) = \frac{1}{2} u_0(y) - \frac{1}{2c} \int_0^y v_0(s)\, \mathrm{d}s \quad \forall y \in \mathbb{R}$$

Sketch of the proof. We already saw the expressions of ϕ and ψ for $y \geq 0$ in 38 D'Alembert formula. For $y < 0$, note that:

$$\psi'(y) = -\phi'(y) + \beta(y/c)$$

because $\beta(t) = u_x(0,t) = \phi'(ct) + \psi'(-ct)$. □

Solution with two fixed endpoints

Consider a string of length L with its two endpoints fixed. In this section we will discuss how to obtain the solutions of its movement solving the following initial-and-boundary conditions problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \tag{11}$$

Definition 46. Let $f : [0,T] \to \mathbb{R}$ be a function. We define the *even periodic extension* of f as the function f_e such that:

- $f_e(x) = f(x)$ for $x \in [0,T]$.
- f_e is even.
- f_e is $2T$-periodic.

We define the *odd periodic extension* of f as the function f_o such that:

- $f_o(x) = f(x)$ for $x \in [0,T]$.
- f_o is odd.
- f_o is $2T$-periodic.

Proposition 47. Consider the odd periodic extensions for u_0 and v_0 of Eq. (11). Then, the solutions of that equation are given by the 38 D'Alembert formula.

Sketch of the proof. Consequence of 38 D'Alembert formula. □

Proposition 48. Suppose we want to know the displacement $u(x,t)$ of the string at the position $A = (x,t) \in [0,L] \times \mathbb{R}_{\geq 0}$ (see Fig. 3). Then:

$$u(A) = -\frac{u_0(D) + u_0(C)}{2} - \frac{1}{2c}\int_C^D v_0(s)\,\mathrm{d}s$$

Sketch of the proof. We will use Eq. (10) to determine $u(A)$. Construct the characteristic lines $x \pm ct$ as shown in Fig. 3. Then, by Eq. (10) we have that the $u(A) = -u(B)$. Since we are provided with the equation at $t = 0$, we can determine $u(B)$ using the points C and D and the 38 D'Alembert formula. □

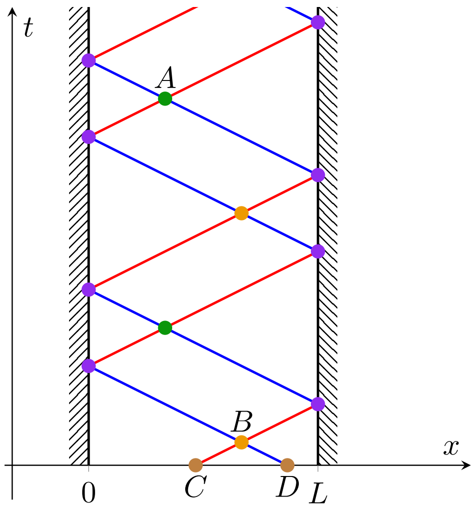

Figure 3: Scheme for Theorem 48 of solving the wave equation

Proposition 49 (Separation of variables). The solution $u(x,t)$ to Eq. (11), using *separation of variables* (i.e. assuming $u(x,t) = X(x)T(t)$), is:

$$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{\pi n x}{L}\right)\left[a_n \cos\left(\frac{\pi n c}{L}t\right) + b_n \sin\left(\frac{\pi n c}{L}t\right)\right]$$

where:

$$a_n = \frac{1}{L}\int_{-L}^{L} u_0(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x$$

$$b_n = \frac{1}{\pi n c}\int_{-L}^{L} v_0(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x$$

Here we have thought u_0 and v_0 as the respective odd periodic extensions.

Sketch of the proof. Assume $u(x,t) = X(x)T(t)$. Then:

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

and $\lambda = \text{const.}$ because the left-hand-side depends entirely on t, whereas the right-hand-side depends entirely on x. From $X'' + \lambda X = 0$, we can deduce that $\lambda > 0$ by multiplying the equation by X and integrating (between 0 and L) by parts the result. Finally, imposing the boundary and initial conditions (using Fourier series) leads to the solution. □

Remark. Note that with the wave equation the derivatives u_x and u_t converge (if they do) more slowly than u. The situation worsen with higher derivatives.

Variable coefficients

Theorem 50 (Sturm-Picone comparison theorem). Let $p_i, q_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be functions such that $0 < p_2 < p_1$ and $q_1 < q_2$. Suppose that the functions $u(x)$ and $v(x)$ satisfy the following differential equations:

$$(p_1(x)u')' + q_1(x)u = 0$$
$$(p_2(x)v')' + q_2(x)v = 0$$

If α_1, α_2 are two successive roots of u, then one of the following holds:

- $\exists \beta \in (\alpha_1, \alpha_2)$ such that $v(\beta) = 0$.
- $\exists \lambda \in \mathbb{R}$ such that $v(x) = \lambda u(x)$ $\forall x \in \mathbb{R}$.

