Harmonic analysis

1. | Introduction

Refer to ?? ?? for a reminder of the introductory concepts of Fourier series.

Uniform convergence

Theorem 1. Let f be a continuous T-periodic function such that f' exists except for a finite number of points and it is continuous and bounded. Then, $S_N f$ converges uniformly to f on $[-T/2, T/2]$.

Proof. We have pointwise convergence towards f. Moreover:

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right| &\leq \left|\widehat{f}(0)\right| + \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}n\left|\widehat{f}(n)\right| \\ &\leq \left|\widehat{f}(0)\right| + \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{n^2}+n^2\left|\widehat{f}(n)\right|^2\right) \\ &= \left|\widehat{f}(0)\right| + \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n^2}+\frac{T^2}{8\pi^2}\sum_{n\in\mathbb{Z}}\left|\widehat{f'}(n)\right|^2 \\ &\leq \left|\widehat{f}(0)\right| + \frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n^2}+\frac{T}{8\pi^2}\|f'\|^2 \\ &< \infty\end{aligned}$$

by ?? ?? and because f' is bounded. Thus, the ?? ?? implies that $S_N f$ converges uniformly to f. □

Corollary 2. Let $f \in \mathcal{C}^{r-1}$ be a T-periodic function such that $f^{(r)}$ exists except for a finite number of points and it is continuous and bounded. Then:

$$\sup_{x\in[-T/2,T/2]}|S_N f(x) - f(x)| \leq \frac{\varepsilon_N}{N^{r-1/2}}$$

for some sequence $(\varepsilon_N) \overset{N\to\infty}{\longrightarrow} 0$.

Proof. By ?? we have:

$$\begin{aligned}|S_N f(x) - f(x)| &\leq \sum_{n>|N|}\frac{1}{n^r}n^r\left|\widehat{f}(n)\right| \\ &\leq \left(\sum_{n>|N|}\frac{1}{n^{2r}}\right)^{\frac{1}{2}}\left(\sum_{n>|N|}n^{2r}\left|\widehat{f}(n)\right|^2\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_N^\infty\frac{1}{x^{2r}}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\sum_{n>|N|}\left|\widehat{f^r}(n)\right|^2\right)^{\frac{1}{2}} \\ &= \frac{\tilde{C}}{N^{r-1/2}}\varepsilon_N\end{aligned}$$

with $\varepsilon_N \overset{N\to\infty}{\longrightarrow} 0$ because it is the tail of a convergent sequence. □

Poisson kernel

For most of the proofs in this section check the analogous ones with the ?? ??.

Definition 3 (Poisson kernel). Let $r \in [0,1]$. We define the *Poisson kernel* as

$$P_r(t) = \sum_{n\in\mathbb{Z}} r^{|n|}\mathrm{e}^{\frac{2\pi i n t}{T}}$$

Lemma 4. Let $r \in [0,1]$. Then:

$$P_r(t) = \frac{1-r^2}{1-2r\cos\left(\frac{2\pi t}{T}\right)+r^2}$$

Sketch of the proof. Use the geometric progression formula. □

Proposition 5. The Poisson kernel has the following properties:

1. P_r is a T-periodic, even and non-negative function.
2. $\displaystyle\frac{1}{T}\int_{-T/2}^{T/2} P_r(t)\,\mathrm{d}t = 1 \quad \forall N.$
3. $\forall\delta > 0,\ \displaystyle\lim_{r\to 1^-}\sup\{|P_r(t)| : \delta \leq |t| \leq T/2\} = 0.$

Theorem 6. Let $f \in L^1([-T/2, T/2])$ be a function having left- and right-sided limits at point x_0. Then:

$$\lim_{r\to 1^-} f * P_r = \frac{f({x_0}^+) + f({x_0}^-)}{2}$$

In particular, if f is continuous at x_0, $\lim_{r\to 1^-} f * P_r = f(x_0)$.

Theorem 7. Let $p \geq 1$ and $f \in L^p([-T/2, T/2])$. Then:

$$\lim_{N\to\infty}\|\sigma_N f - f\|_p = 0$$
$$\lim_{r\to 1^-}\|f * P_r - f\|_p = 0$$

2. | Fourier transform

Definition and first properties

Definition 8. Let $f \in L^1(\mathbb{R})$. We define the *Fourier transform* of f as:

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-2\pi i\xi x}\,\mathrm{d}x$$

The function f is also called *inverse Fourier transform* of $\widehat{f}$.

Proposition 9. Let $f, g \in L^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$. Then:

1. $\widehat{(\alpha f + \beta g)}(\xi) = \alpha\widehat{f}(\xi) + \beta\widehat{g}(\xi)$
2. Let $h \in \mathbb{R}$. We define $T_h f(x) = f(x+h)$. Then:

$$\widehat{T_h f}(\xi) = \mathrm{e}^{2\pi i\xi h}\widehat{f}(\xi)$$

3. If $g(x) = \mathrm{e}^{2\pi \mathrm{i} x h} f(x)$, then:
$$\widehat{g}(\xi) = \widehat{f}(\xi - h)$$
4. If $\lambda \in \mathbb{R}^*$, then:
$$\frac{1}{\lambda}\widehat{f\left(\frac{x}{\lambda}\right)}(\xi) = \widehat{f}(\lambda\xi)$$
5. If $g(x) = \overline{f(x)}$, then:
$$\widehat{g}(\xi) = \overline{\widehat{f}(-\xi)}$$

Sketch of the proof. They follow from the linearity of the integral and some change of variable. □

Definition 10. Let $f \in L^1(\mathbb{R})$. We define the *Fourier transform operator* as $\mathcal{F}f = \widehat{f}$.

Proposition 11. Let $f \in L^1(\mathbb{R})$. Then:

1. $\mathcal{F}f$ is uniformly continuous.
2. $\mathcal{F}$ is a continuous linear operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ and $\|\mathcal{F}f\|_\infty \leq \|f\|_1$.

Proof.

1. Using Item 9-3 we have:
$$|\mathcal{F}f(\xi + h) - \mathcal{F}f(\xi)| \leq \int_{-\infty}^{+\infty} \left|\mathrm{e}^{-2\pi \mathrm{i} x h} - 1\right| |f(x)| \,\mathrm{d}x$$
By the ?? ?? we have that the integral is bounded by $2\|f\|_1$ and so entering the limit we obtain the bound $\varepsilon \|f\|_1$ $\forall \varepsilon > 0$. As the bound does not depend on the point ξ, the convergence is uniform.
2. Clearly $\|\mathcal{F}f\|_\infty \leq \|f\|_1$. Hence the operator is bounded and therefore continuous.

□

Theorem 12 (Riemann-Lebesgue lemma). Let $f \in L^1(\mathbb{R})$. Then:
$$\lim_{|\xi|\to\infty} \left|\widehat{f}(\xi)\right| = 0$$

Sketch of the proof. Note that $2\left|\widehat{f}(\xi)\right| = \left|\widehat{f}(\xi) - \mathrm{e}^{\mathrm{i}\pi}\widehat{f}(\xi)\right|$ and:
$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\pi}\widehat{f}(\xi) &= \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x + \mathrm{i}\pi} \,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} f\left(u + \frac{1}{2\xi}\right)\mathrm{e}^{-2\pi \mathrm{i}\xi u} \,\mathrm{d}u
\end{aligned}$$
So:
$$\left|\widehat{f}(\xi)\right| \leq \frac{1}{2}\int_{-\infty}^{+\infty} \left|f(x) - f\left(x + \frac{1}{2\xi}\right)\right| \mathrm{d}x$$
Now use again the ?? ??. □

Proposition 13. Let $f, g \in L^1(\mathbb{R})$. Then, $f\widehat{g}, \widehat{f}g \in L^1(\mathbb{R})$ and:
$$\int_{-\infty}^{+\infty} \widehat{f}(x)g(x) \,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\widehat{g}(x) \,\mathrm{d}x$$

Sketch of the proof. By Theorem 11, $\widehat{g}$ is bounded. Hence, $f\widehat{g} \in L^1(\mathbb{R})$ and the same applies for $\widehat{f}g$. For the equality, use ?? ??. □

Proposition 14. Let f be a function such that $x^k f \in L^1(\mathbb{R})$ for $k = 0, \ldots, r$. Then, $\widehat{f}$ is r times differentiable and:
$$(\mathcal{F}f)^{(k)} = \mathcal{F}((-2\pi \mathrm{i} x)^k f(x))$$
for $k = 0, 1, \ldots, r$.

Proof. Note that the function $h : \xi \to \mathrm{e}^{-2\pi \mathrm{i}\xi x} f(x)$ is $\mathcal{C}^\infty(\mathbb{R})$ and $h^{(k)}(\xi) = (-2\pi \mathrm{i} x)^k \mathrm{e}^{-2\pi \mathrm{i}\xi x} f(x)$. Since $\left|h^{(k)}(\xi)\right| \leq \left|x^k f(x)\right|$ we can use ?? to conclude the result. □

Proposition 15. Let $f \in L^1(\mathbb{R})$ be such that $f^{(k)} \in L^1(\mathbb{R})$ for $k = 1, \ldots, r$. Then:
$$\widehat{f^{(k)}}(\xi) = (2\pi \mathrm{i}\xi)^k \widehat{f}(\xi)$$
for $k = 0, 1, \ldots, r$.

Proof. We'll prove it by induction on k. The case $k = 0$ is clear. For the other ones note that $\exists (a_n), (b_n) \in \mathbb{R}$ with $\lim_{n\to\infty} a_n = -\infty$ and $\lim_{n\to\infty} b_n = +\infty$ and such that:
$$\lim_{n\to\infty} f^{(k-1)}(a_n) = \lim_{n\to\infty} f^{(k-1)}(b_n) = 0$$
Hence using integration by parts:
$$\begin{aligned}
\widehat{f^{(k)}}(\xi) &= \lim_{n\to\infty} \int_{a_n}^{b_n} f^{(k)}(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x} \,\mathrm{d}x \\
&= \lim_{n\to\infty} f^{(k-1)}(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x}\Big|_{a_n}^{b_n} \,\mathrm{d}x + \\
&\qquad + 2\pi \mathrm{i}\xi \lim_{n\to\infty} \int_{a_n}^{b_n} f^{(k-1)}(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x} \,\mathrm{d}x \\
&= (2\pi \mathrm{i}\xi)\, \widehat{f^{(k-1)}}(n) \\
&= (2\pi \mathrm{i}\xi)^k \widehat{f}(\xi)
\end{aligned}$$
□

Remark. Note that there exists functions $f \in \mathcal{C}(\mathbb{R}) \cap L^1(\mathbb{R})$ for which the limit $\lim_{x\to\infty} f(x)$ does not exist.

Proposition 16. Let $f \in L^1(\mathbb{R})$ be such that it has compact support. Then, $\mathcal{F}f \in \mathcal{C}^\omega(\mathbb{R})$.

Sketch of the proof. Suppose $f(x) \in [-K, K]$, $K > 0$. Then, expanding $\mathcal{F}f$ with the power series of $\mathrm{e}^{-2\pi \mathrm{i}\xi x}$ centered at $a \in \mathbb{R}$ we have:
$$\mathcal{F}f(\xi) = \int_{-K}^{K} f(x) \sum_{n=0}^{\infty} \frac{(-2\pi \mathrm{i} x)^n \mathrm{e}^{-2\pi \mathrm{i} a x}}{n!} (\xi - a)^n \,\mathrm{d}x$$

$$= \sum_{n=0}^{\infty} c_n(\xi - a)^n$$

where $|c_n| \le \frac{(2\pi K)^n}{n!} \|f\|_1$. Finally, use this to show that the radius of convergences (see **??**) is ∞. □

Lemma 17. Let $f(x) = \mathrm{e}^{-ax^2}$. Then, $\mathcal{F}f(\xi) = \sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{(\pi\xi)^2}{a}}$ and moreover $\mathcal{F}^2 f = f$. In particular if $a = \pi$, then $\mathcal{F}f = f$.

Sketch of the proof. f satisfies the ODE $y' = -2axy$. Taking $\widehat{\ }$ on this expression and using Theorems 14 and 15 we obtain that $\widehat{f}$ must satisfy the following ODE:

$$y' = -\frac{2\pi^2\xi}{a}y$$

with initial condition $y(0) = \int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}$. □

Lemma 18. Let $f(x) = \mathrm{e}^{-a|x|}$. Then, $\mathcal{F}f(\xi) = \frac{2a}{a^2+4\pi^2\xi^2}$ and moreover $\mathcal{F}^2 f = f$.

Sketch of the proof.

$$\mathcal{F}f(\xi) = 2\int_{0}^{+\infty} \mathrm{e}^{-ax}\cos(2\pi\xi x)\,\mathrm{d}x = \frac{2a}{a^2+4\pi^2\xi^2}$$

□

Lemma 19. Let $f(x) = \mathbf{1}_{[-a,a]}(x)$, $a > 0$. Then, $\mathcal{F}f(\xi) = \frac{\sin(2\pi a\xi)}{\pi\xi}$.

The inverse Fourier transform

Theorem 20 (Inversion theorem). Let $f \in L^1(\mathbb{R})$ such that $\mathcal{F}f \in L^1(\mathbb{R})$. Then:

$$f(x) \overset{\text{a.e.}}{=} \int_{-\infty}^{+\infty} \widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi$$

Moreover if f is continuous we can remove the “almost everywhere”.

Proof. Consider the integral:

$$I_t(x) = \int_{-\infty}^{+\infty} f(x+y)\frac{1}{t}\mathrm{e}^{-\pi\frac{y^2}{t^2}}\,\mathrm{d}y$$

Note that using Theorem 17 and Item 9-4, we have that $\mathcal{F}\left(\frac{1}{\lambda}\mathrm{e}^{-\pi\frac{x^2}{\lambda^2}}\right) = \mathrm{e}^{-\pi\lambda^2\xi^2}$. On the one hand, using this latter thing and Theorem 13 we have:

$$\begin{aligned} I_t(x) = \int_{-\infty}^{+\infty} f(x+y)\frac{1}{t}\mathrm{e}^{-\pi\frac{y^2}{t^2}}\,\mathrm{d}y &= \int_{-\infty}^{+\infty} f(x+\xi)\widehat{\mathrm{e}^{-\pi t^2\xi^2}}\,\mathrm{d}\xi = \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{2\pi\mathrm{i}\xi x}\widehat{f}(\xi)\mathrm{e}^{-\pi t^2\xi^2}\,\mathrm{d}\xi \end{aligned}$$

which by **?? ??** converges to $\int_{-\infty}^{+\infty} \widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi$ as $t \to 0$.

