Dynamical systems

1. | Dynamical systems in Euclidean spaces

Introduction

Definition 1. Let (X, G, Π) be a dynamical system. In the continuous case, we say that $x \in X$ is an *equilibrium point* if $\Pi(x, t) = x\ \forall t \in G$. In the discrete case, a point $x \in X$ satisfying this property is called a *fixed point*.

Definition 2. Let (X, G, Π) be a dynamical system. A *periodic orbit of period* T is an orbit of the system that satisfies $\Pi(x, t+T) = \Pi(x, t)\ \forall t \in G$ and for some $x \in X$.

Theorem 3. Let (X, G, Π) be a dynamical system and γ_x be an orbit. Then, there are 3 possible cases for γ_x. In the continuous case:

1. γ_x is an equilibrium point.
2. γ_x is a periodic orbit.
3. $\gamma_x \cong \mathbb{R}$.

In the discrete case:

1. γ_x is a fixed point.
2. γ_x is a periodic orbit.
3. $\gamma_x \cong \mathbb{Z}$.

Theorem 4. Let $(X, \mathbb{Z}_{\geq 0}, \Pi)$ be a discrete semidynamical system and γ_x be an orbit. Then, apart form the above possibilities the orbit can also be of the form:

1. $\gamma_x = \{x_1, \ldots, x_{n-1}, x_n, x_n, x_n, x_n, \ldots\}$
2. $\gamma_x = \{x_1, \ldots, x_{n-1}, x_n, \ldots, x_{n+T}, x_n, \ldots, x_{n+T}, \ldots\}$

Definition 5. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two functions and consider the system of ODEs:

$$\begin{cases} x' = p(x, y) \\ y' = q(x, y) \end{cases} \tag{1}$$

We say that the system is *symmetric with respect to the x-axis* if it is invariant under the transformation $(x, y, t) \to (-x, y, -t)$. Analogously, we say that the system is *symmetric with respect to the y-axis* if it is invariant under the transformation $(x, y, t) \to (x, -y, -t)$.

Theorem 6. Consider the system Eq. (1) and suppose that the origin is an equilibrium point. If the system is symmetric with respect to the x-axis or y-axis, and if the origin is a center of the linear part of the system, then the origin is a center of the nonlinear system Eq. (1).

Theorem 7. Consider the differential system in $\mathbb{C}$ defined by:

$$z' = \mathrm{i}z + a_2 z^2 + a_3 z^3 + \cdots =: \mathrm{i}z + f(z)$$

This is a *holomorphic differential equation*. Then, there exists a conjugaction that conjugates this system with $z' = \mathrm{i}z$. This process of finding the conjugacy is called *linearization* of the first system. In particular, the periods of the periodic orbits are always the same.

Theorem 8. Consider the differential system

$$\begin{cases} x' = \lambda x + p(x, y) \\ y' = \mu y + q(x, y) \end{cases}$$

where $p, q \in \mathbb{R}[x, y]$ and $\deg p, \deg q \geq 2$. If $\frac{\lambda}{\mu} \notin \mathbb{Q}$, then there exists a conjugacy of class $\mathcal{C}^\omega$ between them.

Definition 9. Let (X, G, Π) be a dynamical system and $A \subseteq \mathbb{R}^n$ be an invariant subset. We say that A is *minimal* if it doesn't contain any proper invariant subset.

Definition 10. A *Kolmogorov system* is a dynamical system of the form:

$$\begin{cases} {x_1}' = x_1 f_1(\mathbf{x}) \\ \quad\vdots \\ {x_n}' = x_n f_n(\mathbf{x}) \end{cases} \tag{2}$$

Homogeneous systems

Definition 11. We say that function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* k if $\forall \lambda \in \mathbb{R}$ we have:

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$$

Definition 12. We say that differential system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is *homogeneous of degree* k if the components of $\mathbf{f}$ are homogeneous functions of degree k. That is, if $\mathbf{f}(\lambda\mathbf{x}) = \lambda^k \mathbf{f}(\mathbf{x})\ \forall \lambda \in \mathbb{R}$.

Proposition 13. The study of the global dynamics of a homogeneous system with only one equilibrium point (at the origin) is the same as the study of local dynamics at the origin.

Poincaré map

Proposition 14. Let $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^2)$, $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a planar differential system and $\Sigma \subset \mathbb{R}^2$ be a transversal section. Then, the Poincaré map $\mathbf{\Pi} : \Sigma \to \Sigma$ is monotone.

Proposition 15. Let $\mathbf{f} \in \mathcal{C}^k(\mathbb{R}^n)$, $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a differential system and $\Sigma \subset \mathbb{R}^k$ be a transversal section. Then, $\mathbf{\Pi} \in \mathcal{C}^k(\mathbb{R}^n)$.

Discrete maps

Definition 16. A *discrete map of order* m is a recurrence of the form:

$$f(n, x_n, x_{n+1}, \dots, x_{n+m}) = 0 \tag{3}$$

where $f : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is a function. Sometimes we will be able to express it in its explicit form as:

$$x_{n+m} = f(x_n, x_{n+1}, \dots, x_{n+m-1}) \tag{4}$$

Proposition 17. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a function. Consider a discrete map of Eq. (4). Then, we can transform this map into a map of order 1 of m equations:

$$\begin{cases} (y_1)_{n+1} = (y_2)_n \\ (y_2)_{n+1} = (y_3)_n \\ \quad \vdots \\ (y_{m-1})_{n+1} = (y_m)_n \\ (y_m)_{n+1} = f(n, (y_1)_n, (y_2)_n, \dots, (y_{m-1})_n) \end{cases}$$

where $(y_i)_n = x_{n+i-1}$ for $i = 1, \dots, m$.

Proposition 18. Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ be an invertible function and consider the discrete map of Eq. (4). Then, $\Pi(\mathbf{x}, n) := \mathbf{f}^n(\mathbf{x})$ defines a discrete dynamical system.

Proposition 19 (Characteristic equation). Consider the following *linear discrete map*:

$$x_{n+m} + a_{m-1}x_{n+m-1} + \cdots + a_1 x_{n+1} + a_0 x_n = 0$$

Let:

$$p(y) := y^m + a_{m-1}y^{m-1} + \cdots + a_1 y + a_0 = 0$$

Suppose p has r distinct real roots and $2(s-r)$ distinct complex roots.

$$\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \overline{\lambda_{r+1}}, \dots, \lambda_s, \overline{\lambda_s}$$

Here, $\lambda_i \in \mathbb{R}$ $\forall i = 1, \dots, r$ and $\lambda_i = \alpha_i + \mathrm{i}\beta_i \in \mathbb{C}$ $\forall i = r+1, \dots, s$. Assume, each of these roots have multiplicity $k_i \in \mathbb{N}$. Then, the general solution to that discrete map is:

$$\mathbf{x}_n = \sum_{i=1}^{r} \left(c_{i,0} + c_{i,1}n + \cdots + c_{i,k_i-1}n^{k_i-1}\right) {\lambda_i}^n + \\ + \sum_{i=r+1}^{s} \sum_{j=0}^{k_i-1} n^j {\rho_i}^n \left(c_{i,j,1}\cos(\theta_i n) + c_{i,j,2}\sin(\theta_i n)\right)$$

where $c_{i,j,k} \in \mathbb{R}$ are constants and $\lambda_i = \rho_i e^{\mathrm{i}\theta_i}$.

