Algebraic topology

1. | Homotopy and fundamental group

Homotopy between maps and spaces

Definition 1. Let X, Y be two topological spaces [1] and let $f, g : X \to Y$ be two continuous maps. A *homotopy* between f and g is a continuous map $H : X \times [0,1] \to Y$ such that for all $x \in X$

1. $H(x, 0) = f(x)$
2. $H(x, 1) = g(x)$.

We say that f and g are *homotopic* if there exists a homotopy between them, and we denote it by $f \simeq g$.

Lemma 2. Let $X = A \cup B$, where A and B are closed sets. If $\varphi : A \to Y$ and $\psi : B \to Y$ are continuous maps such that $\varphi|_{A\cap B} = \psi|_{A\cap B}$, then the map

$$x \mapsto \begin{cases} \varphi(x) & \text{if } x \in A \\ \psi(x) & \text{otherwise} \end{cases}$$

is also continuous.

Proposition 3. The relation $\simeq$ is an equivalence relation between continuous maps.

Definition 4. A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. We say that X and Y are *homotopy equivalent* or that they have the same *type of homotopy* if there exists a homotopy equivalence between them, and we denote it by $X \simeq Y$.

Proposition 5. The relation $\simeq$ is an equivalence relation between topological spaces.

Definition 6. A space having the homotopy type of a point is called *contractible*.

Paths and foundamental group

Definition 7. A *path* in X is a continuous map $\sigma : [0,1] \to X$.

Definition 8. Let $\sigma, \tau : [0,1] \to X$ be two paths such that $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$. A homotopy between them is a continuous map $H : [0,1] \times [0,1] \to X$ such that

1. $H(s, 0) = \sigma(s)\ \forall s \in [0,1]$
2. $H(s, 1) = \tau(s)\ \forall s \in [0,1]$
3. $H(0, t) = \sigma(0) = \tau(0)\ \forall t \in [0,1]$
4. $H(1, t) = \sigma(1) = \tau(1)\ \forall t \in [0,1]$

When there exists a homotopy between σ and τ we say that they are *homotopic*, and we write $\sigma \simeq \tau$.

[1] From now on X, Y and Z will always be topological spaces

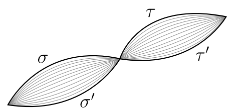
Proposition 9. The relation $\simeq$ is an equivalence relation between paths.

Definition 10. Let $\sigma, \tau : [0,1] \to X$ such that $\sigma(1) = \tau(0)$. We define the product of σ, τ as

$$\begin{aligned} \sigma \cdot \tau : [0,1] &\longrightarrow X \\ x &\longmapsto \begin{cases} \sigma(2s) & 0 \le s \le \frac{1}{2} \\ \tau(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}. \end{aligned}$$

Lemma 11. Let $\sigma, \sigma', \tau, \tau' : [0,1] \to X$ be paths such that

1. $\sigma(0) = \sigma'(0)$
2. $\sigma(1) = \sigma'(1) = \tau(0) = \tau'(0)$
3. $\tau(1) = \tau'(1)$.



If $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$ then $\sigma \cdot \tau \simeq \sigma' \cdot \tau'$.

Definition 12. A *loop at the basepoint* x_0 is a path $\sigma : [0,1] \to X$ such that $\sigma(0) = \sigma(1) = x_0$.

Definition 13. The *foundamental group* of X at the basepoint x_0 is the set of all homotopy classes of loops at the basepoint x_0, and it is denoted by $\pi_1(X, x_0)$. We define $[\sigma] \cdot [\tau] := [\sigma \cdot \tau]$, which is well-defined by Theorem 11.

Proposition 14. $(\pi_1(X, x_0), \cdot)$ is a group.

Proposition 15. Let $x_0, y_0 \in X$. If there exists a path $\gamma : [0,1] \to X$ from x_0 to y_0 then $\pi_1(X, x_0) \cong \pi_1(X, y_0)$.

Proof.

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\longrightarrow \pi_1(X, y_0) \\ [\sigma] &\longmapsto [\gamma \cdot \sigma \cdot \gamma^{-1}] \end{aligned}$$

is an isomorphism. □

Definition 16. Let $f : X \to Y$ be a continuous map, and $x_0 \in X$. We define

$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, f(x_0)) \\ [\alpha] &\longmapsto [f \circ \alpha]. \end{aligned}$$

Proposition 17. Let $f, g : X \to Y$ and $h : Y \to Z$ be continuous maps. Then

1. f_* is well-defined and it is a group homomorphism.
2. If $f \simeq g$ and $f(x_0) = g(x_0)$, then $f_* = g_*$.
3. $(h \circ f)_* = h_* \circ f_*$.

Corollary 18. $X \simeq Y \implies \pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$.