On the other hand with a change of variable we have:

$$I_t(x) = \int_{-\infty}^{+\infty} f(x+ty)\mathrm{e}^{-\pi y^2}\,\mathrm{d}y$$

Using **??** it suffices to prove that $\lim\limits_{t\to 0}\|I_t(x) - f(x)\|_1 = 0$. But using that $\int_{-\infty}^{+\infty}\mathrm{e}^{-\pi y^2}\,\mathrm{d}y = 1$:

$$\begin{aligned} \|I_t(x) - f(x)\|_1 &= \int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}(f(x+ty) - f(x))\mathrm{e}^{-\pi y^2}\,\mathrm{d}y\right|\mathrm{d}x \\ &\le \int_{-\infty}^{+\infty}\mathrm{e}^{-\pi y^2}\int_{-\infty}^{+\infty}|f(x+ty) - f(x)|\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

where we have used **?? ??**. Now use the **?? ??**. □

Corollary 21. Let $f \in L^1(\mathbb{R})$ such that $\mathcal{F}f \overset{\text{a.e.}}{=} 0$. Then, $f \overset{\text{a.e.}}{=} 0$.

Corollary 22. Let $f \in L^1(\mathbb{R})$. Then, $\mathcal{F}^2 f(x) \overset{\text{a.e.}}{=} f(-x)$. Hence, $\mathcal{F}^4 \overset{\text{a.e.}}{=} \mathrm{id}$.

Proof. By the 20 Inversion theorem we have:

$$f(-x) \overset{\text{a.e.}}{=} \int_{-\infty}^{+\infty}\widehat{f}(\xi)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \mathcal{F}\widehat{f}(x) = \mathcal{F}^2 f(x)$$

□

Lemma 23. Let $f, g \in L^1(\mathbb{R})$. Then, $f * g \in L^1(\mathbb{R})$, $\|f*g\|_1 \le \|f\|_1\|g\|_1$ and $\mathcal{F}(f*g) = \mathcal{F}f\mathcal{F}g$. In particular if $g(x) = \overline{f(-x)}$ then $\mathcal{F}(f*g) = \left|\widehat{f}\right|^2$.

Sketch of the proof. Show first that $f(x-y)g(y) \in L^1(\mathbb{R}^2)$ and then use **?? ??**. □

Pointwise convergence

Definition 24. Let $f \in L^1(\mathbb{R})$. We define the *partial inverse Fourier transform* as:

$$S_R f(x) = \int_{-R}^{R}\widehat{f}(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi$$

Definition 25 (Dirichlet kernel). We define the *Dirichlet kernel* of order $R \in \mathbb{R}_{>0}$ as:

$$D_R(t) = \int_{-R}^{R}\mathrm{e}^{-2\pi\mathrm{i}\xi t}\,\mathrm{d}\xi = \frac{\sin(2\pi Rt)}{\pi t}$$

Proposition 26. The Dirichlet kernel has the following properties:

1. D_R is an even function.
2. $\displaystyle\int_{-\infty}^{+\infty} D_R(t)\,\mathrm{d}t = 1$ for all $R > 0$.

3.

$$\begin{aligned} S_R f(x) &= (f * D_R)(x) \\ &= \int_{-\infty}^{+\infty} f(x-t) D_R(t)\,\mathrm{d}t \\ &= \int_{0}^{+\infty} [f(x+t)+f(x-t)] D_R(t)\,\mathrm{d}t \end{aligned}$$

Theorem 27 (Dini's theorem). Let $f \in L^1(\mathbb{R})$ and $x, \ell \in \mathbb{R}$ such that $h(t) := \frac{|f(x+t)+f(x-t)-2\ell|}{t} \in L^1((0,\delta))$ for some $\delta > 0$. Then, $\lim_{R\to\infty} S_R f(x) = \ell$.

Sketch of the proof. Note that

$$S_R f(x) - \ell = \int_0^\infty [f(x+t)+f(x-t)-2\ell] D_R(t)\,\mathrm{d}t$$

Now split this integral as a sum of the following ones:

$$\begin{aligned} I_1 &= \int_0^N [f(x+t)+f(x-t)-2\ell] D_R(t)\,\mathrm{d}t \\ I_2 &= \int_N^\infty [f(x+t)+f(x-t)] D_R(t)\,\mathrm{d}t \\ I_3 &= -2\ell \int_N^\infty D_R(t)\,\mathrm{d}t \end{aligned}$$

Given $\varepsilon > 0$ take N such that $\int_N^\infty \left|\frac{f(x+t)+f(x-t)}{\pi t}\right| \mathrm{d}t < \varepsilon$. Since h is integrable in $(0,N)$, by 12 Riemann-Lebesgue lemma we have that $I_1 \overset{R\to\infty}{\longrightarrow} 0$. Then, as we can write $I_3 = -2\ell \int_{2\pi RN}^\infty \frac{\sin(u)}{\pi u}\,\mathrm{d}u$ we have that $I_3 \overset{R\to\infty}{\longrightarrow} 0$. □

Lemma 28. Let $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$. Then, $\lim_{a\to 0} \|f - T_a f\|_p = 0$.

Sketch of the proof. Clearly is is true if $f \in \mathcal{C}_0^\infty(\mathbb{R})$ using **??** **??**. Now use that since $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $\mathcal{C}_0(\mathbb{R})$, which is dense in $L^p(\mathbb{R})$, $\exists (f_n) \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. □

Uniform convergence

Definition 29. Let $f \in L^1(\mathbb{R})$ and $R > 0$. We define the *Fejér mean* $\sigma_R f(x)$ as:

$$\sigma_R f(x) = \frac{1}{R} \int_0^R S_r f(x)\,\mathrm{d}r$$

Definition 30. Let $f \in L^1(\mathbb{R})$ and $R > 0$. We define the *Fejér kernel* $F_R f(x)$ as:

$$F_R(x) = \frac{1}{R} \int_0^R D_r(x)\,\mathrm{d}r$$

Lemma 31. Let $f \in L^1(\mathbb{R})$ and $R > 0$. Then, $\sigma_R f = f * F_R$ and moreover:

$$F_R(x) = \frac{(\sin(\pi R x))^2}{\pi^2 R x^2}$$

Definition 32. Let $t > 0$. We define the *Poisson kernel* P_t as $P_t(x) := \mathcal{F}^{-1}(\mathrm{e}^{-2\pi t|\xi|})$.

Lemma 33. Let $f \in L^1(\mathbb{R})$ and $t > 0$. Then:

$$\begin{aligned} P_t(x) &= \frac{t}{\pi(t^2+x^2)} \\ (f * P_t)(x) &= \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi t|\xi|} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi \end{aligned}$$

Proof. Check Theorem 18 for the first equality. For the other one:

$$\begin{aligned} (f * P_t)(x) &= \int_{-\infty}^{+\infty} f(y) P_t(x-y)\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y) \mathrm{e}^{-2\pi t|\xi|} \mathrm{e}^{2\pi \mathrm{i}\xi(x-y)}\,\mathrm{d}\xi\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi t|\xi|} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi \end{aligned}$$

□

Definition 34. Let $t > 0$. We define the *Weierstraß kernel* W_t as $W_t(x) := \mathcal{F}^{-1}(\mathrm{e}^{-4\pi^2 t\xi^2})$.

Lemma 35. Let $f \in L^1(\mathbb{R})$ and $t > 0$. Then:

$$\begin{aligned} W_t(x) &= \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}} \\ (f * W_t)(x) &= \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi^2 t\xi^2} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi \end{aligned}$$

Proof. Check Theorem 17 for the first equality. For the other one:

$$\begin{aligned} (f * W_t)(x) &= \int_{-\infty}^{+\infty} f(y) W_t(x-y)\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y) \mathrm{e}^{-4\pi^2 t\xi^2} \mathrm{e}^{2\pi \mathrm{i}\xi(x-y)}\,\mathrm{d}\xi\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi^2 t\xi^2} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi \end{aligned}$$

□

Proposition 36. Let $R > 0$ and $t > 0$. Then:

1. F_R, P_t and W_t are non-negative even functions.
2. $\int_{-\infty}^{+\infty} F_R(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} P_t(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} W_t(x)\,\mathrm{d}x = 1$

3. For all $\delta > 0$, we have:

$$\lim_{R\to\infty} \sup_{|x|\geq\delta} F_R(x) = \lim_{t\to 0} \sup_{|x|\geq\delta} P_t(x) = \\ = \lim_{t\to 0} \sup_{|x|\geq\delta} W_t(x) = 0$$

4. For all $\delta > 0$, we have:

$$\lim_{R\to\infty} \int\limits_{|x|\geq\delta} F_R(x)\,\mathrm{d}x = \lim_{t\to 0} \int\limits_{|x|\geq\delta} P_t(x)\,\mathrm{d}x = \\ = \lim_{t\to 0} \int\limits_{|x|\geq\delta} W_t(x)\,\mathrm{d}x = 0$$

That is, F_R, P_t and W_t are *approximations of the identity.*

Sketch of the proof. The first two properties are straightforward. For the third one, note that:

$$\sup_{|x|\geq\delta} F_R(x) \leq \frac{1}{\pi^2 R\delta^2}$$
$$\sup_{|x|\geq\delta} P_t(x) = \frac{t^2}{\pi(t^2+\delta^2)}$$
$$\sup_{|x|\geq\delta} W_t(x) = \frac{1}{\sqrt{4\pi\delta}}\mathrm{e}^{-\frac{x^2}{4\delta}}$$

The last one is a consequence of the previous ones. □

Theorem 37. Let $f \in L^1(\mathbb{R})$ be a function having left- and right-sided limits at point x_0. Then:

$$\lim_{R\to\infty} \sigma_R f(x_0) = \lim_{t\to 0}(f * P_t)(x_0) = \lim_{t\to 0}(f * W_t)(x_0) = \\ = \frac{f(x_0{}^+) + f(x_0{}^-)}{2}$$

Moreover if f is uniformly continuous, the convergence is uniform.

Sketch of the proof. Copy the proofs of **?? ??** and **?? ??**. □

Lemma 38. Let $E \subseteq \mathbb{R}^n$ be a measurable space, $p \geq 1$, $f \in L^p(E)$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$\|f\|_p = \sup\left\{\int\limits_E fg : \|g\|_q = 1\right\}$$

Proof. On the one hand using **?? ??**:

$$\int\limits_E fg \leq \|fg\|_1 \leq \|f\|_p \|g\|_q = \|f\|_p$$

Now consider $g = \frac{|f|^{p-1}\operatorname{sgn} f}{\|f\|_p{}^{\frac{p}{q}}}$. Then, $\|g\|_q = 1$ and moreover:

$$\int\limits_E fg = \int\limits_E \frac{|f|^p}{\|f\|_p{}^{\frac{p}{q}}} = \|f\|_p{}^{p-\frac{p}{q}} = \|f\|_p$$

□

Lemma 39 (Minkowski's integral inequality). Let $E, F \subseteq \mathbb{R}^n$ be measurable spaces, $p \geq 1$ and $f \in L^p(E \times F)$. Then:

$$\left\|\int\limits_F h(\cdot,\mathbf{y})\,\mathrm{d}\mathbf{y}\right\|_p \leq \int\limits_F \|h(\cdot,\mathbf{y})\|_p\,\mathrm{d}\mathbf{y}$$

Proof. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q(E)$ with $\|g\|_q = 1$. Then, using **?? ??** and **?? ??**:

$$\begin{aligned}\int\limits_E g(\mathbf{x}) \int\limits_F h(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} &= \int\limits_F \int\limits_E h(\mathbf{x},\mathbf{y})g(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &\leq \int\limits_F \|h(\cdot,\mathbf{y})\|_p \|g\|_q\,\mathrm{d}\mathbf{y} \\ &= \int\limits_F \|h(\cdot,\mathbf{y})\|_p\,\mathrm{d}\mathbf{y}\end{aligned}$$

Now use Theorem 38. □

Theorem 40. Let $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and ϕ_ε be an approximation of identity. Then:

$$\lim_{\varepsilon\to 0} \|f * \phi_\varepsilon - f\|_p = 0$$

Sketch of the proof. Using 39 Minkowski's integral inequality, we have:

$$\begin{aligned}\|f * \phi_\varepsilon - f\|_p &= \left\|\int_{-\infty}^{\infty} \phi_\varepsilon(\mathbf{y})(f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x}))\,\mathrm{d}\mathbf{y}\right\|_p \\ &\leq \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \phi_\varepsilon(\mathbf{y})^p |f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}\right]^{\frac{1}{p}}\mathrm{d}\mathbf{y} \\ &= \int_{-\infty}^{\infty} \phi_\varepsilon(\mathbf{y})\,\|f - T_{-\mathbf{y}}f\|_p\,\mathrm{d}\mathbf{y} \\ &\leq \int\limits_{|\mathbf{y}|<\delta} \phi_\varepsilon(\mathbf{y})\,\|f - T_{-\mathbf{y}}f\|_p\,\mathrm{d}\mathbf{y} + \\ &\qquad + 2\|f\|_p \int\limits_{|\mathbf{y}|\geq\delta} \phi_\varepsilon(\mathbf{y})\,\mathrm{d}\mathbf{y}\end{aligned}$$

Given $\varepsilon > 0$, by Theorem 28 $\exists\delta > 0$ such that the first integral is bounded by ε. Now use this δ and Item 36-4 to conclude that the second integral goes to 0 as $R \to \infty$. □

Corollary 41. Let $f \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. Then:

$$\lim_{R\to\infty} \|\sigma_R f - f\|_p = 0$$
$$\lim_{t\to 0} \|f * P_t - f\|_p = 0$$
$$\lim_{t\to 0} \|f * W_t - f\|_p = 0$$

Fourier transform on $L^2(\mathbb{R})$

Lemma 42. Let $f, g \in L^2(\mathbb{R})$. Then, $f * g$ is continuous and bounded. Moreover, $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$.