Sketch of the proof. Suppose v and u are linearly independent and that $u > 0$ and $v > 0$ (if $v < 0$, $-v > 0$ an is also a solution of the same PDE) in (α_1, α_2). Then, multiplying the first equation by $-u$ and the second one by $\frac{u^2}{v}$, adding them and integrating we get:

$$0 = \int_{\alpha_1}^{\alpha_2}\left[-(p_1 u')' u + (p_2(x)v')'\frac{u^2}{v} + (q_2 - q_1)u^2\right]\mathrm{d}x$$
$$= \int_{\alpha_1}^{\alpha_2}\left[p_1 {u'}^2 + p_2\frac{u^2{v'}^2 - 2uu'vv'}{v^2} + (q_2 - q_1)u^2\right]\mathrm{d}x$$

Finally, observe that:

$$\frac{u^2{v'}^2 - 2uu'vv'}{v^2} = \frac{(uv' - u'v)^2}{v^2} - {u'}^2$$

And from the hypothesis we conclude $u = 0$, which is a contradiction. □

Proposition 51. Let $\lambda, \mu \in \mathbb{R}$ such that $\lambda \neq \mu$ and $k, \rho : \mathbb{R} \to \mathbb{R}$. Suppose that the functions $f, g : [0, L] \to \mathbb{R}$ satisfy the following differential equations:

$$\begin{aligned}(k(x)f')' + \lambda\rho(x)f &= 0\\ (k(x)g')' + \mu\rho(x)g &= 0\end{aligned}$$

and $f(0) = f(L) = g(0) = g(L) = 0$. Then, f and g are orthogonal with inner product with weight ρ.

Sketch of the proof. Multiply the first equation by g, the second by f, sum them and integrate (by parts) the resulting equation between 0 and L to conclude:

$$\int_0^L \rho(x)f(x)g(x)\,\mathrm{d}x = 0$$

□

Proposition 52. Consider the following problem of the wave equation of non-constant coefficients:

$$\begin{cases}\rho u_{tt} = (ku_x)_x\\ u(x,0) = u_0(x)\\ u_t(x,0) = v_0(x)\\ u(0,t) = 0\\ u(L,t) = 0\end{cases} \tag{12}$$

Then, the general solution to this problem (assuming that there is a solution for each λ_n number) is:

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)\left[a_n \cos\left(\sqrt{\lambda_n}t\right) + b_n \sin\left(\sqrt{\lambda_n}t\right)\right]$$

where $X_n(x)$ is the solution to the problem

$$\begin{cases}(k{X_n}')' + \lambda_n\rho X_n = 0\\ X_n(0) = 0\\ X_n(L) = 0\end{cases}$$

and:

$$a_n = \frac{\int_0^L u_0(x)X_n(x)\rho(x)\,\mathrm{d}x}{\int_0^L X_n(x)^2\rho(x)\,\mathrm{d}x} \qquad b_n = \frac{\int_0^L v_0(x)X_n(x)\rho(x)\,\mathrm{d}x}{\sqrt{\lambda_n}\int_0^L X_n(x)^2\rho(x)\,\mathrm{d}x}$$

Sketch of the proof. Use separation of variables and Theorem 51. □

4. | Heat equation

Basic solution

Proposition 53. Consider the following boundary problem of the heat equation:

$$\begin{cases}u_t = \alpha u_{xx}\\ u(0,t) = 0\\ u(L,t) = 0\\ u(x,0) = u_0(x)\end{cases} \tag{13}$$

with $\alpha = \text{const}$. The solution $u(x,t)$ to Eq. (13) is:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\frac{\alpha\pi^2 n^2}{L^2}t} \sin\left(\frac{\pi n x}{L}\right)$$

where:

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x$$

Sketch of the proof. Use separation of variables. □

Remark. Note that unlike the wave equation, the heat equation is infinitely many times differentiable for any time $t > 0$ although it is not defined for negative times.

Proposition 54. Consider the simplified Schrödinger equation:

$$\begin{cases}\mathrm{i}u_t = u_{xx}\\ u(0,t) = 0\\ u(L,t) = 0\\ u(x,0) = u_0(x)\end{cases} \tag{14}$$

The solution $u(x,t)$ to Eq. (14) is:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\mathrm{i}\frac{\pi^2 n^2}{L^2}t} \sin\left(\frac{\pi n x}{L}\right)$$

where:

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x$$

Sketch of the proof. Use separation of variables. □

Definition 55. A function $f(x,t)$ is said to be *self-similar* if $\exists \alpha, \beta \in \mathbb{R}$ such that $f(x,t) = t^\beta \varphi\left(\frac{x}{t^\alpha}\right)$ for some function $\varphi : \mathbb{R} \to \mathbb{R}$.

Proposition 56. Consider the heat equation of constant coefficients $u_t = \alpha u_{xx}$ on the whole real line. Then, if we impose u being self-similar satisfying $u(x,t) = u(\lambda x, \lambda^2 t)$ $\forall \lambda > 0$, we obtain:

$$u(x,t) = C_1 \int_0^{\frac{x}{\sqrt{t}}} \mathrm{e}^{-\frac{z^2}{4\alpha}}\,\mathrm{d}z + C_2 \tag{15}$$

for certain constants $C_1, C_2 \in \mathbb{R}$.

Sketch of the proof. Observe that $u(x,t) = f(\frac{x}{\sqrt{t}}) =: f(s)$ and the heat equation is transformed into $-f's = 2\alpha f''$. The solution of this ODE is straightforward. □

Distributions

Definition 57. Let $\Omega \subseteq \mathbb{R}^n$ be a set. We denote $\mathcal{C}_0^\infty(\Omega) =: \mathcal{D}(\Omega)$. We define the space $L^1_{\mathrm{loc}}(\Omega)$ as the set of all locally integrable functions on Ω.

Definition 58 (Distribution). Let $\Omega \subseteq \mathbb{R}^n$ be a set. A *distribution* on Ω is a continuous linear form on $\mathcal{D}(\Omega)$. The vector space of all distributions on Ω is denoted by $\mathcal{D}^*(\Omega)$.