Proposition 20. Consider the discrete map

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$$

and let $\mathbf{p}$ be a fixed point of $\mathbf{f}$.

- If $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$ we have $|\lambda| < 1$, then $\mathbf{p}$ is asymptotically stable.
- If $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$ we have $|\lambda| > 1$, then $\mathbf{p}$ is repelling and negatively stable.
- If $\mathbf{p}$ is positively stable, then $|\lambda| \leq 1$ $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$.
- If $\mathbf{p}$ is negatively stable, then $|\lambda| \geq 1$ $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$.

Theorem 21 (Hartman-Grobman theorem). Consider the discrete map

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\mathbf{p} \in U$ be a fixed point of $\mathbf{f}$ and suppose $\sigma(\mathbf{Df}(\mathbf{p})) = \{\lambda_1, \lambda_2\}$. Then, $\mathbf{p}$ will be a

- stable node if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $|\lambda_1|, |\lambda_2| < 1$.
- unstable node if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $|\lambda_1|, |\lambda_2| > 1$.
- saddle point if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or vice versa).
- stable focus if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $|\lambda_1| < 1$.
- unstable focus if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $|\lambda_1| > 1$.

2. | Study of local dynamics

Stable manifold and center manifold theorems

Theorem 22 (The stable manifold theorem). Let $E \subseteq \mathbb{R}^n$ be an open subset containing the origin, $\mathbf{f} \in \mathcal{C}^1(E)$ and $\boldsymbol{\phi}_t$ be the flow of the following system of n equations:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{5}$$

Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $\mathbf{Df}(\mathbf{0})$ has n_+ eigenvalues with positive real part and $n_- = n - n_+$ eigenvalues with negative real part. Then, there exists a unique n_+-dimensional differentiable manifold S tangent to the stable eigenspace E^{s} of the linear system at $\mathbf{0}$ such that $\forall t \geq 0$, $\boldsymbol{\phi}_t(S) \subseteq S$ and $\forall \mathbf{x}_0 \in S$:

$$\lim_{t \to \infty} \boldsymbol{\phi}_t(\mathbf{x}_0) = \mathbf{0}$$

This manifold is called *stable manifold*. Analogously, there exists a unique n_--dimensional differentiable manifold U tangent to the stable eigenspace E^{u} of the linear system at $\mathbf{0}$ such that $\forall t \leq 0$, $\boldsymbol{\phi}_t(U) \subseteq U$ and $\forall \mathbf{x}_0 \in U$:

$$\lim_{t \to -\infty} \boldsymbol{\phi}_t(\mathbf{x}_0) = \mathbf{0}$$

This manifold is called *unstable manifold*.

Definition 23. Let $E \subseteq \mathbb{R}^n$ be an open subset containing the origin, $\mathbf{f} \in \mathcal{C}^1(E)$ and $\boldsymbol{\phi}_t$ be the flow of the system of Eq. (5). The *global stable manifold* and *global unstable manifold* at $\mathbf{0}$ are defined respectively by:

$$W^{\mathrm{s}}(\mathbf{0}) = \bigcup_{t \leq 0} \boldsymbol{\phi}_t(S) \qquad W^{\mathrm{u}}(\mathbf{0}) = \bigcup_{t \geq 0} \boldsymbol{\phi}_t(U)$$

Proposition 24. Let $E \subseteq \mathbb{R}^n$ be an open subset containing the origin, $\mathbf{f} \in \mathcal{C}^1(E)$ and $\boldsymbol{\phi}_t$ be the flow of the system of Eq. (5). Then, the manifolds W^{s} and W^{u} are unique and invariant with respect to the flow $\boldsymbol{\phi}_t$

Theorem 25 (The center manifold theorem). Let $E \subseteq \mathbb{R}^n$ be an open subset containing the origin, $\mathbf{f} \in \mathcal{C}^r(E)$, $r \geq 1$ and consider the system of Eq. (5). Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $\mathbf{Df}(\mathbf{0})$ has n_+ eigenvalues with positive real part, n_- eigenvalues with negative real part and $n_0 = n - n_+ - n_-$ eigenvalues with zero real part. Then, there exists a n_0-dimensional differentiable manifold W^{c} (called *central manifolds*) of class $\mathcal{C}^{r-1}$ tangent to the center eigenspace E^{c} of the linear system at $\mathbf{0}$; there exists a n_+-dimensional differentiable manifold S tangent to the stable eigenspace E^{s} of the linear system at $\mathbf{0}$, and there exists a n_--dimensional differentiable manifold U tangent to the stable eigenspace E^{u} of the linear system at $\mathbf{0}$. Furthermore, W^{c}, W^{s} and W^{u} are invariant under the flow $\boldsymbol{\phi}_t$. If moreover, $\mathbf{f} \in \mathcal{C}^\omega(E)$, the the manifold W^{c} is also of class $\mathcal{C}^\omega$ and it is unique[1].

Local bifurcation theory

Definition 26. A *local bifurcation* on a differential systems

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) \tag{6}$$

is the study of the changes in the local stability properties of equilibrium points, periodic orbits or other invariant sets as the parameter $\boldsymbol{\mu}$ cross through critical thresholds.

Definition 27. The *normal form* of a dynamical system is a simplified form that can be useful in determining the system's behavior.

Definition 28. The *codimension* of a bifurcation is the number of parameters which must be varied for the bifurcation to occur.

Definition 29 (Saddle-node bifurcation). The normal form of the codimension-one *saddle-node bifurcation* is:

$$x' = \mu + x^2$$

Fig. 1a shows the qualitative behavior of that system[2].

Definition 30 (Transcritical bifurcation). The normal form of the codimension-one *transcritical bifurcation* is:

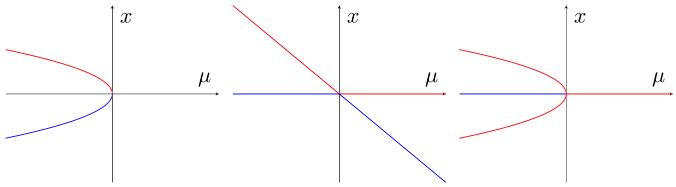
$$x' = \mu x + x^2$$

Fig. 1b shows a qualitative behavior of the stability of the equilibria.

Definition 31 (Pitchfork bifurcation). The normal form of the codimension-one *pitchfork bifurcation* is:

$$x' = \mu x + x^3$$

Fig. 1c shows a qualitative behavior of the stability of the equilibria.