Sketch of the proof. The inequality follows from **??** **??**. Moreover:

$$|(f*g)(x+h) - (f*g)(x)| \leq$$
$$\leq \int_{-\infty}^{+\infty} |f(x+h-y) - f(x-y)||g(y)| \, \mathrm{d}y \leq$$
$$\leq \|g\|_2 \|f - T_{-h}f\|_2$$

So $f * g$ is continuous, by Theorem 28. □

Theorem 43 (Plancherel theorem). Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, $\widehat{f} \in L^2(\mathbb{R})$ and:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{+\infty} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi$$

Proof. Let $\tilde{f}(x) := \overline{f(-x)}$. Then, $\widehat{\tilde{f}}(\xi) = \overline{\widehat{f}(\xi)}$ and so by Theorem 42 we have that $g := f * \tilde{f}$ is continuous and bounded. Moreover $\widehat{g}(\xi) = \widehat{f}(\xi)\widehat{\tilde{f}}(\xi) = \left|\widehat{f}(\xi)\right|^2$ and $g(0) = \int_{-\infty}^{+\infty} \tilde{f}(-y) f(y) \, \mathrm{d}y = \|f\|_2{}^2$. On the other hand, by Theorem 35 we have:

$$(g * W_t)(0) = \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi^2 t\xi^2} \widehat{g}(\xi) \, \mathrm{d}\xi = \int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi^2 t\xi^2} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \tag{1}$$

And by Theorem 40, $\lim_{t\to 0+} (g * W_t)(0) = g(0) = \|f\|_2{}^2$. Thus, by the definition of limit taking $\varepsilon = \|f\|_2{}^2$, we have that $\left|\int_{-\infty}^{+\infty} \mathrm{e}^{-4\pi^2 t\xi^2} \widehat{g}(\xi) \, \mathrm{d}\xi\right| \leq 2\|f\|_2{}^2$ for t small enough. Finally, if t is that small, then $1 \leq 2\mathrm{e}^{-4\pi^2 t\xi^2}$ and so:

$$\int_{-\infty}^{+\infty} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi \leq 2 \int_{-\infty}^{+\infty} \widehat{g}(\xi) \mathrm{e}^{-4\pi^2 t\xi^2} \, \mathrm{d}\xi \leq 4\|f\|_2{}^2 < \infty$$

Now use **??** **??** in Eq. (1) and make $t \to 0$. □

Corollary 44. Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then:

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)} \, \mathrm{d}x = \int_{-\infty}^{+\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \, \mathrm{d}\xi$$

Proof. Use 43 Plancherel theorem and **??** **??**. □

Proposition 45. Let $f \in L^2(\mathbb{R})$. Then, $\exists (f_n) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$.

Sketch of the proof. Take the sequence $f_n(x) = f(x)\mathbf{1}_{[-n,n]}(x)$. □

Proposition 46. Let $f \in L^2(\mathbb{R})$ and $(f_n) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. Then, the limit $\lim_{n\to\infty} \widehat{f_n}(\xi)$ exists and we will call it $\widehat{f}(\xi)$.

Proof. Since $L^2(\mathbb{R})$ is Hilbert, (f_n) is Cauchy. But by 43 Plancherel theorem, $(\widehat{f_n})$ is also Cauchy and so it has limit, because $(\widehat{f_n}) \in L^2(\mathbb{R})$.
To see that the definition is well-defined, suppose $(g_n) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is another sequence such that $\lim_{n\to\infty} \|f - g_n\|_2 = 0$. But in this case:

$$\|g_n - f_n\|_2 \leq \|g_n - f\|_2 + \|f - f_n\|_2 \xrightarrow{n\to\infty} 0$$

□

Remark. Note that the abuse of notation in the definition of the limit make sense as it coincides with the ordinary Fourier transform when $f \in L^1(\mathbb{R})$ (by taking $f_n = f$ $\forall n \in \mathbb{N}$).

Theorem 47. Let $f, g \in L^2(\mathbb{R})$. Then:

1. $\widehat{f}(\xi) \overset{L^2}{=} \lim_{n\to\infty} \int_{-n}^{n} f(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x} \, \mathrm{d}x$
2. $\|f\|_2 = \left\|\widehat{f}\right\|_2$
3. $\int_{-\infty}^{+\infty} f(x)\widehat{g}(x) \, \mathrm{d}x = \int_{-\infty}^{+\infty} \widehat{f}(x)g(x) \, \mathrm{d}x$
4. $\int_{-\infty}^{+\infty} f(x)\overline{g(x)} \, \mathrm{d}x = \int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)} \, \mathrm{d}x$

Proof. The first property follows from its definition. For the second one, if $f(x) \overset{L^2}{=} \lim_{n\to\infty} f_n(x)$, by 43 Plancherel theorem we have $\|f_n\|_2 = \left\|\widehat{f}\right\|_2$. Now use the continuity of the norm. For the other properties, take the function given in the proof of Theorem 45 and use the **??** **??**. □

Proposition 48 (Jensen's inequality). Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function, f be a measurable function, and $\mu : \Omega \to \mathbb{R}$ be measurable with $\int_\Omega \mathrm{d}\mu = 1$. Then:

$$\int_\Omega J(f) \, \mathrm{d}\mu \geq J\left(\int_\Omega f \, \mathrm{d}\mu\right)$$

Sketch of the proof. We assume differentiability on J for simplicity. Since J is convex we have that $\forall a, b \in \mathbb{R}$:

$$J(b) \geq J(a) + J'(a)(b - a)$$

Taking $a = \int_\Omega f \, \mathrm{d}\mu$ and $b = f(x)$, we have:

$$J(f(x)) \geq J\left(\int_\Omega f \, \mathrm{d}\mu\right) + J'\left(\int_\Omega f \, \mathrm{d}\mu\right)\left(f(x) - \int_\Omega f \, \mathrm{d}\mu\right)$$

Multiplying by $\mathrm{d}\mu$ and integrating, yields the result. □

Lemma 49 (Generalized Hölder's inequality). Let $E \subseteq \mathbb{R}^n$ be a measurable set, $1 \leq p_1, \ldots, p_n \leq \infty$ be such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $f_i \in L^{p_i}(E)$. Then:

$$\left\|\prod_{i=1}^n f_i\right\|_1 \leq \prod_{i=1}^n \|f_i\|_{p_i}$$

Proof. We will prove it by induction on n. For $n = 1$ the result is clear. For $n \geq 2$, note that the numbers $q_n = \frac{p_n}{p_n - 1}$ and p_n are Hölder conjugates. Moreover, if we define $r_i = p_i \left(1 - \frac{1}{p_n}\right) = \frac{p_i}{q_n}$ we have that $\sum_{i=1}^{n-1} \frac{1}{r_i} = 1$ and so using ?? ?? we have:

$$\begin{aligned}
\|f_1 \cdots f_n\|_1 &\leq \|f_1 \cdots f_{n-1}\|_{q_n} \|f_n\|_{p_n} \\
&= \||f_1 \cdots f_{n-1}|^{q_n}\|_1^{\frac{1}{q_n}} \|f_n\|_{p_n} \\
&\leq \||f_1|^{q_n}\|_{r_1}^{\frac{1}{q_n}} \cdots \||f_1|^{q_n}\|_{r_1}^{\frac{1}{q_n}} \|f_n\|_{p_n} \\
&= \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}
\end{aligned}$$

where in the penultimate step we have used the induction hypothesis and in the last equality we have used the fact that $r_i q_n = p_i$. □

Lemma 50 (Young's convolution inequality). Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ and take r such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

with $1 \leq p, q, r \leq \infty$. Then:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

Sketch of the proof. Note that:

$$\begin{aligned}
&|(f * g)(\mathbf{x})| \\
&\quad \leq \int_{\mathbb{R}^n} (|f(\mathbf{y})|^p |g(\mathbf{x}-\mathbf{y})|^q)^{\frac{1}{r}} |f(\mathbf{y})|^{1-\frac{p}{r}} |g(\mathbf{x}-\mathbf{y})|^{1-\frac{q}{r}} \,\mathrm{d}\mathbf{y} \\
&\qquad \leq \left\|(|f(\mathbf{y})|^p |g(\mathbf{x}-\mathbf{y})|^q)^{\frac{1}{r}}\right\|_r \left\||f(\mathbf{y})|^{\frac{r-p}{r}}\right\|_{\frac{pr}{r-p}} \cdot \\
&\qquad\qquad \cdot \left\||g(\mathbf{x}-\mathbf{y})|^{\frac{r-q}{r}}\right\|_{\frac{qr}{r-q}}
\end{aligned}$$

where in the second inequality we have used the 49 Generalized Hölder's inequality because:

$$\frac{1}{r} + \frac{r-p}{pr} + \frac{r-q}{qr} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$$

Finally:

$$\begin{aligned}
\|f * g\|_r{}^r &\leq \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\mathbf{y})|^p |g(\mathbf{x}-\mathbf{y})|^q \,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} \\
&= \|f\|_p{}^{r-p} \|g\|_q{}^{r-q} \|f\|_p{}^p \|g\|_q{}^q \\
&= \|f\|_p{}^r \|g\|_q{}^r
\end{aligned}$$

where in the second equality we have used the ?? ??. □

Theorem 51. Let $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then, $\widehat{f * g} \in L^2(\mathbb{R})$ and:

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

Proof. By 50 Young's convolution inequality with $p = r = 2$ and $q = 1$ we have:

$$\|f * g\|_2 \leq \|f\|_2 \|g\|_1 < \infty$$

The equality follows in the same way as in $L^1(\mathbb{R})$. □

Fourier transform on $L^p(\mathbb{R})$

Lemma 52. Let $f \in L^p(\mathbb{R})$ with $1 < p < 2$. Then, there exist functions $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$ such that $f = f_1 + f_2$.

Proof. The set $E := \{|f| \geq 1\}$ has finite measure because $f \in L^p(\mathbb{R})$. Now consider the Functions

$$f_1(x) := \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \qquad f_2(x) := \begin{cases} 0 & \text{if } x \in E \\ f(x) & \text{if } x \notin E \end{cases}$$

By ?? ??, $f_1 \in L^1(\mathbb{R})$ because $|E| < \infty$. On the other hand:

$$\int_{-\infty}^{\infty} |f_2(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}\setminus E} |f(x)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}\setminus E} |f(x)|^p \,\mathrm{d}x < \infty$$

because $|f| < 1$ in $\mathbb{R} \setminus E$. So $f_2 \in L^2(\mathbb{R})$. □

Definition 53. Let $f = f_1 + f_2 \in L^p(\mathbb{R})$ with $1 < p < 2$, $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$. We define the *Fourier transform* of f as:

$$\widehat{f}(\xi) := \widehat{f_1}(\xi) + \widehat{f_2}(\xi)$$

Remark. This definition is well-defined. Indeed, suppose $f = g_1 + g_2$ with $1 < p < 2$ with $g_1 \in L^1(\mathbb{R})$ and $g_2 \in L^2(\mathbb{R})$. Then, $f_1 - g_1 = g_2 - f_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and so

$$\widehat{f_1} - \widehat{g_1} = \widehat{f_1 - g_1} = \widehat{f_2 - g_2} = \widehat{f_2} - \widehat{g_2}$$

Hence, $\widehat{f} = \widehat{f_1} + \widehat{f_2} = \widehat{g_1} + \widehat{g_2}$.

Fourier transform on $\mathbb{R}^n$

In this section we will only expose the most important results of extending the Fourier transform to $L^1(\mathbb{R}^n)$. Moreover we will not prove any of the results of this section as they are completely analogous to the previous ones.

Definition 54. Let $f \in L^1(\mathbb{R}^n)$. We define the *Fourier transform* of f as:

$$\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i}\langle \boldsymbol{\xi}, \mathbf{x}\rangle} \,\mathrm{d}\mathbf{x}$$

The function f is also called *inverse Fourier transform* of $\widehat{f}$.

Proposition 55. Let $f, g \in L^1(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}$. Then:

1. $(\widehat{\alpha f + \beta g})(\boldsymbol{\xi}) = \alpha \widehat{f}(\boldsymbol{\xi}) + \beta \widehat{g}(\boldsymbol{\xi})$
2. Let $\mathbf{h} \in \mathbb{R}^n$. We define $T_{\mathbf{h}} f(x) = f(\mathbf{x} + \mathbf{h})$. Then:
$$\widehat{T_{\mathbf{h}} f}(\boldsymbol{\xi}) = \mathrm{e}^{2\pi \mathrm{i}\langle \boldsymbol{\xi}, \mathbf{h}\rangle} \widehat{f}(\boldsymbol{\xi})$$
3. If $g(\mathbf{x}) = \mathrm{e}^{2\pi \mathrm{i}\langle \mathbf{x}, \mathbf{h}\rangle} f(\mathbf{x})$, then:
$$\widehat{g}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi} - \mathbf{h})$$
4. If $\lambda \in \mathbb{R}^*$, then:
$$\frac{1}{\lambda^n} \widehat{f\left(\frac{\mathbf{x}}{\lambda}\right)}(\boldsymbol{\xi}) = \widehat{f}(\lambda \boldsymbol{\xi})$$

5. If $g(\mathbf{x}) = \overline{f(\mathbf{x})}$, then:
$$\widehat{g}(\boldsymbol{\xi}) = \overline{\widehat{f}(-\boldsymbol{\xi})}$$

Theorem 56. Let $f \in L^1(\mathbb{R}^n)$ and denote also by $\mathcal{F}$ the extension of the Fourier transform operator to $L^1(\mathbb{R}^n)$. Then:

1. $\mathcal{F}f$ is uniformly continuous.
2. $\mathcal{F}$ is a continuous linear operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and $\|\mathcal{F}f\|_\infty \leq \|f\|_1$.

Theorem 57 (Riemann-Lebesgue lemma). Let $f \in L^1(\mathbb{R}^n)$. Then:
$$\lim_{\|\boldsymbol{\xi}\|\to\infty} \left|\widehat{f}(\boldsymbol{\xi})\right| = 0$$

Proposition 58. Let f be a function such that $\xi_j f \in L^1(\mathbb{R}^n)$. Then, $\widehat{f}$ is differentiable with respect to ξ_j and:
$$\frac{\partial(\mathcal{F}f)}{\partial \xi_j}(\boldsymbol{\xi}) = \mathcal{F}((-2\pi\mathrm{i}\xi_j)f(\mathbf{x}))$$

Proposition 59. Let $f \in L^1(\mathbb{R}^n)$ be differentiable with respect to x_j such that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$. Then:
$$\widehat{\frac{\partial f}{\partial x_j}}(\boldsymbol{\xi}) = 2\pi\mathrm{i}\xi_j\widehat{f}(\boldsymbol{\xi})$$

Theorem 60 (Plancherel theorem). Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then, $\widehat{f} \in L^2(\mathbb{R}^n)$ and:
$$\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \left|\widehat{f}(\boldsymbol{\xi})\right|^2 \mathrm{d}\boldsymbol{\xi}$$

Applications of the Fourier transform

Remark. Probably the most important application of Fourier series is the resolution of PDEs and it is a consequence of Theorem 15, which reduces any order of a PDE in the spatial variable to 1. The procedure is to compute the Fourier transform $\mathcal{F}$ of the PDE, solve it, and then get back to the first function using the inverse transform.

Theorem 61 (Uncertainty principle). Let $f \in L^2(\mathbb{R})$ be differentiable such that $x|f|^2 \in L^1(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$. Then:
$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2 \,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \xi^2\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right) \geq \frac{{\|f\|_2}^4}{16\pi^2}$$

and the equality holds if and only if $f(x) = \mathrm{e}^{-\lambda^2x^2}$, $\lambda \in \mathbb{R}$.