Proposition 59. Let $\Omega \subseteq \mathbb{R}^n$ be a set and $f \in L^1_{\mathrm{loc}}(\Omega)$. Then, the map

$$\begin{aligned}\Lambda_f : \mathcal{D}(\Omega) &\longrightarrow \mathbb{R}\\ \varphi &\longmapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\end{aligned}$$

is a distribution. Hence, $\Lambda_f(\varphi)$ is usually denoted by $\langle f, \varphi\rangle$. Sometimes we will do an abuse of notation denoting Λ_f as f (by the Theorem 28).

Proof. Λ_f is clearly linear. Moreover:

$$|\Lambda_f(\varphi)| \leq \int_\Omega |f(\mathbf{x})\varphi(\mathbf{x})| \leq \|f\|_1 \|\varphi\|_\infty$$

Hence, Λ_f is bounded and therefore continuous. □

Proposition 60 (Dirac's δ distribution). Let $\Omega \subseteq \mathbb{R}^n$ be a set and $\mathbf{x}_0 \in \Omega$. Then, the map

$$\begin{aligned}\delta_{\mathbf{x}_0} : \mathcal{D}(\Omega) &\longrightarrow \mathbb{R}\\ \varphi &\longmapsto \varphi(\mathbf{x}_0)\end{aligned}$$

is a distribution. We will denote $\delta_{\mathbf{0}}$ simply by δ.

Proof. Clearly $\delta_{\mathbf{x}_0}$ is linear and bounded because $|\delta_{\mathbf{x}_0}(\varphi)| = |\varphi(\mathbf{x}_0)| \leq \|\varphi\|_\infty$. □

Lemma 61. Let $\Omega \subseteq \mathbb{R}^n$ be a set, $\mathbf{x}_0 \in \Omega$ and $\mu_{\mathbf{x}_0}$ be the measure that equals 1 on the set $\{\mathbf{x}_0\}$ and 0 on the sets disjoint from $\{x_0\}$. Then, $\forall \varphi \in \mathcal{D}(\Omega)$ we have:

$$\delta_{\mathbf{x}_0}(\varphi) = \varphi(\mathbf{x}_0) = \int_\Omega \varphi \,\mathrm{d}\mu_{\mathbf{x}_0}$$

Definition 62. Let $\Omega \subseteq \mathbb{R}^n$ be a set and $n \in \mathbb{N}$. We define the *differentiation operator* $D^n : \mathcal{D}^*(\Omega) \to \mathcal{D}^*(\Omega)$ by:

$$\langle D^n\Lambda, \varphi\rangle = \langle \Lambda, (-1)^n D^n\varphi\rangle$$

for all $\Lambda \in \mathcal{D}^*(\Omega)$ and all $\varphi \in \mathcal{D}(\Omega)$. The distribution $D^n\Lambda$ is called *distributional derivative.*

Definition 63. We define the *Heaviside step function* as the function $H(x) = \mathbf{1}_{x>0}$.

Proposition 64. We have that $\Lambda_H =: H \in \mathcal{D}^*(\mathbb{R})$ and:

$$H' = \delta$$

Proof. For all $\varphi \in \mathcal{D}(\Omega)$ we have:

$$\begin{aligned}\langle H', \varphi\rangle = -\langle H, \varphi'\rangle &= -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,\mathrm{d}x\\ &= -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0) = \delta(\varphi)\end{aligned}$$

because φ has compact support. □

Fundamental solution

Definition 65. A *fundamental solution* (or *heat kernel*) is a solution of the heat equation corresponding to the initial condition of an initial point source of heat at a known position. That is, it is the solution to the problem:

$$\begin{cases} u_t = \alpha u_{xx} \\ \lim\limits_{t\to 0} \Lambda_{u(\cdot,t)}(\varphi) = \delta(\varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R})\end{cases} \tag{16}$$

where δ is the Dirac delta distribution.

Theorem 66. The heat kernel of Eq. (16) is:

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{x^2}{4\alpha t}} \tag{17}$$

Proof. An easy check shows that if u is a solution to the heat equation, so it is u_x. Thus, from this fact and Eq. (15) we get the solution:

$$u(x,t) = \frac{C}{\sqrt{t}}\mathrm{e}^{-\frac{x^2}{4\alpha t}}$$

Imposing $\int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x = 1$ we get the desired result. Let's see now that $\lim\limits_{t\to 0}\Lambda_{u(\cdot,t)}(\varphi) = \delta(\varphi)$ $\forall \varphi \in \mathcal{D}(\mathbb{R})$. Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|\varphi(x) - \varphi(0)| < \frac{\varepsilon}{2K}$ whenever $|x| < \delta$, where $K = \int_{|x|<\delta} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}}$.

$$\begin{aligned} I &= \left|\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}}\varphi(y)\,\mathrm{d}y - \varphi(0)\right| \\ &= \left|\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}}(\varphi(y) - \varphi(0))\,\mathrm{d}y\right| \\ &\leq \int_{|x|<\delta} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}}|\varphi(y) - \varphi(0)|\,\mathrm{d}y \\ &\qquad + \int_{|x|\geq\delta} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}}|\varphi(y) - \varphi(0)|\,\mathrm{d}y\end{aligned}$$