(a) Saddle-node (b) Transcritical (c) Pitchfork

Figure 1: Bifurcation diagrams of the principal bifurcations in 1-dimensional systems

Theorem 32. Consider the system of Eq. (6). Then, it will have a saddle-node bifurcation at the origin if $\mathbf{f}(0,0) = 0$, $\mathbf{f}_x(0,0) = 0$, $\mathbf{f}_{xx}(0,0) \neq 0$ and $\mathbf{f}_\mu(0,0) \neq 0$.

Theorem 33. Consider the system of Eq. (6). Then, it will have a transcritical bifurcation at the origin if $\mathbf{f}(0,0) = 0$, $\mathbf{f}_x(0,0) = 0$, $\mathbf{f}_{xx}(0,0) \neq 0$, $\mathbf{f}_\mu(0,0) = 0$ and $\mathbf{f}_{x\mu}(0,0) \neq 0$.

Theorem 34. Consider the system of Eq. (6). Then, it will have a pitchfork bifurcation at the origin if $\mathbf{f}(0,0) = 0$, $\mathbf{f}_x(0,0) = 0$, $\mathbf{f}_{xx}(0,0) = 0$, $\mathbf{f}_{xxx}(0,0) \neq 0$, $\mathbf{f}_\mu(0,0) = 0$ and $\mathbf{f}_{x\mu}(0,0) \neq 0$.

Non-hyperbolic equilibrium points

Theorem 35 (Lyapunov's theorem). Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. Suppose there exists a function $V : U \to \mathbb{R}$ of class $\mathcal{C}^1$ and a neighbourhood $\tilde{U} \subseteq U$ of $\mathbf{p}$ such that $V(\mathbf{p}) = 0$ and $V(\mathbf{x}) > 0$ $\forall \mathbf{x} \in \tilde{U} \setminus \{\mathbf{p}\}$. Then:

- If $V'(\mathbf{q}) \leq 0$ $\forall \mathbf{q} \in \tilde{U}$, then $\mathbf{p}$ is stable.
- If $V'(\mathbf{q}) < 0$ $\forall \mathbf{q} \in \tilde{U}$, then $\mathbf{p}$ is asymptotically stable.
- If $V'(\mathbf{q}) \leq 0$ $\forall \mathbf{q} \in \tilde{U}$ and $\omega(\tilde{U}) = \{\mathbf{p}\}$, then $\mathbf{p}$ is asymptotically stable.
- If $V'(\mathbf{q}) > 0$ $\forall \mathbf{q} \in \tilde{U}$, then $\mathbf{p}$ is unstable.

Theorem 36 (Lyapunov's theorem). Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and consider the iteration $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$. Let $\mathbf{p} \in U$ be a fixed point of $\mathbf{f}$. Suppose there exists a continuous function $V : U \to \mathbb{R}$ and a neighbourhood $\tilde{U} \subseteq U$ of $\mathbf{p}$ such that $V(\mathbf{p}) = 0$ and $V(\mathbf{x}) > 0$ $\forall \mathbf{x} \in \tilde{U} \setminus \{\mathbf{p}\}$. Then:

- If $V(\mathbf{x}_{n+1}) - V(\mathbf{x}_n) \leq 0$ $\forall \mathbf{x}_n \in \tilde{U}$, then $\mathbf{p}$ is stable.
- If $V(\mathbf{x}_{n+1}) - V(\mathbf{x}_n) < 0$ $\forall \mathbf{x}_n \in \tilde{U}$, then $\mathbf{p}$ is asymptotically stable.

Theorem 37 (Semi-hyperbolic singular points classification theorem). Consider the differential system

$$\begin{cases} x' = X(x,y) \\ y' = y + Y(x,y) \end{cases}$$

[1] Unique in the sense that there is no other manifold of class $\mathcal{C}^\omega$ tangent to E^{c} at $\mathbf{0}$ but it may be other manifolds of class $\mathcal{C}^r$, $r \in \mathbb{N} \cup \{\infty\}$, satisfying this property.

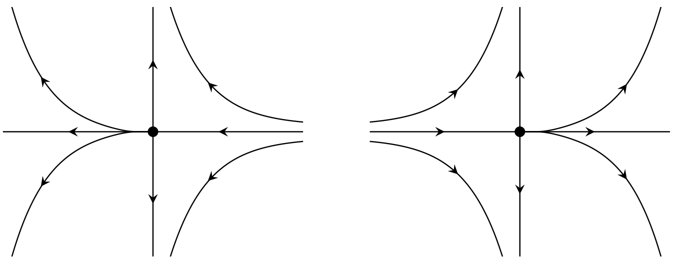
[2] In these images, the red lies means that the point (μ, x) is repelling. The blue lines means that the point (μ, x) is attracting.

where $X, Y \in \mathcal{C}^\omega$ have order ≥ 2 in their Taylor's series. Hence, the differential of the system at the origin is:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose the origin is an equilibium point, let $y = f(x)$ the solution to the equation $y + Y(x, y) = 0$ given by the implicit function theorem and $g(x) = X(x, f(x)) = a_m x^m + \mathrm{O}\left(x^{m+1}\right)$, where $a_m \neq 0$ and $m \geq 2$. Then:

1. If m is odd and $a_m > 0$, the origin is a repelling node.
2. If m is odd and $a_m < 0$, the origin is a saddle.
3. If m is even, the origin is a *saddle-node* (See Fig. 2).



(a) $a_m < 0$ (b) $a_m > 0$

Figure 2: Local phase portraits of the saddle-node system when m is even.

Definition 38. A singular point of a differential system is called *nilpotent* if the differential at this point is nilpotent.

Theorem 39 (Nilpotent singular points classification theorem). Consider the differential system

$$\begin{cases} x' = y + X(x, y) \\ y' = Y(x, y) \end{cases}$$

where $X, Y \in \mathcal{C}^\omega$ have order ≥ 2 in their Taylor's series. Hence, the differential of the system at the origin is:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