Proof. Let I be the left-hand-side term of the inequality. First note that:
$$\int_{-\infty}^{\infty} \xi^2\left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left|\widehat{f'}(\xi)\right|^2 \mathrm{d}\xi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} |f'(\xi)|^2 \,\mathrm{d}\xi$$

where the first equality is by the analogous Theorem 15 in $L^2(\mathbb{R})$ and in the second one we have used 43 Plancherel theorem. Now by the **??**, we have:
$$\begin{aligned}
I &\geq \frac{1}{4\pi^2}\left(\int_{-\infty}^{\infty} x|f(x)f'(x)| \,\mathrm{d}x\right)^2 \\
&\geq \frac{1}{4\pi^2}\left(\int_{-\infty}^{\infty} x\operatorname{Re}(f(x)\overline{f'(x)}) \,\mathrm{d}x\right)^2 \\
&= \frac{1}{16\pi^2}\left(\int_{-\infty}^{\infty} x\frac{\mathrm{d}}{\mathrm{d}x}(|f(x)|^2) \,\mathrm{d}x\right)^2 \\
&= \frac{1}{16\pi^2}\left(\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x\right)^2
\end{aligned}$$

where in the third step we have used that:
$$\frac{\mathrm{d}}{\mathrm{d}x}(|f(x)|^2) = 2\operatorname{Re}(f(x)\overline{f'(x)})$$

and in the last step we have integrated by parts (here we use that $x|f|^2 \in L^1(\mathbb{R})$). □

Theorem 62 (Uncertainty principle in $\mathbb{R}^n$). Let $f \in L^2(\mathbb{R}^n)$ be regular enough. Then:
$$\left(\int_{-\infty}^{\infty} \|\mathbf{x}\|^2 |f(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\right)\left(\int_{-\infty}^{\infty} \|\boldsymbol{\xi}\|^2 \left|\widehat{f}(\boldsymbol{\xi})\right|^2 \mathrm{d}\boldsymbol{\xi}\right) \geq \frac{n^2{\|f\|_2}^4}{16\pi^2}$$

Theorem 63 (Poisson summation formula). Let $f \in \mathcal{C}(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that $\sum_{k\in\mathbb{Z}} f(x+k)$ converges uniformly for $x \in [0,1]$ and such that $\sum_{k\in\mathbb{Z}}\left|\widehat{f}(k)\right| < \infty$. Then:
$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}kx}$$

In particular, for $x = 0$ we have:
$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)$$

Proof. Let $F(x) := \sum_{k\in\mathbb{Z}} f(x+k)$. Note that F is 1-periodic. If we see that $\widehat{F}(n) = \widehat{f}(n)\ \forall n \in \mathbb{Z}$, the continuity of f and the convergence of its Fourier series will imply $F(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{2\pi\mathrm{i}kx}$. But:
$$\begin{aligned}
\widehat{F}(n) &= \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}nx} \,\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 f(x+k)\mathrm{e}^{-2\pi\mathrm{i}nx} \,\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)\mathrm{e}^{-2\pi\mathrm{i}n(x-k)} \,\mathrm{d}x
\end{aligned}$$

$$= \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)\mathrm{e}^{-2\pi \mathrm{i} n x}\,\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi \mathrm{i} n x}\,\mathrm{d}x$$
$$= \widehat{f}(n)$$

□

Definition 64. Let $f \in L^2(\mathbb{R})$. We say that f is *bandlimited* if $\exists B \in \mathbb{R}$ such that $\operatorname{supp} \widehat{f} \subseteq [-B, B]$.

Theorem 65 (Nyquist-Shannon sampling theorem). Let $f \in L^2(\mathbb{R})$ be bandlimited with constant B. Then:

$$f(x) \overset{L^2}{=} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2B}\right) \frac{\sin(\pi(2Bx-k))}{\pi(2Bx-k)}$$

Moreover:

$$\|f\|_2{}^2 = \frac{1}{2B}\sum_{k\in\mathbb{Z}} \left| f\left(\frac{k}{2B}\right)\right|^2$$

Proof. An easy check shows that the Fourier series of $\xi \mapsto \mathrm{e}^{2\pi \mathrm{i} x\xi}$ on $[-B, B]$ is:

$$\mathrm{e}^{2\pi \mathrm{i} x\xi} = \sum_{k\in\mathbb{Z}} \frac{\sin(\pi(2Bx-k))}{\pi(2Bx-k)} \mathrm{e}^{\frac{\pi \mathrm{i} k\xi}{B}}$$

Thus:

$$f(x) = \int_{-B}^{B} \widehat{f}(\xi)\mathrm{e}^{2\pi \mathrm{i}\xi x}\,\mathrm{d}\xi$$
$$= \sum_{k\in\mathbb{Z}} \frac{\sin(\pi(2Bx-k))}{\pi(2Bx-k)} \int_{-B}^{B} \widehat{f}(\xi)\mathrm{e}^{\frac{\pi \mathrm{i} k\xi}{B}}\,\mathrm{d}\xi$$
$$= \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2B}\right)\frac{\sin(\pi(2Bx-k))}{\pi(2Bx-k)}$$

The second equality follows from both 43 Plancherel theorem and **?? ??**:

$$\|f\|_2{}^2 = \left\|\widehat{f}\right\|_2^2 = \frac{1}{2B}\sum_{k\in\mathbb{Z}} \left| f\left(\frac{k}{2B}\right)\right|^2$$

because by a similar argument as before, the Fourier coefficients of $\widehat{f}(k)$ (thought as periodically extended) are $\frac{1}{2B} f\left(\frac{-k}{2B}\right)$. □

Remark. In the context of signal processing, 65 Nyquist-Shannon sampling theorem tells us that if a function f contains no frequencies higher than B hertz, then it can be completely determined from its ordinates at a sequence of points spaced less than $\frac{1}{2B}$ seconds apart.

Discrete Fourier transform

Definition 66. Consider a function f with support $\{0, \ldots, N-1\}$. We can think f as:

$$\begin{aligned} f : \mathbb{Z} &\longrightarrow \mathbb{C} \\ k &\longmapsto f(k \mod N) =: f[k] \end{aligned}$$

Note that with this definition, f is N-periodic. We define the *discrete Fourier transform* (*DFT*) of f as:

$$\widehat{f}[k] := \sum_{n=0}^{N-1} f[n]\mathrm{e}^{-\frac{2\pi \mathrm{i} nk}{N}}$$

If we denote $\omega_N := \mathrm{e}^{-\frac{2\pi \mathrm{i}}{N}}$ we can write:

$$\widehat{f}[k] = \sum_{n=0}^{N-1} f[n]{\omega_N}^{kn}$$

We will denote $\mathbf{f} := (f[0], \ldots, f[N-1])$

Proposition 67. Let $f, g : \mathbb{Z} \to \mathbb{C}$. Then:

1. $\widehat{f}$ is linear.
2. If $n \in \mathbb{Z}$ and $g[k] = f[k-n]$ $\forall k \in \mathbb{Z}$, then:
$$\widehat{g}[k] = \widehat{f}[k]\mathrm{e}^{-\frac{2\pi \mathrm{i} kn}{N}}$$
3. If $g[k] = \overline{f[k]}$ $\forall k \in \mathbb{Z}$, then:
$$\widehat{g}[k] = \overline{\widehat{f}[N-k]}$$

Proposition 68. Let $f : \mathbb{Z} \to \mathbb{C}$. Then, $\widehat{\mathbf{f}} = \mathbf{A}(\omega_N)\mathbf{f}$, where

$$\mathbf{A}(\omega_N) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & {\omega_N}^2 & \cdots & {\omega_N}^{N-1} \\ 1 & {\omega_N}^2 & {\omega_N}^4 & \cdots & {\omega_N}^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & {\omega_N}^{N-1} & {\omega_N}^{2(N-1)} & \cdots & {\omega_N}^{(N-1)(N-1)} \end{pmatrix}$$

is a symmetric matrix.

Lemma 69. Let $N \in \mathbb{N}$. Then:

$$\mathbf{A}(\omega_N)\mathbf{A}(\overline{\omega_N}) = \mathbf{A}(\overline{\omega_N})\mathbf{A}(\omega_N) = N\mathbf{I}_N$$

Sketch of the proof. Remember that both ω_N and $\overline{\omega_N}$ are roots of $1 + x + \cdots + x^{N-1}$. □

Definition 70. Let $f : \mathbb{Z} \to \mathbb{C}$. We define the *inverse discrete Fourier transform* as:

$$\mathbf{f} = \frac{1}{N}\mathbf{A}(\overline{\omega_N})\widehat{\mathbf{f}}$$

Theorem 71 (Plancherel theorem). Let $f : \mathbb{Z} \to \mathbb{C}$. Then:

$$\sum_{k=0}^{N-1} f[k]\overline{g[k]} = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{f}[k]\overline{\widehat{g}[k]}$$

In particular, if $f = g$, we have:

$$\sum_{k=0}^{N-1} |f[k]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \left|\widehat{f}[k]\right|^2$$

Proof. Using vector notation:

$$\begin{aligned}\left\langle \mathbf{f}^{\mathrm{T}}, \overline{\mathbf{g}}\right\rangle &= \left(\frac{1}{N}\mathbf{A}(\overline{\omega_N})\widehat{\mathbf{f}}\right)^{\mathrm{T}}\left(\frac{1}{N}\overline{\mathbf{A}(\overline{\omega_N})\widehat{\mathbf{g}}}\right)\\ &= \frac{1}{N^2}\widehat{\mathbf{f}}^{\mathrm{T}}\mathbf{A}(\overline{\omega_N})^{\mathrm{T}}\mathbf{A}(\omega_N)\overline{\widehat{\mathbf{g}}}\\ &= \frac{1}{N}\left\langle \widehat{\mathbf{f}}^{\mathrm{T}}, \overline{\widehat{\mathbf{g}}}\right\rangle\end{aligned}$$

because $\mathbf{A}(\overline{\omega_N})$ is symmetric. □

Definition 72. Let $f, g : \mathbb{Z} \to \mathbb{C}$. We define the *convolution* of f and g as:

$$(f * g)[k] := \sum_{n=0}^{N-1} f[n]g[k-n]$$

Lemma 73. Let $f, g : \mathbb{Z} \to \mathbb{C}$. Then:

$$\widehat{f * g}[k] = \widehat{f}[k]\widehat{g}[k]$$

Proof.

$$\begin{aligned}\widehat{f * g}[k] &= \sum_{n=0}^{N-1}\sum_{j=0}^{N-1} f[j]g[n-j]{\omega_N}^{nk}\\ &= \sum_{j=0}^{N-1} f[j]{\omega_N}^{jk}\sum_{n=0}^{N-1} g[n-j]{\omega_N}^{(n-j)k}\\ &= \widehat{f}[k]\widehat{g}[k]\end{aligned}$$

□

Theorem 74 (Poisson summation formula). Let $f : \mathbb{Z} \to \mathbb{C}$. Then:

$$\sum_{k=0}^{N-1} \widehat{f}[k] = Nf[0]$$

Proof.

$$\sum_{k=0}^{N-1} \widehat{f}[k] = \sum_{k,n=0}^{N-1} f[n]{\omega_N}^{kn} = Nf[0]$$

because $\sum_{k=0}^{N-1} {\omega_N}^{kn} = N$ if $n = 0$ and 0 otherwise because ${\omega_N}^n$ are roots of $1 + x + \cdots + x^{N-1}$. □

Fast Fourier transform

Definition 75. Let $f : \mathbb{Z} \to \mathbb{C}$. Note that we need $\mathrm{O}\left(N^2\right)$ operations in order to compute $\widehat{f}$. The *fast Fourier transform* (*FFT*) aims to minimize that number by using some tricks.

Definition 76 (Radix-2 DIT Cooley-Tukey FFT algorithm). Let $f : \mathbb{Z} \to \mathbb{C}$ and assume that $N = 2m$. The *radix-2 decimation-in-time (DIT) FFT* is defined as follows. We can write:

$$\begin{aligned}\widehat{f}[k] &= \sum_{n=0}^{N/2-1} f[2n]{\omega_N}^{2nk} + \sum_{n=0}^{N/2-1} f[2n+1]{\omega_N}^{(2n+1)k}\\ &= \sum_{n=0}^{N/2-1} f[2n]\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{N/2}}\right)^{nk} + \mathrm{e}^{-\frac{2\pi\mathrm{i}k}{N}}\sum_{n=0}^{N/2-1} f[2n+1]\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{N/2}}\right)^{nk}\\ &=: E_k + \mathrm{e}^{-\frac{2\pi\mathrm{i}k}{N}}O_k\end{aligned}$$

for $k = 0, \ldots, N/2 - 1$ even though the equality holds for $k = 0, \ldots, N - 1$. For the other cases, we use the periodicity of $\mathrm{e}^{-\frac{2\pi\mathrm{i}k}{N}}$ to get:

$$\widehat{f}[k + N/2] = E_k - \mathrm{e}^{-\frac{2\pi\mathrm{i}k}{N}}O_k$$

for $k = 0, \ldots, N/2-1$. Note that E_k and O_k are both $N/2$-dimensional DFT of the even terms of f and the odd terms of f, respectively. We can thus compute them recursively until the respective m is odd. Using this method we can get the DFT of f in at most (when $N = 2^\ell$) $\mathrm{O}(N \log N)$ time.

3. | Distributions

Introduction

Definition 77. Let $\Omega \subseteq \mathbb{R}^n$ and $(\varphi_n), \varphi \in \mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega)$. The functions on $\mathcal{D}(\Omega)$ are usually called *bump functions* or *test functions*. We say that $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ if:

1. There exists a compact set such that $\operatorname{supp}\varphi_n, \operatorname{supp}\varphi \subseteq K$ $\forall n \in \mathbb{N}$.
2. $\displaystyle\lim_{n\to\infty} \|\partial^\alpha\varphi_n - \partial^\alpha\varphi\|_{L^\infty(K)} = 0$ $\forall \alpha \in (\mathbb{N} \cup \{0\})^d$.

Definition 78 (Distribution). Let $\Omega \subseteq \mathbb{R}^d$ be a set. A *distribution* on Ω is a continuous linear form on $\mathcal{D}(\Omega)$. The vector space of all distributions on Ω is denoted by $\mathcal{D}^*(\Omega)$.

Lemma 79. Let $\Omega \subseteq \mathbb{R}^d$ and $T : \mathcal{D}(\Omega) \to \mathbb{C}$ be linear. Then, T is continuous if and only if $\forall(\varphi_n) \in \mathcal{D}(\Omega)$ with $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$ we have that $T(\varphi_n) \to 0$[1].

Lemma 80 (Fundamental lemma of calculus of variations). Let $\Omega \subseteq \mathbb{R}^d$ be a domain and $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then, $f \overset{\text{a.e.}}{=} 0$ in Ω.