The first integral is bounded by $\frac{\varepsilon}{2}$, while for the second one, given that δ we can find $t > 0$ such that $\int_{|x|\geq\delta} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}} \leq \frac{\varepsilon}{4\|\varphi\|_\infty}$. Finally, for $t \to 0$:

$$I \leq \frac{\varepsilon}{2} + 2\|\varphi\|_\infty \int_{|x|\geq\delta} \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}} < \varepsilon$$

This is valid $\forall \varphi \in \mathcal{D}(\mathbb{R})$. Hence, $\lim\limits_{t\to 0}\Lambda_{u(\cdot,t)}(\varphi) = \delta(\varphi)$ $\forall \varphi \in \mathcal{D}(\mathbb{R})$. □

Corollary 67. Let $[H(t)](x)$ be the heat kernel of Eq. (17) at a fixed point $t > 0$. Then, the general solution to the problem

$$\begin{cases} u_t = \alpha u_{xx} \\ u(x,0) = f(x) \end{cases} \tag{18}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded is:

$$u(x,t) = [H(t) * f](x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha t}} \mathrm{e}^{-\frac{(x-y)^2}{4\alpha t}} f(y)\,\mathrm{d}y$$

Sketch of the proof. Clearly the heat equation holds by construction. The proof of $\lim_{t\to 0} \|H(t) * f - f\|_\infty = 0$ follows in the same way as the one in Theorem 66. □

Proposition 68. Let $[H(t)](x)$ be the heat kernel of Eq. (17) at a fixed point $t \geq 0$. Then, $\forall s, t > 0$ we have:

$$H(s+t) = H(s) * H(t)$$

Proof. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}
&[H(s) * H(t)](x) = \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\alpha s}} \mathrm{e}^{-\frac{(x-y)^2}{4\alpha s}} \frac{1}{\sqrt{4\pi\alpha t}} \mathrm{e}^{-\frac{y^2}{4\alpha t}}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi\alpha s}} \frac{1}{\sqrt{4\pi\alpha t}} \mathrm{e}^{-\frac{x^2}{4\alpha(s+t)}} \int_{-\infty}^{+\infty} \mathrm{e}^{\frac{x^2}{4\alpha(s+t)} - \frac{(x-y)^2}{4\alpha s} - \frac{y^2}{4\alpha t}}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi\alpha s}} \frac{1}{\sqrt{4\pi\alpha t}} \mathrm{e}^{-\frac{x^2}{4\alpha(s+t)}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(s+t)\left(y - \frac{xt}{s+t}\right)^2}{4\alpha s t}}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi\alpha s}} \frac{1}{\sqrt{4\pi\alpha t}} \mathrm{e}^{-\frac{x^2}{4\alpha(s+t)}} \sqrt{4\pi\alpha \frac{st}{s+t}} \\
&= \frac{1}{\sqrt{4\pi\alpha(s+t)}} \mathrm{e}^{-\frac{x^2}{4\alpha(s+t)}} \\
&= [H(s+t)](x)
\end{aligned}$$

□

Proposition 69. Consider the generalized n-th dimensional heat equation:

$$u_t = \alpha \Delta u \tag{19}$$

Then, the *generalized heat kernel* for this equation is:

$$[H(t)](\mathbf{x}) = \frac{1}{(4\pi\alpha t)^{n/2}} \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{4\alpha t}}$$

Its associated integral form can be written as the operator:

$$T(t)u_0 = \int_{\mathbb{R}^n} \frac{1}{(4\pi\alpha t)^{n/2}} \mathrm{e}^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4\alpha t}} u_0(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{20}$$

?? **??** gives a representation of $T(t)$ in the form $T(t) = \mathrm{e}^{t\Delta}$.

Sketch of the proof. An easy check shows that $[H(t)](\mathbf{x})$ solves the heat equation $u_t = \alpha \Delta u$. To show that the initial condition holds, use the 1-dimensional case (Theorem 66) and **??** **??**. □

Proposition 70. Consider the operator $T(t)$ defined on Eq. (20). Then, $\{T(t) : t \in \mathbb{R}_{\geq 0}\}$ is a semigroup with the composition. That is, $T(0) = \mathrm{id}$ and $T(s) \circ T(t) = T(s+t)$.

Sketch of the proof. It is a consequence of the generalization of $\lim_{t\to 0} \|H(t) * f - f\|_\infty = 0$, which it can be proven using the 1-dimensional case (Theorem 67) and **??** **??**. □

Lemma 71. The function

$$u(x,t) = \begin{cases} \frac{1}{\sqrt{\alpha |t|}} \mathrm{e}^{-\frac{x^2}{4\alpha t}} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \text{ and } x \neq 0 \end{cases}$$

is a solution to the heat equation for $t > 0$ and also for $t < 0$.

Operators

Definition 72 (Explicit scheme in finite differences). Let E be a Banach space and $A : E \to E$ be an linear operator. Consider the following ivp:

$$\begin{cases} u_t = Au \\ u(x,0) = u_0(x) \end{cases} \tag{21}$$

We would like to extend the notion of **??**. Thus for $n \gg t$ we can rewrite the previous equation as:

$$u(t + t/n) \simeq \left(I + \frac{t}{n} A\right) u(t)$$

Thus, taking the limit as $n \to \infty$ we can conclude:

$$u(t) = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^n u(0) =: \mathrm{e}^{tA} u(0)$$

Note that for this to be well-defined we need that A must be a bounded operator. And in that case, the following identity also holds:

$$\mathrm{e}^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

Definition 73 (Implicit scheme in finite differences). Let E be a Banach space and $A : E \to E$ be an linear operator. Consider the ivp of Eq. (21) and rewrite it this time as:

$$u(t) \simeq \left(I - \frac{t}{n} A\right)^{-1} u(t - t/n)$$

for $n \gg t$. Thus, taking the limit as $n \to \infty$ we can conclude:

$$u(t) = \lim_{n\to\infty} \left(I - \frac{t}{n} A\right)^{-n} u(0) =: \mathrm{e}^{tA} u(0)$$

Note that for this to be well-defined we need that A^{-1} must be a bounded operator[2].