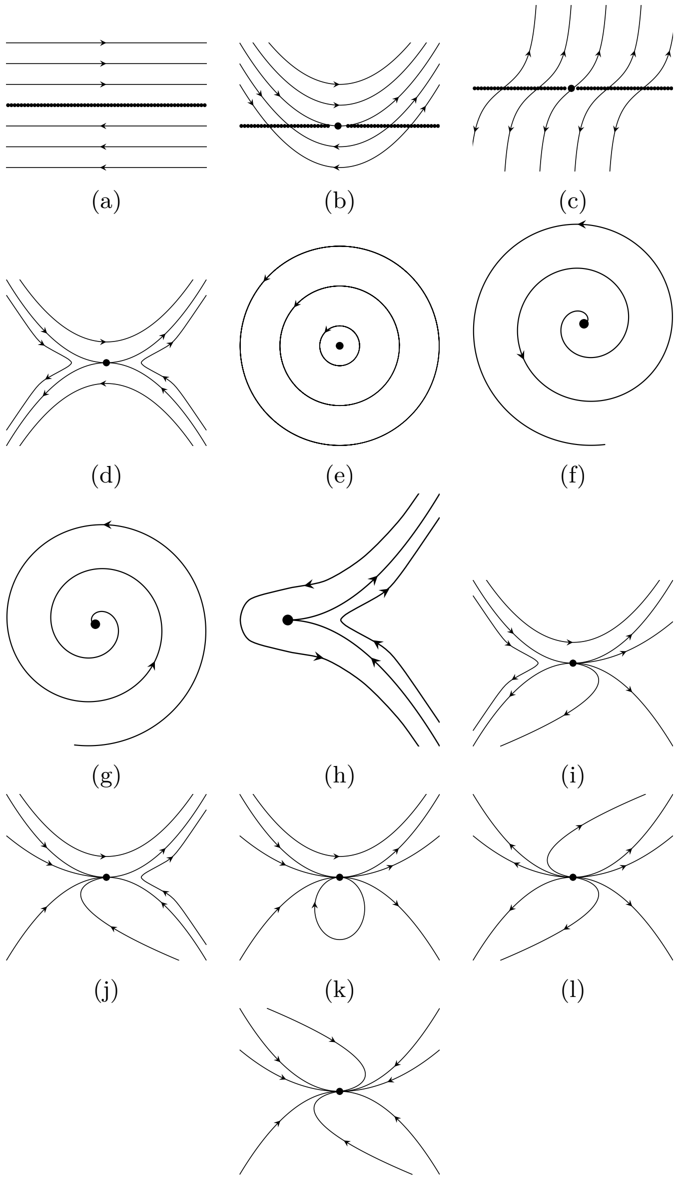
Suppose the origin is an equilibium point, let $y = f(x)$ the solution to the equation $y + X(x, y) = 0$ given by the implicit function theorem and $g(x) = Y(x, f(x))$. Moreover, define $\Phi(x) = \mathbf{div}(X, Y)(x, f(x))$. Then:

1. If $g(x) = \Phi(x) = 0$, the phase portrait at the origin is given by Fig. 3a.
2. If $g(x) = 0$ and $\Phi(x) = bx^n + \mathrm{O}\left(x^{n+1}\right)$ with $n \geq 1$ and $b \neq 0$, then the phase portrait at the origin is given by Figs. 3b or 3c.
3. If $\Phi(x) = 0$ and $g(x) = ax^m + \mathrm{O}\left(x^{m+1}\right)$ with $m \geq 2$ and $a \neq 0$, then:
   i) If m is odd and
      a) $a > 0$, the origin is a saddle (Fig. 3d).
      b) $a < 0$, the origin is a center or a focus (Figs. 3e to 3g).

   ii) If m is even, the origin is a *cusp* (it consists of two hyperbolic sectors, Fig. 3h).
4. If $g(x) = ax^m + \mathrm{O}\left(x^{m+1}\right)$ and $\Phi(x) = bx^n + \mathrm{O}\left(x^{n+1}\right)$ with $m \geq 2$, $n \geq 1$ and $a, b \neq 0$, then:
   i) If m is even and
      a) $m > 2n + 1$, the origin is a saddle-node (Figs. 3i or 3j).
      b) $m < 2n + 1$, the origin is a cusp (Fig. 3h).

   ii) If m is odd and $a > 0$, the origin is a saddle (Fig. 3d).

   iii) If m is odd, $a < 0$ and
      a) n is even and either $m > 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) \geq 0$,, then the origin is a node. The node is attracting if $b < 0$ (Fig. 3m) and repelling if $b > 0$ (Fig. 3l).
      b) n is odd and either $m > 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) \geq 0$, then the origin consists of an elliptic sector and a hyperbolic sector (Fig. 3k).
      c) either $m < 2n + 1$, or $m = 2n + 1$ and $b^2 + 4a(n + 1) < 0$, then the origin is a center or a focus (Figs. 3e to 3g).



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

(m)

Figure 3: Phase portraits of nilpotent singular points

Theorem 40 (Blow-up in polar coordinates). Consider a planar differential system with an equilibrium at the origin such that its differential at $(0,0)$ is:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

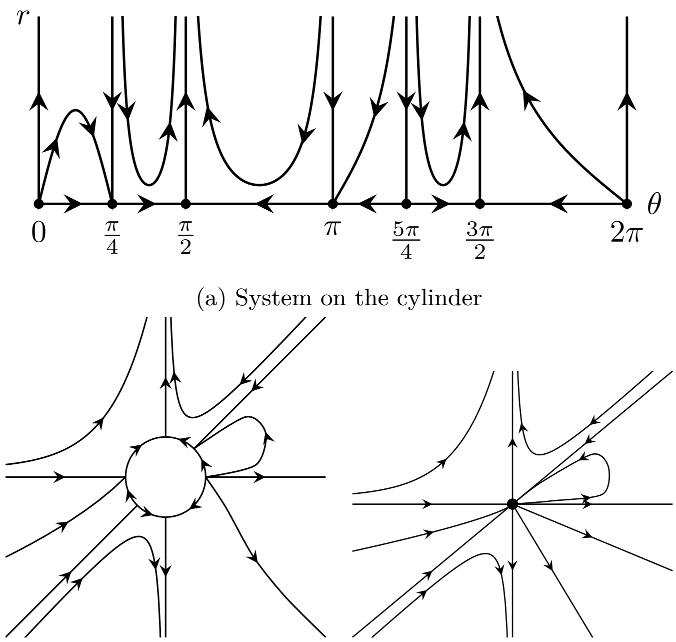
Hence, a change form cartesian coordinates to polar coordinates will transform the equations into the following ones:

$$\begin{cases} \dot{r} = \displaystyle\sum_{k=3}^{\infty} r^{k-1} f_k(\theta) \\ \dot{\theta} = \displaystyle\sum_{k=3}^{\infty} r^{k-2} g_k(\theta) \end{cases}$$

where $f_k(\theta)$ and $g_k(\theta)$ are trigonometric polynomials of degree k. The above system has the same phase portrait as:

$$\begin{cases} \dot{r} = \displaystyle\sum_{k=3}^{\infty} r^{k-2} f_k(\theta) \\ \dot{\theta} = \displaystyle\sum_{k=3}^{\infty} r^{k-3} g_k(\theta) \end{cases}$$

We are interested in the local behavior of the origin so we should first think this problem in the cylinder $(r, \theta) \in [0, \infty] \times [0, 2\pi)$. The equilibrium points of that system at $r = 0$ will be the isolated zeros of $g_3(\theta)$. Let $Z(g_3)$ be the set of such zeros. Now we want to study the behavior on each of these points. To do so, we can apply either Hartman's theorem, Theorem 37 or Theorem 39. If none of these are applicable of some $\theta^* \in Z(g_3)$, we should repeat the procedure by doing another polar transformation centered at θ^*[3]. We can then represent all these local behavior by *blowing-up* the origin to S^1 and once done, we should *blow-down* until contract S^1 into the origin (See the example Fig. 4 for a better understanding).



(a) System on the cylinder

(b) Blow-up (c) Blow-down

Figure 4: Blowing-up of a non-hyperbolic critical point. The kinds of equilibrium points in the cylinder (here $Z(g_3) = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}\}$) are three saddles, two saddle-node and one node. These are transformed in parabolic, hyperbolic and elliptic sectors when blowing-down the origin.