Proposition 81. Let $\Omega \subseteq \mathbb{R}^d$ and $T : \mathcal{D}(\Omega) \to \mathbb{C}$ be linear. Then, $T \in \mathcal{D}^*(\Omega)$ if and only if for all compact set $K \subseteq \Omega$, there exist $C > 0$ and $m \in \mathbb{N} \cup \{0\}$ such that $\forall\varphi \in \mathcal{D}(K)$ we have:

$$|T(\varphi)| \leq C \sum_{|\alpha|\leq m} \|\partial^\alpha\varphi\|_{L^\infty(K)}$$

Proof. The right-to-left implication is clear. For the other one, suppose that there exists a compact set K such that $\forall C > 0$ and all $m \in \mathbb{N} \cup \{0\}$ there exists a sequence $(\varphi_k) \in \mathcal{D}(\Omega)$ such that:

$$|T(\varphi_k)| > C \sum_{|\alpha|\leq m} \|\partial^\alpha\varphi_k\|_{L^\infty(K)} =: C\|\varphi_k\|_{m,K}$$

Now consider $\psi_k := \frac{\varphi_k}{k\|\varphi_k\|_{m,K}}$. Clearly $\forall\alpha \in (\mathbb{N} \cup \{0\})^d$ $\|\partial^\alpha\psi_k\|_{L^\infty(K)} \leq \frac{1}{k} \xrightarrow{k\to\infty} 0$ but $|T(\psi_k)| = \frac{|T(\varphi_k)|}{k\|\varphi_k\|_{m,K}} > 1$ by considering the particular case of $C = k$. Hence, T cannot be continuous, which is a contradiction. □

[1] Sometimes we will denote $T(\varphi)$ as $\langle T, \varphi\rangle$.

Proposition 82. Let $\Omega \subseteq \mathbb{R}^n$ and $f \in L^1_{\text{loc}}(\Omega)$. Then, the map

$$\begin{aligned} T_f : \mathcal{D}(\Omega) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} \end{aligned} \tag{2}$$

is a distribution. Hence, $T_f(\varphi)$ is usually denoted by $\langle f, \varphi\rangle$. Sometimes we will do an abuse of notation denoting T_f as f (in view of the 80 Fundamental lemma of calculus of variations).

Proof. T_f is clearly linear. Moreover:

$$|T_f(\varphi)| \leq \int_\Omega |f(\mathbf{x})\varphi(\mathbf{x})| \leq \|f\|_1 \|\varphi\|_\infty$$

Hence, T_f is bounded and therefore continuous. □

Definition 83. The distributions that can be expressed as in Eq. (2) are called *regular distributions*.

Proposition 84 (Dirac's δ distribution). Let $\Omega \subseteq \mathbb{R}^d$ be a set and $\mathbf{x}_0 \in \Omega$. Then, the map

$$\begin{aligned} \delta_{\mathbf{x}_0} : \mathcal{D}(\Omega) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \varphi(\mathbf{x}_0) \end{aligned}$$

is a distribution and it is called *Dirac's δ distribution*. We will denote $\delta_{\mathbf{0}}$ simply by δ.

Proof. Clearly $\delta_{\mathbf{x}_0}$ is linear and bounded because $|\delta_{\mathbf{x}_0}(\varphi)| = |\varphi(\mathbf{x}_0)| \leq \|\varphi\|_\infty$. □

Lemma 85. The Dirac's $\delta_{\mathbf{0}}$ distribution is not regular.

Proof. Suppose it is regular. Then, $\exists f \in L^1_{\text{loc}}(\Omega)$ such that $\delta = T_f$. Hence, $\varphi(\mathbf{0}) = \delta(\varphi) = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}$ for all $\varphi \in \mathcal{D}(\Omega)$. Then, if we take $\varphi_n(\mathbf{x}) := \varphi(n\mathbf{x})$, where:

$$\varphi(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-\|\mathbf{x}\|^2}} & \text{if } \|\mathbf{x}\| \leq 1 \\ 0 & \text{if } \|\mathbf{x}\| > 1 \end{cases}$$

then $\varphi_n \in \mathcal{D}(\Omega)$ and have support $\overline{B(\mathbf{0}, 1/n)}$. So:

$$\mathrm{e}^{-1} = \left| \int_{\Omega \cap \overline{B(\mathbf{0},1/n)}} f(\mathbf{x})\varphi_n(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \leq \int_{\|x\| < \frac{1}{n}} |f(\mathbf{x})|\,\mathrm{d}\mathbf{x} \overset{n\to\infty}{\longrightarrow} 0$$

□

Proposition 86 (Cauchy principal value). We define the *Cauchy principal value* $T := \text{p.v.}\left(\frac{1}{x}\right)$ as the distribution

$$T(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x$$

Proof. First of all note that it is well defined because we can write:

$$T(\varphi) = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{\varphi(x) - \varphi(-x)}{x}\,\mathrm{d}x = \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}\,\mathrm{d}x$$

which is well-defined because φ has compact support and in a neighborhood of 0 the integrand is bounded (by the **??** **??**). Moreover it is clearly linear and continuous because

$$|T(\varphi)| \leq 2|K| \, \|\varphi'\|_\infty$$

where $|K|$ is the measure of the support of φ. □

Definition 87. Let $\Omega \subseteq \mathbb{R}^n$. We say that a distribution $T \in \mathcal{D}^*(\Omega)$ is a *distribution of order* $N \in \mathbb{N} \cup \{0\}$ if $\exists N \in \mathbb{N} \cup \{0\}$ such that for all compact set K $\exists C_K > 0$ with

$$|T(\varphi)| \leq C_K \|\varphi\|_{N,K}$$

for all $\varphi \in \mathcal{D}(\Omega)$. We say that T is has *infinite order* if it is not of order N for any $N \in \mathbb{N}$.

Definition 88. Let $\Omega \subseteq \mathbb{R}^n$, $T, S \in \mathcal{D}^*(\Omega)$, $a \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. We define the distributions $T + S$, aT and fT as:

$$\begin{aligned} \langle T + S, \varphi\rangle &:= \langle T, \varphi\rangle + \langle S, \varphi\rangle \\ \langle aT, \varphi\rangle &:= \langle T, a\varphi\rangle \\ \langle fT, \varphi\rangle &:= \langle T, f\varphi\rangle \end{aligned}$$

Remark. In general the product of two distributions is not associative. For example, one can check that $\delta x = 0$ and $x\text{p.v.}\left(\frac{1}{x}\right) = 1$. So:

$$(\delta x)\,\text{p.v.}\left(\frac{1}{x}\right) \neq \delta\left(x\text{p.v.}\left(\frac{1}{x}\right)\right)$$

Convergence of distributions

Definition 89. Let $\Omega \subseteq \mathbb{R}^n$ be a set and $(T_n) \in \mathcal{D}^*(\Omega)$. We say that (T_n) converges to $T \in \mathcal{D}^*(\Omega)$ if $T_n(\varphi) \overset{n\to\infty}{\longrightarrow} T(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$.

Definition 90. Let $\Omega \subseteq \mathbb{R}^n$ be a set. We say that a sequence of functions $(\phi_\varepsilon) \in L^1_{\text{loc}}(\Omega)$ is an *approximation of identity* if

1. $\displaystyle\int_\Omega \phi_\varepsilon = 1$
2. $\displaystyle\int_\Omega |\phi_\varepsilon| \leq M \ \forall \varepsilon > 0$
3. $\displaystyle\lim_{\varepsilon\to 0} \int_{\|\mathbf{x}\| \geq \delta} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \ \forall \delta > 0.$

Proposition 91. Let $\Omega \subseteq \mathbb{R}^n$ be a set and $\phi \in L^1(\Omega)$ such that $\int_\Omega \phi = 1$. Let $\phi_\varepsilon := \frac{1}{\varepsilon^n}\phi(\frac{\mathbf{x}}{\varepsilon})$. Then, (ϕ_ε) is an approximation of identity, $\phi_\varepsilon \in L^1_{\text{loc}}(\Omega)$ $\forall \varepsilon > 0$ and $\phi_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \delta_{\mathbf{0}}$ in $\mathcal{D}^*(\Omega)$.

Sketch of the proof. Let $\varphi \in \mathcal{D}(\Omega)$. Then:

$$|\phi_\varepsilon(\varphi) - \delta_{\mathbf{0}}(\varphi)| \leq \int_\Omega |\phi_\varepsilon(x)||\varphi(\mathbf{x}) - \varphi(\mathbf{0})|\,\mathrm{d}\mathbf{x}$$

$$= \int\limits_{\|\mathbf{x}\|<\delta} |\phi_\varepsilon(x)||\varphi(\mathbf{x}) - \varphi(\mathbf{0})| \, \mathrm{d}\mathbf{x} + \\ + \int\limits_{\|\mathbf{x}\|\geq\delta} |\phi_\varepsilon(x)||\varphi(\mathbf{x}) - \varphi(\mathbf{0})| \, \mathrm{d}\mathbf{x}$$

Now use the properties of approximation of identity to see that each interval goes to zero as $\varepsilon \to 0$. □

Theorem 92. Let $\Omega \subseteq \mathbb{R}^n$ and $(f_n) \in L^p_{\mathrm{loc}}(\Omega)$ such that $f_n \xrightarrow{L^p_{\mathrm{loc}}} f$ (which means that $\|f_n - f\|_{L^p(K)} \to 0$ for any compact set $K \subseteq \Omega$). Then, T_{f_n} converges to T_f in $\mathcal{D}(\Omega)$.

Proof. Use **?? ??**. □

Remark. Clearly if f_n converge uniformly to f, the condition of the theorem holds and we get the same result. But it can be seen that only with pointwise convergence is not enough (consider $f_n(x) = n^k x^n (1-x)\mathbf{1}_{[0,1]}$ for $k \in \mathbb{N}$). Moreover, T_{f_n} converges to T_f in $\mathcal{D}(\Omega)$ does not imply pointwise convergence of f_n towards f.

Support of a distribution

Definition 93. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$. We define the *support* of T, $\operatorname{supp} T$, as the intersection of all closed sets K such that if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ has support in $\mathbb{R}^n \setminus K$, then $\langle T, \varphi \rangle = 0$.

Lemma 94. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\operatorname{supp} \varphi \cap \operatorname{supp} T = \varnothing$. Then, $\langle T, \varphi \rangle = 0$.

Proof. Assume $\operatorname{supp} T = \bigcap_{i \in I} C_i$. Then, by the compactness of $\operatorname{supp} \varphi$, there exists $i_1, \ldots, i_n \in I$ such that:

$$\operatorname{supp} \varphi \subseteq \bigcup_{j=1}^{n} (\mathbb{R}^n \setminus C_{i_j})$$

Now take a partition of unity $\psi_1, \ldots, \psi_n \in \mathcal{D}(\mathbb{R}^n)$ subordinated to the open cover $\{\mathbb{R}^n \setminus C_{i_j} : j = 1, \ldots, n\}$ (check **??**). These ψ_j satisfy (by definition) that $\operatorname{supp} \psi_j \subseteq \mathbb{R}^n \setminus C_{i_j}$ for all $j = 1, \ldots, n$ and $\sum_{j=1}^n \psi_j = 1$ on $\operatorname{supp} \varphi$. Therefore, defining $\psi := \sum_{j=1}^n \psi_j$ we have that $\psi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi\psi = \varphi$ on $\operatorname{supp} \varphi$. Therefore:

$$\langle T, \varphi \rangle = \langle T, \varphi\psi \rangle = \sum_{j=1}^{n} \langle T, \varphi\psi_j \rangle = 0$$

□

Definition 95. We denote $\mathcal{E}(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)$ and $\mathcal{E}^*(\mathbb{R}^n)$ its dual space.

Definition 96. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ with compact support. We can extent the definition of T to $\mathcal{E}(\mathbb{R}^n)$ in the following way. Let $\varphi \in \mathcal{E}(\mathbb{R}^n)$ and take $\rho \in \mathcal{D}(\mathbb{R}^n)$ such that $\rho = 1$ on $\operatorname{supp} T$. Then, we define:

$$\langle T, \varphi \rangle := \langle T, \rho\varphi \rangle$$

Remark. Note that in view of Theorem 94, this definition is well defined because if $\rho, \omega \in \mathcal{D}(\mathbb{R}^n)$ are two different test functions such that $\rho, \omega = 1$ on $\operatorname{supp} T$, then $\varphi(\rho - \omega) = 0$ on $\operatorname{supp} T$ and therefore $\langle T, \varphi(\rho - \omega) \rangle = 0$.

Proposition 97. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ with compact support. Then, $T \in \mathcal{E}^*(\mathbb{R}^n)$ if and only if $\exists C > 0$, $N \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that:

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup_{\|\mathbf{x}\| \leq N} |\partial^\alpha \varphi(\mathbf{x})|$$

for all $\varphi \in \mathcal{E}(\mathbb{R}^n)$.

Proof. The implication to the left is clear. For the other one, from the continuity in $\mathcal{D}^*(\mathbb{R}^n)$ we know that for all compact K, there exist $C > 0$ and $m \in \mathbb{N} \cup \{0\}$ such that:

$$|\langle T, \varphi \rangle| = |\langle T, \rho\varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup_{\mathbf{x} \in K} |\partial^\alpha (\rho\varphi)(\mathbf{x})|$$

Now take $N > 0$ such that $\operatorname{supp} \rho \subsetneq \operatorname{supp} T \subsetneq B(0, N)$. Thus:

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup_{\mathbf{x} \in \operatorname{supp} \rho \leq N} |\partial^\alpha \varphi(\mathbf{x})| \leq \\ \leq C \sum_{|\alpha| \leq m} \sup_{\|\mathbf{x}\| \leq N} |\partial^\alpha \varphi(\mathbf{x})|$$

□

Differentiation of distributions

Definition 98. Let $\Omega \subseteq \mathbb{R}^n$ be a set, $T \in \mathcal{D}^*(\Omega)$ and α be a multiindex. We define the distribution $\partial^\alpha T$ as:

$$\langle \partial^\alpha T, \varphi \rangle = \left\langle T, (-1)^{|\alpha|} \partial^\alpha \varphi \right\rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$. The distribution $\partial^\alpha T$ is called *distributional derivative*.

Definition 99. We define the *Heaviside step function* as the function $H(x) = \mathbf{1}_{x>0}$.

Proposition 100. We have that $T_H =: H \in \mathcal{D}^*(\mathbb{R})$ and:

$$H' = \delta$$

Proof. For all $\varphi \in \mathcal{D}(\Omega)$ we have:

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x) \, \mathrm{d}x = \varphi(0) = \delta(\varphi)$$

because φ has compact support. □

Lemma 101. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, $(T_f)' = T_{f'}$.