[2] It can be proved that this definition of exponential matrix for an operator is more appropriate for the differential operators than the previous one. Also, computationally is more efficient.

Proposition 74 (Duhamel principle). Let E be a Banach space, $D : E \to E$ be a linear differential operator that involves no time derivatives and $F : E \to E$ be a functional. Consider the following ivp:

$$\begin{cases} u_t = Du + F(u) \\ u(x,0) = u_0(x) \end{cases} \tag{22}$$

Then, the general solution to this problem is the solution to the following integral equation:

$$u(t) = \mathrm{e}^{tD}u_0 + \int_0^t \mathrm{e}^{(t-s)D}F(u(s))\,\mathrm{d}s$$

Sketch of the proof. The solution of the homogeneous system is $\mathrm{e}^{tD}u_0$. Let $u(t) = \mathrm{e}^{tD}g$. We will use the variation of constants method to find the solution. Imposing that $u(t)$ has to be the solution we have:

$$D\mathrm{e}^{tD}g + \mathrm{e}^{tD}g' = D\mathrm{e}^{tD}g + F(u) \iff \mathrm{e}^{tD}g' = F(u)$$

And integrating we get:

$$g(t) = u_0 + \int_0^t \mathrm{e}^{-sD}F(u(s))\,\mathrm{d}s$$

□

Maximum and minimum principles

Definition 75. Let $U \subset \mathbb{R}^n$ be open and bounded and fix a time $t = T$. We define the *parabolic cylinder* as $U_T := U \times (0,T]$. We define the *parabolic boundary* as $\Gamma_T = \overline{U_T} \setminus U_T = \partial U_T \setminus (U \times \{T\})$.

Theorem 76 (Maximum principle). Let $U \subset \mathbb{R}^n$ be open and bounded and fix a time $t = T$. Suppose $u \in \mathcal{C}^2_1(U_T) \cap \mathcal{C}(\overline{U_T})$ solve the heat equation in U_T. Then:

$$\max\{u(\mathbf{x},t) : (\mathbf{x},t) \in \overline{U_T}\} = \max\{u(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\}$$

Proof. Let $v \in \mathcal{C}^2_1(U_T) \cap \mathcal{C}(\overline{U_T})$ such that $v_t - \alpha\Delta v < 0$. Then, $\max\{v(\mathbf{x},t) : (\mathbf{x},t) \in \overline{U_T}\} = \max\{v(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\}$. Indeed, if the maximum was in U_T or $U \times \{T\}$ we would have $v_t \geq 0$ and $\Delta v \leq 0$, which contradicts $v_t - \alpha\Delta v < 0$ because $\alpha > 0$.
Now take $v = u - \varepsilon t$ with $\varepsilon > 0$. We have that:

$$v_t - \alpha\Delta v = u_t - \Delta u - \alpha\varepsilon = -\varepsilon < 0$$

Thus:

$$\begin{aligned} u &= v + \varepsilon t \\ &\leq \max\{v(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\} + \varepsilon t \\ &\leq \max\{u(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\} + \varepsilon t \end{aligned}$$

for all $\varepsilon > 0$ and all $t \in [0,T]$. □

Theorem 77 (Minimum principle). Let $U \subset \mathbb{R}^n$ be open and bounded and fix a time $t = T$. Suppose $u \in \mathcal{C}^2_1(U_T) \cap \mathcal{C}(\overline{U_T})$[3] solves the heat equation in U_T. Then:

$$\min\{u(\mathbf{x},t) : (\mathbf{x},t) \in \overline{U_T}\} = \min\{u(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\}$$

Sketch of the proof. Apply the 76 Maximum principle to the function $-u(\mathbf{x},t)$. □

Theorem 78 (Uniqueness of the heat equation). Let $U \subset \mathbb{R}^n$ be open and bounded, $g \in \mathcal{C}(\Gamma_T)$ and $f \in \mathcal{C}(U_T)$. Then, there exists at most one solution $u \in \mathcal{C}^2_1(U_T) \cap \mathcal{C}(\overline{U_T})$ of the problem:

$$\begin{cases} u_t - \alpha\Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Sketch of the proof. Suppose u_1 and u_2 are two solutions of this problem. Apply both 76 Maximum principle and 77 Minimum principle to the function $u_1 - u_2$. □

Theorem 79 (Maximum principle on unbounded domains). Let $g \in \mathcal{C}(\mathbb{R}^n)$. Suppose $u \in \mathcal{C}^2(\mathbb{R}^n \times (0,T]) \cap \mathcal{C}(\mathbb{R}^n \times [0,T])$ solves the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

and satisfies that $u(\mathbf{x},t) \leq A\mathrm{e}^{a\|\mathbf{x}\|^2}$ $\forall(\mathbf{x},t) \in \mathbb{R}^n \times [0,T]$ and for some constants $a, A > 0$. Then:

$$\sup\{u(\mathbf{x},t) : (\mathbf{x},t) \in \mathbb{R}^n \times [0,T]\} = \sup\{g(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$$