Corollary 41 (Blow-up in generalized polar coordinates). If, in the above theorem, in a neighbourhood of the origin we have $x^\alpha \sim y^\beta$ where $\alpha, \beta > 0$, then a better change of variables instead of the ordinary change to polars would be:

$$x = r^\beta \cos\theta$$
$$y = r^\alpha \sin\theta$$

[3]See Theorem 42 for a bettern solution to this problem.

Theorem 42 (Blow-up in cartesian coordinates). Consider a planar differential system with an equilibrium at the origin such that its differential at $(0,0)$ is:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We would like to do not a blow-up equally-weighted in all directions (as in the polar blow-up) but two blow-ups: one by leaving one axis undefined and the other one covering that axis left. Thus, if at the origin $x^\alpha \sim y^\beta$ where $\alpha, \beta > 0$ we seek transformations of the form $(x, y) \to \left(x, \frac{y^\beta}{x^\alpha}\right)$, which is undefined at $x = 0$, and transformations of the form $(x, y) \to \left(\frac{x^\alpha}{y^\beta}, y\right)$, which is undefined at $y = 0$. From here, we proceed as in Theorem 40.

Proposition 43 (Lyapunov's method). Consider the differential system:

$$\begin{cases} x' = \mu x - y + X(x, y) \\ y' = x + \mu y + Y(x, y) \end{cases} \tag{7}$$

where $X, Y \in \mathcal{C}^\omega$ have order ≥ 2 in their Taylor's series. In polar coordinates this system is transformed into:

$$\begin{cases} r' = \mu r + \displaystyle\sum_{k=3}^{\infty} f_k(\theta) r^{k-1} \\ \theta' = 1 + \displaystyle\sum_{k=3}^{\infty} g_k(\theta) r^{k-1} \end{cases}$$

where $f_k(\theta)$ and $g_k(\theta)$ are trigonometric homogeneous polynomials of degree k. And so, in a neighbourhood of the origin, we can express $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ as:

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mu r + \sum_{k=3}^{\infty} h_k(\theta) r^{k-1}$$

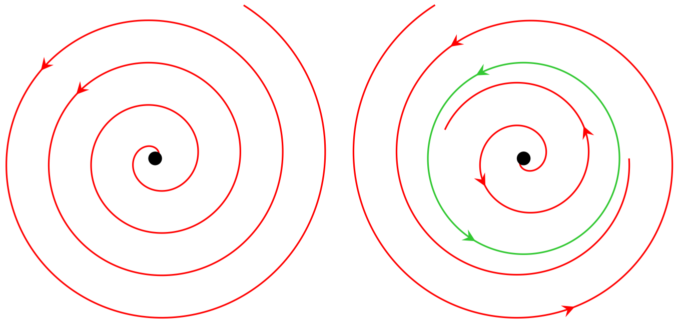
where $h_k(\theta)$ are trigonometric polynomials of degree k. If we impose $r(0) = \rho \simeq 0$ and assume $\mu = 0$, then the solution $r(\theta, \rho)$ can be written as:

$$r(\theta, \rho) = \rho + v_2(\theta)\rho^2 + v_3(\theta)\rho^3 + \cdots$$ [4]

We define the *n-th Lyapunov constant* as:

$$\mathrm{L}_n := v_{2n+1}(2\pi)$$ [5][6]

Theorem 44 (Hopf bifurcation theorem). Consider the system of Eq. (7) with $\mu = 0$ and suppose $\mathrm{L}_1 \neq 0$. Then, for $\mu \gtrsim 0$ there exists a unique periodic orbit that bifurcates at the origin. If $\mathrm{L}_1 > 0$ the periodic orbit is an unstable limit cycle, and if $\mathrm{L}_1 < 0$ the periodic orbit is an stable limit cycle (see Fig. 5). This codimension-one bifurcation is called *Hopf-bifurcation*.



(a) $\mu \leq 0$ (b) $\mu \gtrsim 0$

Figure 5: Hopf bifurcation with $\mathrm{L}_1 < 0$

Theorem 45. Consider the system of Eq. (7) with $\mu = 0$ and the functions $v_k(\theta)$. We have that $v_{2n}(2\pi) = 0$.

Corollary 46. Consider the system of Eq. (7) with $\mu = 0$. Then, the origin is a center if and only if:

$$\mathrm{L}_n = 0 \qquad \forall n \in \mathbb{N}$$

Theorem 47 (Poincaré's method). Consider the system of Eq. (7) with $\mu = 0$. Then, there exists a function $H(x, y)$[7] of the form

$$H(x, y) = x^2 + y^2 + H_3(x, y) + H_4(x, y) + \cdots$$

where each $H_j(x, y)$ is an homogeneous polynomial of degree j, such that:

$$H' = \sum_{k=1}^{\infty} \mathrm{P}_k (x^2 + y^2)^{k+1}$$ [8]

Note that due to the given stability of the problem, if $k = k_0$ is the index of the first non-zero Lyapunov coefficient, we must have $\mathrm{L}_k = a_k \mathrm{P}_k$, with $a_k > 0$.

3. | Global dynamics in continuous systems

Bifurcation of periodic orbits

Theorem 48 (Bautin's theorem). Consider the differential system:

$$\begin{cases} x' = \alpha x - y + p_2(x, y, \boldsymbol{\mu}) \\ y' = x + \alpha y + q_2(x, y, \boldsymbol{\mu}) \end{cases} \tag{8}$$

where p_2 and q_2 are homogeneous polynomials of degree 2. Then, in a neighbourhood of the origin and for $(\alpha, \boldsymbol{\mu}) \simeq 0$ there are at most 3 limit cycles that born from the origin.

Definition 49 (Homoclinic bifurcation). A *homoclinic bifurcation* is a bifurcation in which a limit cycle collides with a saddle point.

[4] Because of the differentiable dependence on initial conditions.
[5] Note that this has a strong relation with the Poincaré map $\Pi(\rho) = r(2\pi, \rho)$ given the section $\{(r, 0) : r \simeq 0\}$.
[6] In practice we will only compute the L_n coefficient if $v_k(2\pi) = 0$ for $k = 2, \ldots, 2n$.
[7] This H will be either a first integral of the system or a Lyapunov function. In the first case, we obtain a center for the system; in the second one, a focus.
[8] Note that finding such a function H is computationally more efficient that finding the values of $v_{2n+1}(2\pi)$ of Lyapunov's method.

Theorem 50 (Bogdanov-Takens bifurcation). The normal form of the codimension-two *Bogdanov-Takens bifurcation* is:

$$\begin{cases} x' = y \\ y' = \beta_1 + \beta_2 x + x^2 - xy \end{cases} \tag{9}$$

Observe that three codimension-one bifurcations occur nearby: a saddle-node bifurcation, a Hopf bifurcation and a homoclinic bifurcation.