Proposition 102 (Schwarz theorem). Let $\Omega \subseteq \mathbb{R}^n$ be a set and $T \in \mathcal{D}^*(\Omega)$. Then:

$$\frac{\partial^2 T}{\partial x_i \partial x_j} = \frac{\partial^2 T}{\partial x_j \partial x_i}$$

Proposition 103 (Leibnitz rule). Let $\Omega \subseteq \mathbb{R}^n$ be a set, $T \in \mathcal{D}^*(\Omega)$, $f \in \mathcal{C}^\infty(\Omega)$ and α be a multiindex. Then:

$$\partial^\alpha (fT) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha - \beta} T$$

Proposition 104. Let $T \in \mathcal{D}^*(\mathbb{R})$ be such that $T' = 0$. Then, T is constant (in the sense of distributions).

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi = 0$. Then, $\phi(x) := \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t \in \mathcal{D}(\mathbb{R})$ and $\phi' = \varphi$. Thus:

$$\langle T, \varphi \rangle = \langle T, \phi' \rangle = \langle T', \phi \rangle = 0$$

Now consider a general $\varphi \in \mathcal{D}(\mathbb{R})$ and $\omega \in \mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}} \omega = 1$. Then, $\phi(x) := \varphi - \omega \int_{\mathbb{R}} \varphi$ integrates 0 and thus:

$$\langle T, \varphi \rangle = \int\limits_{\mathbb{R}} \varphi \langle T, \omega \rangle = \langle C, \varphi \rangle$$

with $C := \langle T, \omega \rangle$. ☐

Proposition 105. Let $T \in \mathcal{D}^*(\mathbb{R})$ be such that $x^m T = 0$ for some $m \in \mathbb{N}$. Then, $T = \sum_{j=0}^{m-1} a_j \delta^{(j)}$ for some $a_j \in \mathbb{R}$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with Taylor polynomial:

$$P_\varphi(x) = \sum_{j=0}^{m-1} \frac{\varphi^{(j)}(0)}{j!} x^j + \frac{\varphi^{(m)}(\xi_x)}{m!} x^m$$

with $\xi_x \in (0, x)$. Then:

$$\begin{aligned}\langle T, \varphi \rangle &= \sum_{j=0}^{m-1} \frac{\varphi^{(j)}(0)}{j!} \langle T, x^j \rangle + \frac{1}{m!} \langle x^m T, \varphi^{(m)}(\xi_x) \rangle \\ &= \sum_{j=0}^{m-1} a_j \langle \delta^{(j)}, \varphi \rangle\end{aligned}$$

with $a_j = \frac{(-1)^j}{j!} \langle T, x^j \rangle$. ☐

Schwartz class of functions

Definition 106. Let $d \in \mathbb{N}$. The *Schwartz space* or *space of rapidly decreasing functions* on $\mathbb{R}^n$ is defined as:

$$\mathcal{S}(\mathbb{R}^d) := \{ f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \ \forall \alpha, \beta \in (\mathbb{N} \cup \{0\})^d \}$$

where:

$$\|f\|_{\alpha,\beta} := \sup_{\mathbf{x} \in \mathbb{R}^n} \left| \mathbf{x}^\alpha (\partial^\beta f)(\mathbf{x}) \right|$$

Lemma 107. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then, $\mathbf{x}^\alpha f, \partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ for all $\alpha \in (\mathbb{N} \cup \{0\})^d$.

Lemma 108. Let $d \in \mathbb{N}$. Then, $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset \mathcal{E}(\mathbb{R}^d)$.

Definition 109. Let $f, (f_n) \in \mathcal{S}(\mathbb{R}^d)$. We say that $f_n \xrightarrow{\mathcal{S}} f$ if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$.

Proposition 110. Let $d \in \mathbb{N}$. Then, $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.

Proof. For $p = \infty$ the result is clear. Now suppose that $p \in [1, \infty)$ and let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then:

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |\phi|^p &= \int\limits_{B(0,1)} |\phi|^p + \int\limits_{\mathbb{R}^d \setminus B(0,1)} \frac{|\phi|^p \|\mathbf{x}\|^{kp}}{\|\mathbf{x}\|^{kp}} \\ &\leq C_1 + C_2 \int\limits_{\mathbb{R}^d \setminus B(0,1)} \frac{1}{\|\mathbf{x}\|^{kp}}\end{aligned}$$

for some $k \in \mathbb{N}$ yet to be determined. Here in the last step we have used $\left| |\phi| \, \|\mathbf{x}\|^k \right| \leq \|f\|_{k,0} =: C_2$. Now if $R_j := \{\mathbf{x} \in \mathbb{R}^d : 2^j \leq \|\mathbf{x}\| \leq 2^{j+1}\}$, $j \in \mathbb{N} \cup \{0\}$, then:

$$\begin{aligned}\int\limits_{\mathbb{R}^d \setminus B(0,1)} \frac{1}{\|\mathbf{x}\|^{kp}} &\leq \sum_{j=0}^{\infty} \int\limits_{R_j} \frac{1}{\|\mathbf{x}\|^{kp}} \\ &\leq \sum_{j=0}^{\infty} \frac{C 2^{(j+1)d}}{2^{kpj}} \\ &< \infty\end{aligned}$$

if and only if $kp - d > 0$. So take $k > \frac{d}{p}$. ☐

Remark. In $\mathbb{R}^n$, the integrals of the form $\int\limits_{B(0,1)} \frac{1}{\|\mathbf{x}\|^k} \mathrm{d}\mathbf{x}$ converge if and only if $k < n = \dim \mathbb{R}^n$ whereas the integrals of the form $\int\limits_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{\|\mathbf{x}\|^k} \mathrm{d}\mathbf{x}$ converge if and only if $k > n$. The limit case $k = n$ is diverges in both cases.

Lemma 111. Let f be a function that has Fourier transform. Then, $f \in \mathcal{S}(\mathbb{R}^d) \iff \widehat{f} \in \mathcal{S}(\mathbb{R}^d)$.

Proof. By symmetry, it suffices to do one implication. Moreover we will only do the case $d = 1$ in order to keep the notation simple. Let $f \in \mathcal{S}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Then, using Theorems 14 and 15:

$$\begin{aligned}\left| \boldsymbol{\xi}^\alpha \partial^\beta \widehat{f}(\boldsymbol{\xi}) \right| &= \left| \boldsymbol{\xi}^\alpha \mathcal{F}((-2\pi \mathrm{i}\mathbf{x})^\beta f)(\boldsymbol{\xi}) \right| \\ &= \frac{1}{|2\pi \mathrm{i}|^\alpha} \left| \mathcal{F}[\partial^\alpha((-2\pi \mathrm{i}\mathbf{x})^\beta f)](\boldsymbol{\xi}) \right| \\ &\leq \left\| \partial^\alpha((-2\pi \mathrm{i}\mathbf{x})^\beta f) \right\|_1 \\ &< \infty\end{aligned}$$

where in the last inequality we have used that $\|\widehat{g}\|_\infty \leq \|g\|_1$. ☐

Tempered distributions

Definition 112. A *tempered distribution* is a linear and continuous operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. The space of all tempered distributions is denoted by $\mathcal{S}^*(\mathbb{R}^d)$.

Lemma 113. Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ be linear. Then, $T \in \mathcal{S}^*(\mathbb{R}^d)$ if and only if there exists $C > 0$ and $m \in \mathbb{N} \cup \{0\}$ such that $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$ we have:

$$|T(\varphi)| \leq C \sum_{|\alpha| + |\beta| \leq m} \|\varphi\|_{\alpha,\beta}$$

Lemma 114. $L^p(\mathbb{R}^d) \subset \mathcal{S}^*(\mathbb{R}^d) \subset \mathcal{D}^*(\mathbb{R}^d)$.

Lemma 115. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$. Then, $\partial^\alpha T \in \mathcal{S}^*(\mathbb{R}^d)$ for all $\alpha \in (\mathbb{N} \cup \{0\})^d$.

Definition 116. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. We define the *convolution* $T * \psi$ as:

$$\langle T * \psi, \varphi \rangle := \langle T, \tilde{\psi} * \varphi \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Here $\tilde{\psi}(\mathbf{x}) := \psi(-\mathbf{x})$.

Lemma 117. Let $a, b \geq 0$ and $m \in \mathbb{N} \cup \{0\}$. Then:

$$(a+b)^m \leq 2^{m-1}(a^m + b^m)$$

and the equality holds if and only if $a = b$ or $m = 0, 1$.

Proof. For $m = 0, 1$, the equality is true. Now suppose $m \geq 2$ and $b = \lambda a$ with $\lambda \in [0, \infty)$. We need to show that:

$$(1+\lambda)^m \leq 2^{m-1}(1+\lambda^m)$$

Consider $f(\lambda) := 2^{m-1}(1+\lambda^m) - (1+\lambda)^m$. Then:

$$f'(\lambda) = m\left[(2\lambda)^{m-1} - (1+\lambda)^{m-1}\right]$$

Note that $\forall m \geq 2$, $f'(\lambda) < 0$ for $\lambda \in [0,1)$, $f'(1) = 0$ and $f'(\lambda) > 0$ for $\lambda \in (1, \infty)$. Moreover $f(1) = 0$. So $f(\lambda) \geq 0$ for all $\lambda \in [0, \infty)$ and the equality holds if and only if $\lambda = 1$. □

Lemma 118. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then, $T * \psi \in \mathcal{S}^*(\mathbb{R}^d)$.

Proof. Clearly $T * \psi$ is linear. Let $\varphi_n \xrightarrow{\mathcal{S}} 0$. Then, it suffices to see that $\tilde{\psi} * \varphi_n \xrightarrow{\mathcal{S}} 0$. For the sake of simplicity we only do the case $d = 1$. For all $\alpha, \beta \in \mathbb{N} \cup \{0\}$ we have:

$$\begin{aligned}
\left|\mathbf{x}^\alpha \partial^\beta (\tilde{\psi} * \varphi_n)(\mathbf{x})\right| &= \left|\mathbf{x}^\alpha (\partial^\beta \tilde{\psi} * \varphi_n)(\mathbf{x})\right| \\
&\leq \int_{\mathbb{R}^d} \left|\mathbf{x}^\alpha \partial^\beta \psi(\mathbf{y}) \varphi_n(\mathbf{x}-\mathbf{y})\right| \mathrm{d}\mathbf{y} \\
&\leq 2^m \int_{\mathbb{R}^d} |\mathbf{x}-\mathbf{y}|^\alpha \left|\partial^\beta \psi(\mathbf{y})\right| |\varphi_n(\mathbf{x}-\mathbf{y})| \mathrm{d}\mathbf{y} \\
&\quad + 2^m \int_{\mathbb{R}^d} |\mathbf{y}|^\alpha \left|\partial^\beta \psi(\mathbf{y})\right| |\varphi_n(\mathbf{x}-\mathbf{y})| \mathrm{d}\mathbf{y} \\
&\leq 2^m \sup_{\mathbf{x} \in \mathbb{R}^d} |\mathbf{x}|^\alpha |\varphi_n(\mathbf{y})| \int_{\mathbb{R}^d} \left|\partial^\beta \psi(\mathbf{y})\right| \mathrm{d}\mathbf{y} \\
&\quad + 2^m \sup_{\mathbf{x} \in \mathbb{R}^d} |\varphi_n(\mathbf{y})| \int_{\mathbb{R}^d} |\mathbf{y}|^\alpha \left|\partial^\beta \psi(\mathbf{y})\right| \mathrm{d}\mathbf{y}
\end{aligned}$$

where in the second inequality we have used Theorem 117 with $m = |\alpha| + 1$. Note that this latter terms tend to zero as $n \to \infty$ because of the properties of the Schwartz space. □

Lemma 119. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$. Then:

$$\partial^\alpha (T * \psi) = \partial^\alpha T * \psi = T * \partial^\alpha \psi$$

Proof.

$$\begin{aligned}
\langle \partial^\alpha (T*\psi), \varphi \rangle &= (-1)^{|\alpha|} \langle T*\psi, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle T, \tilde{\psi} * \partial^\alpha \varphi \rangle \\
&= (-1)^{|\alpha|} \langle T, \partial^\alpha (\tilde{\psi} * \varphi) \rangle = \langle \partial^\alpha T, \tilde{\psi} * \varphi \rangle = \langle \partial^\alpha T * \psi, \varphi \rangle
\end{aligned}$$

The other equality is analogous. □

Fourier transform of distributions

Definition 120. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$. We define the *Fourier transform* $\widehat{T}$ (or $\mathcal{F}T$) of T as

$$\langle \widehat{T}, \varphi \rangle := \langle T, \widehat{\varphi} \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We define the *inverse Fourier transform* $\mathcal{F}^{-1}T$ of T as

$$\langle \mathcal{F}^{-1}T, \varphi \rangle := \langle T, \mathcal{F}^{-1}\varphi \rangle$$

Lemma 121. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$. Then, $\mathcal{F}T, \mathcal{F}^{-1}T \in \mathcal{S}^*(\mathbb{R}^d)$.

Proposition 122. Let $T \in \mathcal{S}^*(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$. Then:

1. $\partial^\alpha \widehat{T} = \mathcal{F}((-2\pi \mathrm{i}\mathbf{x})^\alpha T)$
2. $\widehat{\partial^\alpha T} = (2\pi \mathrm{i}\boldsymbol{\xi})^\alpha \widehat{T}$
3. $\widehat{T * \psi} = \widehat{T}\widehat{\psi}$

Proof. We prove the third one. The other are similar. We have:

$$\begin{aligned}
\langle \widehat{T*\psi}, \varphi \rangle = \langle T*\psi, \widehat{\varphi} \rangle = \langle T, \tilde{\psi} * \widehat{\varphi} \rangle = \langle \widehat{T}, \mathcal{F}^{-1}(\tilde{\psi}\widehat{\varphi}) \rangle = \\
= \langle \widehat{T}, \mathcal{F}^{-1}(\tilde{\psi})\varphi \rangle = \langle \widehat{T}, \widehat{\psi}\varphi \rangle = \langle \widehat{T}\widehat{\psi}, \varphi \rangle
\end{aligned}$$

□

Lemma 123. We have that:

1. $\widehat{\delta_\mathbf{a}} = \mathrm{e}^{-2\pi \mathrm{i}\mathbf{a}\cdot\mathbf{x}}$
2. $\widehat{\mathbf{1}_{(0,\infty)}} = \frac{1}{2\pi \mathrm{i}} \mathrm{p.v.}\left(\frac{1}{x}\right) + \frac{1}{2}\delta_0$
3. $\widehat{\mathrm{p.v.}\left(\frac{1}{x}\right)} = -\pi \mathrm{i}\, \mathrm{sgn}(\xi)$
4. $\delta * f = f \ \forall f \in \mathcal{D}(\mathbb{R}^d)$

Proof. We prove the second one, the others are easier. Note that $(\mathbf{1}_{(0,\infty)})' = \delta$. Taking Fourier transform and using Theorem 122, we get $2\pi \mathrm{i} x \widehat{\mathbf{1}_{(0,\infty)}} = 1$. Hence, since $x\mathrm{p.v.}\left(\frac{1}{x}\right) = 1$, we have:

$$x\left(2\pi \mathrm{i}\widehat{\mathbf{1}_{(0,\infty)}} - \mathrm{p.v.}\left(\frac{1}{x}\right)\right) = 0$$

By Theorem 105, we get that $\widehat{\mathbf{1}_{(0,\infty)}} = \frac{1}{2\pi \mathrm{i}} \mathrm{p.v.}\left(\frac{1}{x}\right) + C\delta_0$, for some $C \in \mathbb{R}$. To find the constant change $x \to -x$ in the equation or alternatively apply the distribution to the function $\mathrm{e}^{-\pi x^2}$. □

Definition 124. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ and $S \in \mathcal{D}^*(\mathbb{R}^m)$. We define the *direct product* TS as the distribution in $\mathcal{D}^*(\mathbb{R}^{n+m})$ given by:

$$\langle TS, \varphi \rangle = \langle T, \langle S, \varphi(\mathbf{x}, \cdot) \rangle \rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$. Usually we will dentote

$$\langle TS, \varphi(\mathbf{x}, \mathbf{y}) \rangle = \langle T(\mathbf{x}), \langle S(\mathbf{y}), \varphi(\mathbf{x}, \mathbf{y}) \rangle \rangle$$

in order to distinguish the variables

Lemma 125. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ and $S \in \mathcal{D}^*(\mathbb{R}^m)$. Then:

1. $\phi(\mathbf{x}) := \langle S(\mathbf{y}), \varphi(\mathbf{x}, \mathbf{y})\rangle \in \mathcal{D}(\mathbb{R}^n)$ and $\forall \alpha \in (\mathbb{N} \cup \{0\})^n$ we have
$$\partial^\alpha_\mathbf{x} \phi(\mathbf{x}) = \langle S(\mathbf{y}), \partial^\alpha_\mathbf{x} \varphi(\mathbf{x}, \mathbf{y})\rangle$$
2. TS is indeed a distribution.
3. $TS = ST$

Proposition 126. Let $T \in \mathcal{D}^*(\mathbb{R}^n)$ and $S \in \mathcal{D}^*(\mathbb{R}^m)$. Then, $\widehat{TS} = \widehat{T}\widehat{S}$.