Proof. First divide $[0,T]$ into subintervals with size $\ell < \frac{1}{4a}$. It suffices to prove the claim on one of such subintervals. So from now on assume $T < \frac{1}{4a}$. Let $\mathbf{y} \in \mathbb{R}^n$ and consider the function

$$v(\mathbf{x},t) = u(\mathbf{x},t) - \frac{\delta}{(T+\varepsilon-t)^{n/2}}\mathrm{e}^{\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(T+\varepsilon-t)}}$$

with $\varepsilon > 0$ such that $T + \varepsilon < \frac{1}{4a}$ and $\delta > 0$. It can be checked that $v_t - \Delta v = 0$. Now fix $r > 0$ and consider the set $U_T = B(\mathbf{y},r) \times (0,T]$. By the 76 Maximum principle, we have that:

$$\begin{aligned} &\max\{v(\mathbf{x},t) : (\mathbf{x},t) \in \overline{U_T}\} = \\ &\qquad = \max\{v(\mathbf{x},t) : (\mathbf{x},t) \in \Gamma_T\} =: M \end{aligned}$$

Now let's prove that we can bound M by $\sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})$. If $\mathbf{x} \in \mathbb{R}^n$ and $t = 0$, then:

$$v(\mathbf{x},0) = u(\mathbf{x},0) - \frac{\delta}{(T+\varepsilon)^{n/2}}\mathrm{e}^{\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(T+\varepsilon)}} \leq u(\mathbf{x},0) = g(\mathbf{x})$$

And if $\|\mathbf{x}-\mathbf{y}\| = r$ and $t \in [0,T]$, then:

$$\begin{aligned} v(\mathbf{x},t) &= u(\mathbf{x},t) - \frac{\delta}{(T+\varepsilon-t)^{n/2}}\mathrm{e}^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq A\mathrm{e}^{a\|\mathbf{x}\|^2} - \frac{\delta}{(T+\varepsilon)^{n/2}}\mathrm{e}^{\frac{r^2}{4(T+\varepsilon)}} \\ &\leq A\mathrm{e}^{a\|\mathbf{y}\|^2+ar^2} - \frac{\delta}{(T+\varepsilon)^{n/2}}\mathrm{e}^{\frac{r^2}{4(T+\varepsilon)}} \\ &\leq A\mathrm{e}^{a\|\mathbf{y}\|^2+ar^2} - \frac{\delta}{(T+\varepsilon)^{n/2}}\mathrm{e}^{ar^2+\gamma r^2} \end{aligned}$$

[3] Here the subindex 1 in $\mathcal{C}^2_1(U_T)$ indicates that the differentiability is with respect to the first component of u, that is, with respect to $\mathbf{x}$.

where $\gamma > 0$ satisfies $a + \gamma = \frac{1}{4(T+\varepsilon)}$. Letting $r \to \infty$ this last inequality is bounded by $\sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})$. So $\forall \mathbf{y} \in \mathbb{R}^n$ and all $t \in [0, T]$ we have

$$v(\mathbf{y}, t) \leq \sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})$$

Finally, letting $\delta \to 0$ we get $\forall \mathbf{y} \in \mathbb{R}^n$ and all $t \in [0, T]$ the desired result:

$$u(\mathbf{y}, t) \leq \sup_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x})$$

□

Theorem 80 (Minimum principle on unbounded domains). Let $g \in \mathcal{C}(\mathbb{R}^n)$. Suppose $u \in \mathcal{C}^2(\mathbb{R}^n \times (0, T]) \cap \mathcal{C}(\mathbb{R}^n \times [0, T])$ solves the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

and satisfies that $u(\mathbf{x}, t) \geq -Ae^{a\|\mathbf{x}\|^2}$ $\forall(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$ and for some constants $a, A > 0$. Then:

$$\inf\{u(\mathbf{x}, t) : (\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]\} = \inf\{g(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$$

Sketch of the proof. Apply the 79 Maximum principle on unbounded domains to the function $-u(\mathbf{x}, t)$. □

Theorem 81 (Uniqueness of the heat equation on the unbounded domains). Let $g \in \mathcal{C}(\mathbb{R}^n)$ and $f \in \mathcal{C}(\mathbb{R}^n \times [0, T])$. Then, there exists at most one solution $u \in \mathcal{C}^2(\mathbb{R}^n \times (0, T]) \cap \mathcal{C}(\mathbb{R}^n \times [0, T])$ of the problem:

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

satisfying $|u(\mathbf{x}, t)| \leq Ae^{a\|\mathbf{x}\|^2}$ $\forall(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$ and for some constants $a, A > 0$.

Sketch of the proof. Suppose u_1 and u_2 are two solutions of this problem. Apply both the 79 Maximum principle on unbounded domains and 80 Minimum principle on unbounded domains to the function $u_1 - u_2$. □

5. | Laplace equation

General properties and solutions

Definition 82 (Laplace equation). Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an unknown function. The *Laplace equation* is the PDE defined by:

$$\Delta u = 0$$

Proposition 83 (Dirichlet problem in the disc). Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function such that $f(0) = f(2\pi)$. Then, there exists a continuous function $v : \overline{D(0, \rho)} \to \mathbb{R}$ that $v \in \mathcal{C}^2(D(0, \rho) \setminus \{0\})$ and such that:

1. $v(r, 0) = v(r, 2\pi)$ $\forall r \in [0, \rho]$
2. $\Delta v = 0$.
3. $v(\rho, \theta) = f(\theta)$ $\forall \theta \in [0, 2\pi]$