Proposition 51 (Routh-Hurwitz stability criterion). Consider the following polynomial:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$

The *Routh-Hurwitz stability criterion* gives a method to conclude whether all the real parts of the roots of p are negative or not. Construct a table as follows:

a_n	a_{n-2}	a_{n-4}	$\cdots$
a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
b_{11}	b_{12}	b_{13}	$\cdots$
b_{21}	b_{22}	b_{23}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1

where the coefficient b_{ij} is obtained by computing the determinant of the matrix whose first column is formed by the the first two elements of the two rows just above b_{ij}, and the second column is formed by the two elements of the $(i+1)$-th column in the two rows just above b_{ij}. Finally this determinant is divided by the first by minus the first coefficient of the row just above b_{ij}. That is:

$$\begin{aligned} b_{1j} &= -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2j} \\ a_{n-1} & a_{n-2j-1} \end{vmatrix} \\ b_{2j} &= -\frac{1}{b_{11}} \begin{vmatrix} a_{n-1} & a_{n-2j-1} \\ b_{11} & b_{1(j+1)} \end{vmatrix} \\ b_{ij} &= -\frac{1}{b_{(i-1)1}} \begin{vmatrix} b_{(i-2)1} & b_{(i-2)(j+1)} \\ b_{(i-1)1} & b_{(i-1)(i+1)} \end{vmatrix} \quad \forall i \geq 3 \end{aligned}$$

When completed, the number of sign changes in the first column of Table 1 will be the number of non-negative roots.

Rotated vector fields

Definition 52. Let $\mathbf{f}(\mathbf{x}) = (X(\mathbf{x}), Y(\mathbf{x}))$ be a vector field in $\mathbb{R}$. We say that $\{\mathbf{f}(\cdot, \mu) : \mu \in \mathbb{R}\}$ is a *one-parameter family of rotated vector fields* if the equilibrium points of $\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mu)$ are isolated and:

$$\mathbf{f} \times \frac{\partial \mathbf{f}}{\partial \mu} = \begin{vmatrix} X & Y \\ X_\mu & Y_\mu \end{vmatrix} \neq 0$$

The vector field is *positively rotated* if $\mathbf{f} \times \frac{\partial \mathbf{f}}{\partial \mu} > 0$. Otherwise it is *negatively rotated*.

Remark. The word "rotated" can be explained by the following the expression of the rate of rotation in terms of μ is:

$$\frac{\partial \theta}{\partial \mu} = \frac{XY_\mu - YX_\mu}{X^2 + Y^2} = \frac{\mathbf{f} \times \frac{\partial \mathbf{f}}{\partial \mu}}{X^2 + Y^2}$$

Theorem 53. Stable and unstable limit cycles of a family of rotated vector fields expand or contract monotonically as the parameter μ varies in a fixed sense and the motion covers an annular neighbourhood of the initial position.

Theorem 54. A semiestable limit cycle Γ_μ of a family of rotated vector fields splits into two simple limit cycles, one stable and one unstable, as the parameter μ is varied in one sense and it disappears as μ is varied in the opposite sense.

Theorem 55 (Melnikov's method). Let $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^2)$, $\mathbf{g} \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}^m)$ and $\varepsilon \simeq 0$. Consider the following ODE:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \boldsymbol{\mu}) \tag{10}$$

Suppose that for $\varepsilon = 0$ the system has a one-parameter family of periodic orbits $\boldsymbol{\gamma}_h(t)$ of period T_h. Then for any simple zero $(\boldsymbol{\mu}_0, h_0)$ of the function

$$M(\boldsymbol{\mu}, h) = \int_0^{T_h} \mathbf{f}(\boldsymbol{\gamma}_h(t)) \times \mathbf{g}(\boldsymbol{\gamma}_h(t)) \,\mathrm{d}t$$

there exists a unique limit cycle $\boldsymbol{\Gamma}_\varepsilon$ for $\varepsilon \simeq 0$ such that $\lim\limits_{\varepsilon \to 0} \boldsymbol{\Gamma}_\varepsilon = \boldsymbol{\gamma}_{h_0}$. On the other hand, if $M(\boldsymbol{\mu}_0, h_0) \neq 0$, for sufficiently small ε, the system of Eq. (10) with $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ has no limit cycle in any sufficiently small neighborhood of $\boldsymbol{\gamma}_{h_0}$.

Corollary 56 (Melnikov's method). Let $H \in \mathcal{C}^2(\mathbb{R}^2)$, $P, Q \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}^m)$ and $\varepsilon \simeq 0$. Consider the following system of ODEs:

$$\begin{cases} x' = -H_y - \varepsilon Q(x, y, \boldsymbol{\mu}) \\ y' = H_x + \varepsilon P(x, y, \boldsymbol{\mu}) \end{cases}$$

Suppose the system has a center and let $\boldsymbol{\gamma}_h = \{H(x, y) = h\}$ be a one-parameter family of periodic orbits of it. Then:

$$M(\boldsymbol{\mu}, h) = \int_{\boldsymbol{\gamma}_h} P \,\mathrm{d}x + Q \,\mathrm{d}y$$

Graphs

Definition 57. A *graph* in a continuous dynamical system is the collection of a set of equilibrium points joined together with orbits which have them as α- and ω-limits.

Definition 58. We say that a graph in the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is *non-degenerated* if all the equilibrium points on it are *linear saddles*, that is, saddles $p \in \mathbb{R}^n$ such that $\det \mathbf{Df}(p) \neq 0$.

Proposition 59. Consider a system with a non-degenerated graph Γ with n equilibria and such that the differential at each of the n saddles has eigenvalues $\lambda_i > 0$ and $\mu_i < 0$. Then:

1. If $\displaystyle\prod_{i=1}^n \left|\frac{\mu_i}{\lambda_i}\right| < 1$, then Γ is unstable.
2. If $\displaystyle\prod_{i=1}^n \left|\frac{\mu_i}{\lambda_i}\right| > 1$, then Γ is stable.

Corollary 60. Suppose that the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has a homoclinic orbit $\boldsymbol{\gamma}$ which has p as α- and ω-limit. Then:

1. If $\mathbf{div}\,\mathbf{f}(p) > 0$, then $\boldsymbol{\gamma}$ is unstable.
2. If $\mathbf{div}\,\mathbf{f}(p) < 0$, then $\boldsymbol{\gamma}$ is stable.

Liénard system

Definition 61. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions of class $\mathcal{C}^1$. A *Liénard system* is a system of the form:

$$x'' + f(x)x' + g(x) = 0 \iff \begin{cases} x' = y - F(x) \\ y' = -g(x) \end{cases} \tag{11}$$

where we have denoted $F(x) := \int_0^x f(\xi)\,\mathrm{d}\xi$.

Theorem 62 (Liénard's theorem). Let $F, g \in \mathcal{C}^1(\mathbb{R})$ be odd functions such that:

- $xg(x) > 0$ for $x \neq 0$.
- $F'(0) < 0$.
- F has a single positive zero at $x = a$.
- F increases monotonically to infinity for $x \geq a$ as $x \to \infty$.

Then, the Liénard system of Eq. (11) has exactly one limit cycle and it is stable.