Proof. Given $\varphi(\mathbf{x}, \mathbf{y}) \in \mathcal{D}(\mathbb{R}^{n+m})$ we have:
$$\begin{aligned}\mathcal{F}(\varphi(\mathbf{x}, \mathbf{y})) &= \int_{\mathbb{R}^{n+m}} \varphi(\mathbf{x}, \mathbf{y}) e^{-2\pi\mathrm{i}(\mathbf{x},\mathbf{y})\cdot(\boldsymbol{\xi},\boldsymbol{\eta})} \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^n} e^{-2\pi\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}} \left(\int_{\mathbb{R}^m} \varphi(\mathbf{x}, \mathbf{y}) e^{-2\pi\mathrm{i}\mathbf{y}\cdot\boldsymbol{\eta}} \,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} \\ &=: \mathcal{F}_\mathbf{x}(\mathcal{F}_\mathbf{y}(\varphi)) = \mathcal{F}_\mathbf{y}\mathcal{F}_\mathbf{x}\varphi\end{aligned}$$
by **?? ??**. Therefore:
$$\begin{aligned}\langle \mathcal{F}(TS), \varphi\rangle &= \langle T, \langle S, \mathcal{F}_\mathbf{y}\mathcal{F}_\mathbf{x}\varphi\rangle\rangle = \langle T, \langle \widehat{S}, \mathcal{F}_\mathbf{x}\varphi\rangle\rangle = \\ &= \langle \widehat{S}T, \mathcal{F}_\mathbf{x}\varphi\rangle = \langle \widehat{S}, \langle T, \mathcal{F}_\mathbf{x}\varphi\rangle\rangle = \langle \widehat{S}, \langle \widehat{T}, \varphi\rangle\rangle = \langle \widehat{S}\widehat{T}, \varphi\rangle\end{aligned}$$
□

Homogeneous distributions

Definition 127 (Homogeneous distribution). A distribution $T \in \mathcal{S}^*(\mathbb{R}^n)$ is said to be *homogeneous of degree* $r \in \mathbb{R}$ if:
$$\langle T, \varphi(\lambda\mathbf{x})\rangle = \lambda^{-n-r}\langle T, \varphi(\mathbf{x})\rangle$$
for all $\lambda > 0$ and all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 128. Let $T \in \mathcal{S}^*(\mathbb{R}^n)$ be a homogeneous distribution of degree $r \in \mathbb{R}$. Then, $\partial^\alpha T$ is homogeneous of degree $r - |\alpha|$ for all $\alpha \in (\mathbb{N} \cup \{0\})^n$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$. Then:
$$\begin{aligned}\langle \partial^\alpha T, \varphi(\lambda\mathbf{x})\rangle &= (-1)^{|\alpha|}\langle T, (\partial^\alpha\varphi)(\lambda\mathbf{x})\lambda^{|\alpha|}\rangle = \\ &= (-1)^{|\alpha|}\lambda^{-n-r+|\alpha|}\langle T, \partial^\alpha\varphi\rangle = \lambda^{-n-(r-|\alpha|)}\langle \partial^\alpha T, \varphi\rangle\end{aligned}$$
□

Proposition 129. Let $T \in \mathcal{S}^*(\mathbb{R}^n)$ and $r \in \mathbb{R}$. Then, T is homogeneous of degree r if and only if $\widehat{T}$ is homogeneous of degree $-n - r$.

Proof. We only check one implication, the other is analogous. Using Theorem 55 we have:
$$\begin{aligned}\langle \widehat{T}, \varphi(\lambda\mathbf{x})\rangle &= \left\langle T, \widehat{\varphi}(\boldsymbol{\xi}/\lambda)\frac{1}{\lambda^n}\right\rangle = \\ &= \lambda^{n+r-n}\langle T, \widehat{\varphi}\rangle = \lambda^{-n-(-r-n)}\langle \widehat{T}, \varphi\rangle\end{aligned}$$
□

Corollary 130. Let $k \in \mathbb{R}$, $n \in \mathbb{N}$ with $k < n$. Then, $\frac{1}{\|\mathbf{x}\|^k} \in \mathcal{S}^*(\mathbb{R}^n)$ and it is homogeneous of degree $-k$. Moreover, $\mathcal{F}\left(\frac{1}{\|\mathbf{x}\|^k}\right) = C_{k,n}\frac{1}{\|\boldsymbol{\xi}\|^{n-k}}$ with $C_{k,n} = \frac{(2\pi)^{\frac{n}{2}}}{2^{\frac{\alpha}{2}}}\frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$.

Differential operators over distributions

Definition 131. A *differential operator over distributions* is an operator of the form:
$$L(x, \partial) := \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$$
If the coefficients a_α are constant, we will omit the x and write $L(\partial)$ instead of $L(x, \partial)$.

Definition 132. Let $L(x, \partial)$ be a differential operator on an open set $U \subseteq \mathbb{R}^d$ and $f \in \mathcal{D}^*(U)$. We say that $u \in \mathcal{D}^*(\mathbb{R}^n)$ is a *generalized solution* of $L(\mathbf{x}, \partial)u = f$ in U if
$$\langle L(\mathbf{x}, \partial)u, \varphi\rangle = \langle f, \varphi\rangle$$
for all $\varphi \in \mathcal{D}(U)$.

Definition 133. Let $L(\partial)$ be a differential operator. We say that $E \in \mathcal{D}^*(\mathbb{R}^n)$ is a *fundamental solution* of $L(\partial)$ if $L(\partial)E = \delta$.

Theorem 134. Let E be a fundamental solution of $L(\partial)u = f$, $f \in \mathcal{S}(\mathbb{R}^n)$. Then, $E * f$ is a generalized solution of $L(\partial)u = f$.

Proof.
$$\begin{aligned}L(\partial)(E * f) = \sum_{|\alpha|\leq m} a_\alpha\partial^\alpha(E * f) = \sum_{|\alpha|\leq m} a_\alpha(\partial^\alpha E) * f = \\ = \delta * f = f\end{aligned}$$
where in the last equality we have used Theorem 123. □

Remark. In general we don't have unicity of fundamental solutions. Indeed if E_0 solves $L(\partial)u = 0$ and E is a fundamental solution, then $E + E_0$ is also a fundamental solution.

Theorem 135. Let $L(\partial)$ be a differential operator and $E \in \mathcal{S}^*(\mathbb{R}^n)$. Then, E is a fundamental solution of $L(\partial)$ if and only if $L(2\pi\mathrm{i}\boldsymbol{\xi})\widehat{E} = 1$.

Proof. Suppose E is a fundamental solution. Then, $L(\partial)E = \delta$. Taking Fourier transforms we have:
$$\sum_{|\alpha|\leq m} a_\alpha\mathcal{F}(\partial^\alpha E) = \sum_{|\alpha|\leq m} a_\alpha(2\pi\mathrm{i}\boldsymbol{\xi})^\alpha\widehat{E} = L(2\pi\mathrm{i}\boldsymbol{\xi})\widehat{E} = 1$$
The other implication is similar using $\mathcal{F}^{-1}$ instead. □

Definition 136. Let $L(\partial)$ be a differential operator. We say that E is a *fundamental solution* of the Cauchy problem
$$\begin{cases} L(\partial)u(t, \mathbf{x}) = 0 \\ u(0, \mathbf{x}) = f(\mathbf{x}) \end{cases} \tag{3}$$
if $L(\partial)E(t, \mathbf{x}) = 0$ and $E(0, \mathbf{x}) = \delta(\mathbf{x})$.

Theorem 137. Let $L(\partial)$ be a differential operator and E be a fundamental solution of the Cauchy problem of Eq. (3). Then, $E * f$ is a solution of it.

Applications to some PDEs

Proposition 138. Consider the operator:

$$L(\partial)=\partial_t-a^2\sum_{j=1}^n\partial_{x_j}^2$$

Then,

$$E(t)=\frac{\mathbf{1}_{[0,\infty)}(t)}{(4\pi a^2t)^{n/2}}\mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{4a^2t}}$$

is a fundamental solution the heat equation $L(\partial)u=0$.

Proof. Taking $\mathcal{F}_{\mathbf{x}}$ on the equation $L(\partial)E=\delta$ we can transform it to:

$$\partial_t\widehat{E}+4\pi^2a^2\left\|\boldsymbol{\xi}\right\|^2\widehat{E}=\delta_t$$

because $\delta=\delta_{\mathbf{x}}\delta_t$. It can be seen that a solution of this ODE is:

$$\widehat{E}(t,\xi)=\mathbf{1}_{[0,\infty)}(t)\mathrm{e}^{-4\pi^2a^2\|\boldsymbol{\xi}\|^2t}$$

Taking the Fourier transform (in this case $\mathcal{F}^{-1}=\mathcal{F}$) we have:

$$E(t,x)=\frac{\mathbf{1}_{[0,\infty)}(t)}{(4\pi a^2t)^{n/2}}e^{-\frac{\|\mathbf{x}\|^2}{4a^2t}}$$

where we have used Item 9-4 and Theorem 17. □

Proposition 139. Consider the Laplace operator:

$$L(\partial)=\sum_{j=1}^n\partial_{x_j}^2$$

Then,

$$E(t)=\begin{cases}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\pi^{\frac{n}{2}-2}}\frac{1}{\|\mathbf{x}\|^{n-2}} & \text{if } n\geq 3\\ \frac{\log\|\mathbf{x}\|}{2\pi} & \text{if } n=2\end{cases}$$

is a fundamental solution the Laplace equation $L(\partial)u=0$.

Proof. Taking $\mathcal{F}_{\mathbf{x}}$ on the equation $L(\partial)E=\delta$ we see that $\widehat{E}$ satisfies:

$$\widehat{E}(\boldsymbol{\xi})=\frac{-1}{4\pi^2\left\|\boldsymbol{\xi}\right\|^2}$$

Let's study the integrability of this latter function in a neighbourhood of 0. Let $R_j:=\{\mathbf{x}\in\mathbb{R}^n:2^{-j}\leq\|x\|\leq 2^{-j+1}\}$. Then:

$$\int\limits_{B(0,1)}\frac{1}{\|\mathbf{x}\|^2}=\sum_{j=1}^\infty\int\limits_{R_j}\frac{1}{\|\mathbf{x}\|^2}\lesssim\sum_{j=1}^\infty\frac{(2^{-j})^n}{2^{-2j}}=$$

$$=\sum_{j=1}^\infty 2^{j(n-2)}<\infty\iff n\geq 3$$

Let's study first the case $n\geq 3$. We need to compute $\mathcal{F}^{-1}(\widehat{E})=\mathcal{F}(\widehat{E})$. Recall that $\mathcal{F}(\mathrm{e}^{-k\|\mathbf{x}\|^2})=\left(\frac{\pi}{k}\right)^{\frac{n}{2}}\mathrm{e}^{-\frac{\pi^2\|\boldsymbol{\xi}\|^2}{k}}$ (try to generalize Theorem 17). Therefore:

$$\left(\frac{\pi}{k}\right)^{\frac{n}{2}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{-\frac{\pi^2\|\boldsymbol{\xi}\|^2}{k}}\varphi(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}=\langle\mathcal{F}(\mathrm{e}^{-k\|\mathbf{x}\|^2}),\varphi\rangle=$$

$$=\int\limits_{\mathbb{R}^n}\mathrm{e}^{-k\|\mathbf{x}\|^2}\widehat{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

Integrating both sides with respect to k and using **?? ??** we have that, on the one hand:

$$\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{\xi})\int\limits_0^\infty\mathrm{e}^{-k\|\boldsymbol{\xi}\|^2}\,\mathrm{d}k\,\mathrm{d}\boldsymbol{\xi}=\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{\xi})\frac{1}{\|\boldsymbol{\xi}\|^2}\,\mathrm{d}\boldsymbol{\xi}=\left\langle\frac{1}{\|\boldsymbol{\xi}\|^2},\widehat{\varphi}\right\rangle$$