An example of such function is:

$$v(r, \theta) = \sum_{n=0}^{\infty} \frac{r^n}{\rho^n} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

where:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, \mathrm{d}\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, \mathrm{d}\theta$$

Sketch of the proof. The Laplacian in polar coordinates is:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Now use separation of variables $v(r, \theta) = R(r)\Theta(\theta)$ imposing that $\Theta(\theta)$ must be 2π-periodic. □

Definition 84 (Dirichlet problem). Let $U \subseteq \mathbb{R}^n$ be an open bounded set such that ∂U is of class $\mathcal{C}^1$, $f \in \mathcal{C}(\Omega)$ and $g \in \mathcal{C}(\partial\Omega)$. The *Dirichlet problem* is defined as the following ivp:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{23}$$

Proposition 85 (Uniqueness of Dirichlet problem). Let $U \subseteq \mathbb{R}^n$ be an open bounded set such that ∂U is of class $\mathcal{C}^1$, $f \in \mathcal{C}^2(\Omega)$ and $g \in \mathcal{C}^2(\partial\Omega)$. Then, there exists at most one solution of Eq. (23).

Sketch of the proof. Suppose u_1 and u_2 are two solutions of the problem. Apply the first of the 30 Green identities to the functions $f = g = u_1 - u_2$. □

Definition 86. Let $U \subseteq \mathbb{R}^n$ be an open bounded set such that ∂U is of class $\mathcal{C}^1$, $f \in \mathcal{C}^2(\Omega)$ and $g \in \mathcal{C}^2(\partial\Omega)$. Considering Eq. (23) we define the *energy functional* as the operator

$$Ew = \int_U \frac{1}{2}\|\boldsymbol{\nabla} w\|^2 - wf$$

defined on the set $\{w \in \mathcal{C}^2(\overline{U}) : w = g \text{ on } \partial U\}$.

Theorem 87 (Dirichlet's principle). Let $U \subseteq \mathbb{R}^n$ be an open bounded set such that ∂U is of class $\mathcal{C}^1$, $f \in \mathcal{C}^2(\Omega)$ and $g \in \mathcal{C}^2(\partial\Omega)$. Then, $u \in \mathcal{C}^2(\overline{U})$ solves Eq. (23) if and only if u minimizes E.

Proof.

$\Longrightarrow$) Let u be a solution of Eq. (23) and take $w \in \mathrm{dom}(E)$ such that $w = u + v$. Thus, $v = 0$ on ∂U. We need to show that $Ew \geq Eu$. A calculation shows that:

$$Ew = Eu + \int_U \frac{1}{2}\|\boldsymbol{\nabla} v\|^2$$

where we have use the 30 Green identities to conclude that:

$$\int_U \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v - vf = 0$$

Hence $Ew \geq Eu$.

$\Longleftarrow$) Let u be a minimizer of E and $\lambda(t) = E(u+vt)$, $t \in \mathbb{R}$. By the definition of a minimum, we have that $\lambda'(0) = 0$ and so:

$$0 = \lambda'(0) = \int_U \nabla u \cdot \nabla v - vf = \int_U v(-\Delta u - f)$$

again by the 30 Green identities. Since, this is valid $\forall v \in \operatorname{dom}(E)$ it follows that $-\Delta u = f$ by Theorem 28.

□

6. | Sobolev spaces

Definition 88. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. We define the *Sobolev space* $H^1(\Omega)$ (or $W^{1,2}(\Omega)$) as the following space:

$$H^1(\Omega) := \{f \in L^2(\Omega) : D_i f \in L^2(\Omega), i = 1, \dots, n\}$$

Here D_i denotes the distributional derivative with respect to the i-th component.

Proposition 89. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. Then, $H^1(\Omega)$ with the inner product

$$\begin{aligned}\langle f, g\rangle_{H^1} :=& \langle f, g\rangle_2 + \sum_{i=1}^n \langle D_i f, D_i g\rangle_2 \\ =& \langle f, g\rangle_2 + \int_\Omega \nabla f \cdot \nabla g\end{aligned}$$

and associated norm

$$\|f\|_{H^1}{}^2 = \|f\|_2{}^2 + \sum_{i=1}^n \|D_i f\|_2{}^2$$

is a Hilbert space.

Sketch of the proof. Clearly, $H^1(\Omega)$ is pre-Hilbert. It's missing to show that $H^1(\Omega)$ is complete. Let $(f_n) \in H^1(\Omega)$ be Cauchy. Then, $(f_n), (D_i f_n) \in L^2(\Omega)$ are also Cauchy as $\|f\|_2 \leq \|f\|_{H^1}$ and $\|D_i f\|_2 \leq \|f\|_{H^1}$ $\forall i = 1, \dots, n$. Hence, $\lim_{n\to\infty} f_n \overset{L^2}{=} G$ and $\lim_{n\to\infty} D_i f_n \overset{L^2}{=} g_i$ for some $G, g_i \in L^2(\Omega)$, $\forall i = 1, \dots, n$. If we prove that $D_i G = g_i$, we will be done. But this is clear from the definition of distributional derivative as $\forall \varphi \in \mathcal{D}(\Omega)$ we have:

$$\int_\Omega D_i f_n \varphi = \int_\Omega f_n D_i \varphi$$

And the ?? ?? allow us to conclude that:

$$\int_\Omega g_i \varphi = \int_\Omega G D_i \varphi$$

□

Definition 90. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. We define the space $H_0^1(\Omega) := \mathrm{Cl}_{H^1(\Omega)}(\mathcal{D}(\Omega))$.