Definition 63. Let $\mu \in \mathbb{R}$. A *Van der Pol oscillator* is a Liénard system of the form:

$$x'' + \mu(x^2 - 1)x' + x = 0 \iff \begin{cases} x' = y \\ y' = -x + \mu(1 - x^2) \end{cases} \tag{12}$$

Corollary 64. For $\mu > 0$, Van der Pol oscillator has a unique limit cycle and it is stable.

Dynamics on $\mathbb{R}^3$

Proposition 65. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and $\boldsymbol{\gamma}(t)$ be a periodic orbit of period T of it. Consider a transversal section $\Sigma \subset \mathbb{R}^3$ that cuts $\boldsymbol{\gamma}$ and let $\boldsymbol{\Pi} : \Sigma \to \Sigma$ be the Poincaré map on it. If $\mathbf{M}(t)$ is the solution to the variational equation

$$\begin{cases} \mathbf{M}' = \mathbf{Df}(\boldsymbol{\gamma}(t))\mathbf{M} \\ \mathbf{M}(0) = \mathbf{I}_n \end{cases}$$

then, $\sigma(\mathbf{M}(T)) = \sigma(\mathbf{D}\boldsymbol{\Pi}) \cup \{1\}$. Hence, the stability of the orbit can be studied from $\mathbf{M}(T)$.

Lorenz System

Definition 66. The *Lorenz system* is defined as:

$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases}$$

4. | Global dynamics in discrete systems

Periodic orbits

Definition 67. Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a function and consider the discrete map

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \tag{13}$$

A *periodic point* $\mathbf{y}$ of period m is a solution to the equation $\mathbf{f}^m(\mathbf{x}) = \mathbf{x}$. If moreover it satisfies that $\mathbf{f}^k(\mathbf{y}) \neq \mathbf{y}$ $\forall k = 1, \ldots, m-1$, then m is called *prime period* of $\mathbf{y}$. We denote the *set of periodic points* of period (not necessarily prime) n by $\mathrm{Per}_n(\mathbf{f})$. Finally we define the *set of periods* $\mathrm{Per}(\mathbf{f}) \subseteq \mathbb{N}$ as follows:

$$m \in \mathrm{Per}(\mathbf{f}) \iff \exists \mathbf{y} \in \mathbb{R}^m \text{ such that } \mathbf{y} \text{ is a periodic point of prime period } m \text{ under } \mathbf{f}$$

Proposition 68. Consider the one-dimensional discrete map $x_{n+1} = f(x_n)$ and let $\gamma = \{x^1, \ldots, x^k\}$ be a periodic orbit of period k of f. Then, the orbit is stable if:

$$\prod_{i=1}^k |f'(x^i)| < 1$$

Analogously, it is unstable if:

$$\prod_{i=1}^k |f'(x^i)| > 1$$

Proof. The orbit γ is stable if $\left|(f^k(x^1))'\right| < 1$ as x^1 is a fixed point of f^k. But by the ?? ?? we have:

$$(f^k(x^1))' = \prod_{i=1}^k f'(f^{k-i}(x^1)) = \prod_{i=1}^k f'(x^{k+1-i}) = \prod_{i=1}^k f'(x^i)$$

The unstable case is analogous. □

Logistic map

Definition 69. The *logistic map* is the one-dimensional iteration

$$x_{n+1} = \mu x_n(1 - x_n) =: f_\mu(x_n) \tag{14}$$

where $\mu \in \mathbb{R}^*$.

Proposition 70. The logistic map of Eq. (14) satisfies that $f_\mu(0) = f_\mu(1) = 0$ and $f_\mu(p_\mu) = p_\mu$, where $p_\mu = 1 - \frac{1}{\mu}$. Moreover for $\mu > 1$ we have the following properties:

- $0 < p_\mu < 1$
- If $x > 1$ or $x < 0$, then $\lim_{n\to\infty} \omega(x) = -\infty$.

Proposition 71. The logistic map of Eq. (14) has a transcritical bifurcation at $\mu = 1$.

Proposition 72. Consider the logistic map of Eq. (14) with $\mu \in (1, 3)$. Then:

- p_μ is attracting and 0 i repelling.
- If $0 < x < 1$, then $\lim_{n\to\infty} \omega(x) = p_\mu$.

Definition 73 (Period-doubling bifurcation). A *period-doubling bifurcation* occurs when a slight change in a system's parameters causes a new periodic trajectory to emerge from an existing periodic trajectory, the new one having double the period of the original. A *period-halving bifurcation* occurs when a system switches to a new behavior with half the period of the original system.

Definition 74. A *period-doubling cascade* is an infinite sequence of period-doubling bifurcations.

Theorem 75. The logistic map of Eq. (14) has a period-doubling cascade starting at $\mu = 3$.

Definition 76 (Sharkovskii's order). Consider the following order relation $\geq_s$ on $\mathbb{N}$:

$$\begin{array}{ccccccccc} & & 3 & \geq_s & 5 & \geq_s \cdots \geq_s & (2n+1) & \geq_s & \cdots \\ \cdots & \geq_s & 2\cdot 3 & \geq_s & 2\cdot 5 & \geq_s \cdots \geq_s & 2\cdot(2n+1) & \geq_s & \cdots \\ \cdots & \geq_s & 2^2\cdot 3 & \geq_s & 2^2\cdot 5 & \geq_s \cdots \geq_s & 2^2\cdot(2n+1) & \geq_s & \cdots \\ & & \vdots & & \vdots & & \vdots & & \\ \cdots & \geq_s & 2^k\cdot 3 & \geq_s & 2^k\cdot 5 & \geq_s \cdots \geq_s & 2^k\cdot(2n+1) & \geq_s & \cdots \\ \cdots & \geq_s & 2^k & \geq_s & \cdots & \geq_s\ 2^2\ \geq_s & 2 & \geq_s & 1 \end{array}$$

Lemma 77. $(\mathbb{N}, \geq_s)$ is a totally ordered set.

Proposition 78 (Expansive fixed point theorem). Let $I \subseteq \mathbb{R}$ be a closed bounded interval and $f : I \to \mathbb{R}$ be a continuous function such that $I \subseteq f(I)$. Then, f has a fixed point.

Proof. Suppose $I = [a_1, a_2]$. Then, $\exists b_1, b_2 \in I$ such that $f(b_1) = a_1$ and $f(b_2) = a_2$. Now use the **??** **??**. □

Lemma 79 (Itinerary lemma). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that there exists closed bounded intervals $I_0, I_1, \ldots, I_{n-1}$ such that:

$$f(I_0) \subseteq I_1, f(I_1) \subseteq I_2, \ldots, f(I_{n-2}) \subseteq I_{n-1}, f(I_{n-1}) \subseteq I_0$$

Then, $\exists x \in I_0$ such that $f^n(x) = x$ and $f^j(x) \in I_j\ \forall j \in \mathbb{N}$.

Theorem 80 (Sharkovskii's theorem).