On the other hand:

$$\int\limits_{\mathbb{R}^n}\varphi(\mathbf{x})\int\limits_0^\infty\left(\frac{\pi}{k}\right)^{\frac{n}{2}}\mathrm{e}^{-\frac{\pi^2\|\mathbf{x}\|^2}{k}}\,\mathrm{d}k\,\mathrm{d}\mathbf{x}=\frac{\Gamma\left(\frac{n}{2}-1\right)}{\pi^{\frac{n}{2}-2}}\int\limits_{\mathbb{R}^n}\frac{\varphi(\mathbf{x})}{\|\mathbf{x}\|^{n-2}}$$

where we have used the change of variable $r=\frac{\pi^2\boldsymbol{\xi}^2}{k}$. Let's do now the case $n=2$. Consider $E_n=\frac{1}{2}\log\left(\|\mathbf{x}\|^2+1/n^2\right)\xrightarrow{\mathcal{S}}\log\|\mathbf{x}\|$ (by the **?? ??**). Hence, we have that $\Delta E_n=\frac{2n^2}{(n^2\|\mathbf{x}\|^2+1)^2}\xrightarrow{\mathcal{S}}\Delta\log\|\mathbf{x}\|$. Thus, $\forall\varphi\in\mathcal{S}(\mathbb{R}^2)$:

$$\begin{aligned}\langle\Delta\log\|\mathbf{x}\|,\varphi\rangle&=\lim_{n\to\infty}\langle\Delta E_n,\varphi\rangle\\&=\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\frac{2n^2}{(n^2\|\mathbf{x}\|^2+1)^2}\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&=\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\frac{2}{(\|\mathbf{x}\|^2+1)^2}\varphi(\mathbf{x}/n)\,\mathrm{d}\mathbf{x}\\&=\varphi(0)\int\limits_{\mathbb{R}^2}\frac{2}{(\|\mathbf{x}\|^2+1)^2}\,\mathrm{d}\mathbf{x}\\&=2\pi\varphi(0)\end{aligned}$$

where in the forth equality we have used the **?? ??** and at the end we have calculated the integral using polar coordinates. □

Corollary 140. Consider the Cauchy-Riemann operators:

$$\partial_z=\frac{1}{2}\left(\partial_x-i\partial_y\right)\qquad\partial_{\overline{z}}=\frac{1}{2}\left(\partial_x+i\partial_y\right)$$

The fundamental solutions to $\partial_zu=f$ and $\partial_{\overline{z}}u=f$ are respectively:

$$E=\frac{1}{\pi}\frac{1}{\overline{z}}\qquad E=\frac{1}{\pi}\frac{1}{z}$$

Sketch of the proof. Recall that $\partial_z\partial_{\overline{z}}=\partial_{\overline{z}}\partial_z=\frac{1}{4}\Delta$. $E=\frac{\log(z\overline{z})}{4\pi}$ is a fundamental solution of the Laplace equation and $\partial_zE=\frac{1}{4\pi z}$, $\partial_{\overline{z}}E=\frac{1}{4\pi\overline{z}}$. □

Proposition 141. Consider the Cauchy problem:

$$\begin{cases}u_t=a^2\Delta u & \text{in }(0,\infty)\times\mathbb{R}^n\\ u(0,\mathbf{x})=f(\mathbf{x}) & \text{in }\mathbb{R}^n\end{cases}$$

Then, a fundamental solution is given by:

$$E(t,\mathbf{x})=\frac{1}{(4\pi a^2t)^{\frac{n}{2}}}\mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{4a^2t}}$$

And the general solution is:

$$u(t,\mathbf{x})=(E*f)(t,\mathbf{x})=\int\limits_{\mathbb{R}^n}\frac{f(\mathbf{y})}{(4\pi a^2t)^{\frac{n}{2}}}\mathrm{e}^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4a^2t}}\,\mathrm{d}\mathbf{y}$$

Proof. Taking $\mathcal{F}_{\mathbf{x}}$ on the equation we obtain:

$$\widehat{E}_t = -4\pi^2 \left\|\boldsymbol{\xi}\right\|^2 \widehat{E}$$

Solving it we obtain, $\widehat{E} = C\mathrm{e}^{-4\pi^2\|\boldsymbol{\xi}\|^2 t}$. Using the initial condition we see that $C = 1$. Now proceeding as in the proof of the Theorem 138 we obtain the result. □

Theorem 142 (Malgrange-Ehrenpreis theorem). Every non-zero linear partial differential operator with constant coefficients has a fundamental solution.

4. | Singular intergals

Hilbert transform

Definition 143. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. The *truncated Hilbert transform* is defined as:

$$\mathcal{H}^\varepsilon f(x) = \frac{1}{\pi} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,\mathrm{d}y$$

Definition 144. Let $f \in \mathcal{S}(\mathbb{R})$. We define the *Hilbert transform* of f as:

$$\mathcal{H}f(x) = \frac{1}{\pi}\left(\text{p.v.}\left(\frac{1}{x}\right) * f\right)(x) = \lim_{\varepsilon\to 0} \mathcal{H}^\varepsilon f(x)$$

Remark. We can extend the definition of $\mathcal{H}$ to functions that satisfy locally a *Hölder condition*: $\forall x \in \mathbb{R}$, $\exists C_x, \alpha_x, \delta_x > 0$ such that

$$|f(x) - f(y)| \leq C_x|x-y|^{\alpha_x} \qquad \text{for all}|x-y| < \delta_x$$

In that case we write:

$$\mathcal{H}^\varepsilon f(x) = \frac{1}{\pi} \int\limits_{\varepsilon<|x-y|<\delta_x} \frac{f(y)-f(x)}{x-y}\,\mathrm{d}y + \frac{1}{\pi} \int\limits_{|x-y|>\delta_x} \frac{f(y)}{x-y}\,\mathrm{d}y$$

Lemma 145. Let $a, b \in \mathbb{R}$, $a < b$. Then:

$$\mathcal{H}(\mathbf{1}_{[a,b]})(x) = \frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|$$

Proposition 146. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then:

$$\mathcal{F}(\mathcal{H}f)(\xi) = -\mathrm{i}\,\mathrm{sgn}(\xi)\mathcal{F}f(\xi) =: m(\xi)\mathcal{F}f(\xi)$$

Proof.

$$\begin{aligned}\mathcal{F}(\mathcal{H}f)(\xi) &= \frac{1}{\pi}\mathcal{F}\left(\text{p.v.}\left(\frac{1}{x}\right) * f\right)(\xi) = \\ &= \frac{1}{\pi}\mathrm{i}\widehat{\text{p.v.}\left(\frac{1}{x}\right)}\widehat{f}(\xi) = m(\xi)\widehat{f}(\xi)\end{aligned}$$

□

Lemma 147. Let $f \in L^2(\mathbb{R})$. Then, $\mathcal{H}f \in L^2(\mathbb{R})$ and $\|\mathcal{H}f\|_2 = \|f\|_2$.

Proof. Using 43 Plancherel theorem:

$$\|\mathcal{H}f\|_2 = \left\|\widehat{\mathcal{H}f}\right\|_2 = \left\|m\widehat{f}\right\|_2 = \left\|\widehat{f}\right\|_2 = \|f\|_2$$

□

Lemma 148. We have that $\mathcal{H}^2 = -\mathrm{id}$ on $L^p(\mathbb{R})$, $1 \leq p < \infty$.

Proof. $\mathcal{H}^2 f = \mathcal{H}(\mathcal{F}^{-1}(m\widehat{f})) = \mathcal{F}^{-1}(m^2\widehat{f}) = -f$ □

Lemma 149. Let $f \in \mathcal{S}(\mathbb{R})$. Then, $(\mathcal{H}f)^2 = f^2 + 2\mathcal{H}(f\mathcal{H}f)$.

Proof. We'll prove the equality using the Fourier transform and the uniqueness of it will imply the result.
In general we have that $\widehat{fg} = \widehat{f} * \widehat{g}$ because:

$$\mathcal{F}^{-1}(\widehat{f} * \widehat{g}) = \mathcal{F}^3(\widehat{f} * \widehat{g}) = \mathcal{F}^2(\mathcal{F}^2 f\mathcal{F}^2 g) = fg$$

Thus:

$$\widehat{f^2} = \widehat{f} * \widehat{f} \qquad 2\mathcal{F}(\mathcal{H}(f\mathcal{H}f)) = 2m\widehat{f} * (m\widehat{f})$$

The first term is $\int_{\mathbb{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)\,\mathrm{d}\eta$ whereas the second one is $2\int_{\mathbb{R}} m(\xi)\widehat{f}(\eta)m(\xi-\eta)\widehat{f}(\xi-\eta)\,\mathrm{d}\eta = 2\int_{\mathbb{R}} m(\xi)\widehat{f}(\xi-\eta)\widehat{f}(\eta)m(\eta)\,\mathrm{d}\eta$. Averaging those terms we have:

$$\begin{aligned}\widehat{f^2} + 2\mathcal{F}(\mathcal{H}(f\mathcal{H}f)) &= \int\limits_{\mathbb{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)[1+m(\xi)\cdot \\ &\qquad \cdot (m(\xi-\eta)+m(\eta))]\,\mathrm{d}\eta \\ &= \int\limits_{\mathbb{R}} \widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\xi-\eta)m(\eta)\,\mathrm{d}\eta \\ &= \widehat{\mathcal{H}f} * \widehat{\mathcal{H}f} = \widehat{(\mathcal{H}f)^2}\end{aligned}$$

where the second equality follows for all $\xi, \eta \in \mathbb{R}^2 \setminus \{(0,0)\}$ □

Theorem 150 (Riesz theorem). Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then, $\exists C_p > 0$ such that:

$$\|\mathcal{H}f\|_p \leq C_p\|f\|_p$$

Proof. We will prove only the cases $p = 2^k$, $k \in \mathbb{N}$ and we'll do it by induction. The case $k = 1$ is clear. Using Theorem 149 we have:

$$\begin{aligned}{\|\mathcal{H}f\|_{2p}}^2 &= \left\|(\mathcal{H}f)^2\right\|_p \leq \left\|f^2\right\|_p + 2\left\|\mathcal{H}(f\mathcal{H}f)\right\|_p \leq \\ &\leq {\|f\|_{2p}}^2 + 2C_p\|f\mathcal{H}f\|_p \leq {\|f\|_{2p}}^2 + 2C_p\|f\|_{2p}\|\mathcal{H}f\|_{2p}\end{aligned}$$

where the last inequality follows from the **??**. This reduces to find for which y we have $y^2 - 2C_p\alpha y - \alpha^2 \leq 0$, where $\alpha = {\|f\|_{2p}}^2$. An easy check shows that:

$$\|\mathcal{H}f\|_{2p} \leq \left(C_p + \sqrt{C_p^2+1}\right)\|f\|_{2p}$$

□

Lemma 151. Let P_y be the Poisson kernel and $f \in L^p(\mathbb{R})$. Then, if $z = x + \mathrm{i}y$:

$$(P_y * f)(z) = \mathrm{Re}\left(\frac{\mathrm{i}}{\pi}\int\limits_{\mathbb{R}} \frac{f(t)}{z-t}\,\mathrm{d}t\right) =: \mathrm{Re}\,F_f(z)$$

Moreover, $F_f \in \mathcal{H}(\{\mathrm{Im}\,f > 0\})$.

Sketch of the proof. The first part follows from:

$$(P_y * f)(z) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2} \, \mathrm{d}t$$

To show the last part, note that F_f is $\mathbb{R}$-differentiable and $\overline{\partial} F_f = 0$. □

Definition 152. We define the *conjugate Poisson kernel* Q_y as:

$$Q_y(x) = \frac{x}{\pi(x^2 + y^2)}$$

Lemma 153. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then, if $z = x + \mathrm{i}y$:

$$\operatorname{Im} F_f(z) = \int_{\mathbb{R}} \frac{f(t)(x-t)}{(x-t)^2 + y^2} \, \mathrm{d}t$$

Theorem 154. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then:

$$f * Q_\varepsilon - \mathcal{H}^\varepsilon f \xrightarrow[\varepsilon \to 0]{L^p} 0$$

In particular:

$$F_\varphi(x + \mathrm{i}y) \overset{y \to 0}{\longrightarrow} \varphi(x) + \mathrm{i}\mathcal{H}\varphi(x)$$

Riesz transform

Definition 155. We define $W_j := \text{p.v.}\left(\frac{x_j}{\|\mathbf{x}\|^{n+1}}\right)$, $j = 1, \ldots, n$.

Lemma 156. For each $n \in \mathbb{N}$ and $j = 1, \ldots, n$, $W_j \in \mathcal{S}^*(\mathbb{R}^n)$.

Definition 157. We define the *Riesz transform* $R_j f$ as:

$$R_j f(\mathbf{x}) := c_n (W_j * f)(\mathbf{x}) = $$
$$= \lim_{\varepsilon \to 0} c_n \int_{\|\mathbf{x}-\mathbf{y}\| > \varepsilon} \frac{x_j - y_j}{\|\mathbf{x} - \mathbf{y}\|^{n+1}} f(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

with $c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$.

Lemma 158. $\partial_j \left(\frac{1}{\|\mathbf{x}\|^{n-1}}\right) = (1-n) W_j$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and write it as $\varphi = \varphi_{\mathrm{e}} + \varphi_{\mathrm{o}}$, where φ_{e} is even and φ_{o} is odd. Then:

$$\begin{aligned}
\left\langle \partial_j \left(\frac{1}{\|\mathbf{x}\|^{n-1}}\right), \varphi \right\rangle &= -\int_{\mathbb{R}^n} \frac{\partial_j \varphi_{\mathrm{o}}}{\|\mathbf{x}\|^{n-1}} \, \mathrm{d}\mathbf{x} \\
&= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\partial_j \varphi_{\mathrm{o}}}{\|\mathbf{x}\|^{n-1}} \, \mathrm{d}\mathbf{x} \\
&= (1-n) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{x_j \varphi_{\mathrm{o}}}{\|\mathbf{x}\|^{n+1}} \, \mathrm{d}\mathbf{x} \\
&= (1-n) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{x_j \varphi}{\|\mathbf{x}\|^{n+1}} \, \mathrm{d}\mathbf{x}
\end{aligned}$$

□

Theorem 159. For each $j = 1, \ldots, n$ we have:

$$\widehat{W_j} = -\frac{\mathrm{i}}{c_n} \frac{\xi_j}{\|\boldsymbol{\xi}\|}$$

Proof. Using Theorems 130 and 158 we have:

$$\widehat{W_j} = \frac{1}{1-n} \mathcal{F}(\partial_j \|\mathbf{x}\|^{1-n}) = \frac{2\pi \mathrm{i} \xi_j}{1-n} \widehat{\|\mathbf{x}\|^{1-n}} =$$
$$= \frac{2\pi \mathrm{i} \xi_j}{1-n} \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{\|\boldsymbol{\xi}\|} = \frac{\mathrm{i}\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\xi_j}{\|\boldsymbol{\xi}\|}$$

□

Corollary 160. For each $j = 1, \ldots, n$ we have:

$$\mathcal{F}(R_j f)(\boldsymbol{\xi}) = -\mathrm{i} \frac{\xi_j}{\|\boldsymbol{\xi}\|} \widehat{f}(\boldsymbol{\xi})$$

Proposition 161.

1. $\displaystyle\sum_{j=1}^{n} R_j^2 = -\mathrm{id}$
2. For all $1 \leq j, k \leq n$, $\partial_j \partial_k = R_j R_k \Delta$

Sketch of the proof. Apply $\mathcal{F}$ on each of the equations and use the Fourier transform properties. □