Definition 91. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset and $u \in L^2(\Omega)$. We define the *average* of u over Ω as:

$$\overline{u} := \frac{1}{|\Omega|} \int_\Omega u$$

Theorem 92 (Trace theorem). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset such that $\partial\Omega$ is of class $\mathcal{C}^1$. Then, there exists a bounded linear operator

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that:

$$Tu = u|_{\partial\Omega} \qquad \forall u \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$$

We call Tu the *trace* of u on $\partial\Omega$[4].

Theorem 93. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset such that $\partial\Omega$ is of class $\mathcal{C}^1$ and let $u \in H^1(\Omega)$. Then:

$$u \in H_0^1(\Omega) \iff u|_{\partial\Omega} = 0$$

Proposition 94 (Poincaré inequality). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset and $u \in H^1(\Omega)$. Then, there exists $C \in \mathbb{R}$ such that:

$$\int_\Omega (u - \overline{u})^2 \leq C \int_\Omega \|\nabla u\|^2$$

Proposition 95. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. Consider the map:

$$\begin{aligned} Q : H^1(\Omega) &\longrightarrow H^1(\Omega) \ominus \mathbb{R} \\ u &\longmapsto u - \overline{u} \end{aligned}$$

Then, the space $H(\Omega) := Q(H^1(\Omega))$ equipped with the inner product

$$\langle f, g\rangle_H = \int_\Omega \nabla f \cdot \nabla g$$

is Hilbert.

Proof. First of all the map is well-defined. Indeed, if $Q(u) = \text{const.}$, then $u = \text{const.}$ But in this case $u = \overline{u}$ and so $Q(u) = 0$. Let's see now that $H(\Omega)$ is Hilbert. Clearly $H(\Omega)$ is pre-Hilbert as Q is linear and continuous. To show the completeness note that the norms on $H(\Omega)$ and $H^1(\Omega)$ are equivalent. Indeed by the 94 Poincaré inequality we have that $\forall \tilde{u} \in H(\Omega)$:

$$\int_\Omega \|\nabla u\|^2 \leq \int_\Omega (u^2 + \|\nabla u\|^2) \leq (C+1) \int_\Omega \|\nabla u\|^2$$

for certain $C \in \mathbb{R}$ and because $\overline{\tilde{u}} = 0$. Thus, the Cauchy convergence is the same with both norms. And since $H^1(\Omega)$ is complete, so it is $H(\Omega)$. □

Proposition 96. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. Then, $H_0(\Omega) := Q(H_0^1(\Omega))$ is closed in $H(\Omega)$.

[4] From now on we will call Tu as $u|_{\partial\Omega}$.

Proof. Let $(w_n) \in Q(H_0^1(\Omega))$ be a sequence that converges in $H_0^1(\Omega)$ to w. We need to show that $w \in Q(H_0^1(\Omega))$. Note that $w_n = Q(u_n) = u_n - \overline{u_n}$ for certain $(u_n) \in H_0^1(\Omega)$. Thus, $\overline{u_n} = u_n - w_n$. By Theorem 93 we have that $\overline{u_n} = -w_n|_{\partial\Omega}$ which converges in $L^2(\partial\Omega)$ by the continuity of the trace and therefore, as $L^2(\partial\Omega)$, $\overline{u_n}$ converges on $\mathbb{R}$. Let $c := \lim_{n\to\infty} \overline{u_n}$. Now, $u = w_n + \overline{u_n}$ converges in $H_0^1(\Omega)$ to $w + c =: u$. We claim that $c = \overline{u}$, which is clear by the continuity of the average. Hence, $w = u - \overline{u} = Q(u)$ with $u \in H_0^1(\Omega)$ because $H_0^1(\Omega)$ is closed. So $w \in Q(H_0^1(\Omega))$. □

Proposition 97. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset and $\tilde{g} \in H(\Omega)$. Then, $\exists! \tilde{u} \in H_0(\Omega)^\perp = H(\Omega) \ominus H_0(\Omega)$ such that

$$\tilde{u} = \underset{w \in H_0(\Omega)}{\arg\min} \left\{ \int_\Omega \frac{1}{2} \|\nabla w\|^2 : \tilde{g} - w \in H_0(\Omega) \right\}$$

Sketch of the proof. Use the **?? ??**. □

Theorem 98. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset. Consider the Dirichlet problem of Eq. (23) with $f = 0$ and $g \in H^1(\Omega)$. Then, this problem has existence and uniqueness of solutions.

Proof. Let $\tilde{g} = Q(g) = g - \overline{g} \in H(\Omega)$ and $\tilde{u} \in H_0(\Omega)^\perp$ be the minimizer of Theorem 97 given $\tilde{g}$. Thus, $\tilde{u} - \tilde{g} \in H_0(\Omega)$ and so $\tilde{u} - \tilde{g} = v - \overline{v}$, $v \in H_0^1(\Omega)$. Define $u := \tilde{u} + \overline{g} + \overline{v}$. Note that $u - g = v \in H_0^1(\Omega)$ and Theorem 93 implies $u = g$ on $\partial\Omega$. It's missing to show that u minimizes E. But this is clear from the fact that $u - \tilde{u} \in \mathbb{R}$ and so $\|u\|_{H(\Omega)} = \|\tilde{u}\|_{H(\Omega)}$ and the existence and uniqueness of $\tilde{u}$. □