1. Let $I \subseteq \mathbb{R}$ be a closed bounded interval and $f : I \to I$ be a continuous function. If $n \in \mathrm{Per}(f)$, then $m \in \mathrm{Per}(f)\ \forall m \leq_s n$.

2. Given $n \in \mathbb{N}$, there exists a continuous function $f : I \to I$ defined on a closed bounded interval $I \subseteq \mathbb{R}$ such that $n \in \mathrm{Per}(f)$ and $\forall m \geq_s n$, $m \notin \mathrm{Per}(f)$.

Corollary 81 (Period three theorem). Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to I$ be a continuous function. If $3 \in \mathrm{Per}(f)$, then $\mathrm{Per}(f) = \mathbb{N}$.

Definition 82. Consider the logistic map of Eq. (14) with $\mu > 4$ and define the following set:

$$A_0 := \{x \in [0,1] : f_\mu(x) \notin [0,1]\}$$

Note that A_0 breaks the interval $[0,1]$ in itself and two other intervals: one on the right of it, I_0, and one on its left, I_1. Thus, $[0,1] = I_0 \cup A_0 \cup I_1$.

Proposition 83. Consider the logistic map of Eq. (14) with $\mu > 4$ and let $(s_0, s_1, s_2, \ldots) \in \{0,1\}$ be a sequence of zeros and ones. We define the following sets

$$\begin{aligned} I_{s_0 s_1 \cdots s_n} &:= \\ &:= \{x \in I : x \in I_{s_0}, f_\mu(x) \in I_{s_1}, \ldots, {f_\mu}^n(x) \in I_{s_n}\} \end{aligned}$$

Then, $\Lambda_\mu := \bigcap_{n=0}^\infty I_{s_0 s_1 \cdots s_n}$ is homeomorphic to the Cantor set.

Proposition 84. Consider the logistic map of Eq. (14) with $\mu > 4$. Then, Λ_μ is invariant under f_μ.

Symbolic dynamics

Definition 85. Consider the set

$$\Sigma_2 := \{(s_0, s_1, s_2, \ldots) : s_n \in \{0,1\}\ \forall n \in \mathbb{N} \cup \{0\}\}$$

and define the following distance on Σ_2:

$$d_2((s_0, s_1, s_2, \ldots), (t_0, t_1, t_2, \ldots)) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n}$$

In order to simplify the notation we will write from now on $s = (s_0, s_1, s_2, \ldots)$ and $t = (t_0, t_1, t_2, \ldots)$.

Lemma 86. (Σ_2, d_2) is a metric space.

Proposition 87 (Proximity theorem). Let $s, t \in \Sigma_2$. Then, $s_i = t_i$ for $i = 0, \ldots, n$ if and only if $d_2(s,t) \leq \frac{1}{2^n}$.

Definition 88 (Shift map). The *shift map* is defined as:

$$\begin{aligned} \sigma :\ & \Sigma_2 \longrightarrow \Sigma_2 \\ & (s_0, s_1, s_2, \ldots) \longmapsto (s_1, s_2, s_3, \ldots) \end{aligned}$$

Proposition 89. The shift map is continuous.

Proof. Let $\varepsilon > 0$ and $s, t \in \Sigma_2$ with $d_2(s,t) < \delta$. Then:

$$d_2(\sigma(s), \sigma(t)) = |s_0 - t_0| + 2d_2(\sigma(s), \sigma(t)) < \varepsilon$$

if $\delta < \frac{\varepsilon - |s_0 - t_0|}{2}$. □

Proposition 90. The triplet $(\Sigma_2, \mathbb{Z}_{\geq 0}, \sigma)$ is a discrete semidynamical system.

Proposition 91.

1. $|\mathrm{Per}_n(\sigma)| = 2^n$

2. There exists a dense orbit for σ in Σ_2.

3. The set of periodic orbits of σ is dense in the set of orbits of σ.

Sketch of the proof.

1. Just note that there are 2^n different sequences in Σ_2 of the form:

$$(s_0, \ldots, s_{n-1}, s_0, \ldots, s_{n-1}, s_0, \ldots, s_{n-1}, \ldots)$$

2. Consider the sequence of "all sequences":

$$(0,1,0,0,0,1,1,0,1,1,0,0,0,0,0,1,0,1,0,1,0,0, \\ 0,1,1,1,0,1,1,1,0,1,1,1,\ldots)$$

3. Let $\varepsilon > 0$, take $k \in \mathbb{N}$ such that $\sum_{n=k}^{\infty} \frac{1}{2^n} < \varepsilon$ and let $s = (s_n) \in \Sigma_2$. The element

$$t_s := (s_0, \ldots, s_{k-1}, s_0, \ldots, s_{k-1}, s_0, \ldots, s_{k-1}, \ldots)$$

is periodic and satisfies $d_2(s, t_s) < \varepsilon$.

□

Definition 92. Consider the logistic map of Eq. (14). We define the *itinerary* of $x \in \Lambda_\mu$ as the function $S_\mu : \Lambda_\mu \to \Sigma_2$ defined by $S_\mu(x) = (s_n)$, where:

$$s_n = \begin{cases} 0 & \text{if } {f_\mu}^n \in I_0 \\ 1 & \text{if } {f_\mu}^n \in I_1 \end{cases}$$

Proposition 93. For $\mu > 2 + \sqrt{5}$, the itinerary map is a homeomorphism.

Theorem 94. Consider the logistic map of Eq. (14). Then, $S_\mu \circ f_\mu = \sigma \circ S_\mu$.

Corollary 95. Consider the logistic map of Eq. (14) for $\mu > 2 + \sqrt{5}$. Then:

1. $|\mathrm{Per}_n(f_\mu)| = 2^n$
2. There exists a dense orbit for f_μ in Λ_μ.
3. The set of periodic orbits of f_μ is dense in Σ_2.

Introduction to chaos

Definition 96. Let $f : I \to I$ be a function. The iteration $x_{n+1} = f(x_n)$ is *topologically transitive* if for any pair of open subsets $U, V \subseteq I$, $\exists k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$.

Definition 97. Let $f : I \to I$ be a function. The iteration $x_{n+1} = f(x_n)$ has *sensitive dependence on initial conditions* on I if $\exists \delta > 0$ such that for each $x \in I$ and any neighborhood N of x, exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

Lemma 98. Let $f : I \to I$ be a function such that the iteration $x_{n+1} = f(x_n)$ is topologically transitive. Then, it has a dense orbit.

Definition 99 (Chaos). Let $f : I \to I$ be a function. The iteration $x_{n+1} = f(x_n)$ is said to be *chaotic* on I if f has the following properties:

1. Periodic points are dense in I.
2. f is topologically transitive.
3. f has sensitive dependence on initial conditions[9].

Theorem 100. The logistic maps $x_{n+1} = \mu x_n(1 - x_n)$ are chaotic on Λ for $\mu > 2 + \sqrt{5}$.

Theorem 101. The logistic map $x_{n+1} = 4x_n(1 - x_n)$ is chaotic on $[0, 1]$.

[9]Although this is the classical definition of chaos, it has been shown that the first two properties imply the third one.