Foundamental group of S^1

Lemma 19. Let (X,d) be a compact metric space, and $\{U_1, \ldots, U_n\}$ a finite cover of open sets of X. Then there exists $\varepsilon > 0$ such that $\forall x \in X$, $B(x, \varepsilon) \subset U_i$ for some i. Such a number ε is called a Lebesgue number of the cover.

Proof. Let $C_i = X \setminus U_i$. Then the map

$$\begin{aligned} f : X &\longrightarrow [0, +\infty) \\ x &\longmapsto \frac{1}{n} \sum d(x, C_i) \end{aligned}$$

is a continuous map on a compact set and therefore reaches the minimum at X. The number $\varepsilon = \min f$ satisfies the desired properties. □

Lemma 20. Let $\sigma : [0,1] \to S^1$ be a loop such that $\sigma(0) = (1,0)$. Then, there is a unique path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and $\exp \circ \tilde{\sigma} = \sigma$, where $\exp(x) = \mathrm{e}^{2\pi x i}$. Such a path $\tilde{\sigma}$ is called a *lift* of σ. Morover, if $\sigma \simeq \tau$ then $\tilde{\sigma}(1) = \tilde{\tau}(1)$, and $\tilde{\sigma}(1)$ is called the degree of σ.

Proof. Let $U = S^1 \setminus \{(1,0)\}$ and $V = S^1 \setminus \{(-1,0)\}$. Let $k \in \mathbb{N}$ be such that $1/k < \varepsilon$, where ε is a Lebesgue number of the cover $\{\sigma^{-1}(U), \sigma^{-1}(V)\} \subset [0,1]$. For every $n \in \mathbb{N}$ $\exp : (n, n+1) \to U$ and $\exp : (n - \frac{1}{2}, n + \frac{1}{2}) \to V$ are homeomorphisms and, therefore, for every $1 \leq i \leq k$ there exists a unique $\tilde{\sigma}_i : [\frac{i-1}{k}, \frac{i}{k}] \to \mathbb{R}$ such that $\exp \circ \tilde{\sigma}_i = \sigma|_{[\frac{i-1}{k}, \frac{i}{k}]}$, and $\tilde{\sigma}_{i+1}(\frac{i}{k}) = \tilde{\sigma}_i(\frac{i}{k})$ if $i \geq 1$ and $\tilde{\sigma}_1(0) = 0$. Finally, for $t \in [\frac{i-1}{k}, \frac{i}{k}]$ we define $\tilde{\sigma}(t) = \tilde{\sigma}_i(t)$. Now let us suppose that $H : [0,1] \times [0,1] \to S^1$ is a homotopy between σ and τ. A very similar argument shows that there exists a continuous map $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}$ such that $\exp \circ \tilde{H} = H$. Thus, $t \mapsto \tilde{H}(1,t)$ is an integer-valued continuous map and, therefore, it is constant, which means that $\tilde{\sigma}(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{\tau}(1)$. □

Theorem 21. The foundamental group of S^1 is isomorphic to $\mathbb{Z}$

Proof.

$$\begin{aligned} \varphi : \pi_1(S^1, (1,0)) &\longrightarrow \mathbb{Z} \\ [\sigma] &\longmapsto \deg(\sigma) \end{aligned}$$

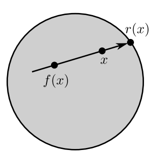
is a well-defined isomorphism. □

Definition 22. Let $A \subseteq X$ be a subspace of X. A *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. If such a retraction exists, A is called a *retract* of X.

Corollary 23. $S^1 \subseteq \mathbb{D}^2$ is not a retract of $\mathbb{D}^2$.

Theorem 24 (Brower theorem). Every continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.

Proof.
If $f(x) \neq x$ for every $x \in \mathbb{D}^2$, then the map $r : \mathbb{D}^2 \to S^1$ defined by $r(x)$ being the intersection of the line passing through x and $f(x)$ and S^1 closest to x is a retraction of $\mathbb{D}^2$ to S^1, which contradicts Theorem 23.



□

Lemma 25. Let $\alpha : [0,1] \to S^1$ be such that $\alpha(t + \frac{1}{2}) = -\alpha(t)$ for all $t \in [0, \frac{1}{2}]$. Then, the degree of α is odd.

Theorem 26 (Borsuk-Ulam theorem). If $f : S^2 \to \mathbb{R}^2$ is a continuous map, then there exists $x \in S^2$ such that $f(x) = f(-x)$.

Proof. If $f(x) \neq f(-x)$ for all $x \in S^2$, then $g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$ is a continuous map from S^2 to S^1. Consider $h : [0,1] \to S^2$ defined by $h(t) = (\cos(2\pi t), \sin(2\pi t), 0)$, and $\alpha = g \circ h$. Then, since h is contractile, α is also contractile, but $g(-x) = -g(x) \implies \alpha(t + 1/2) = -\alpha(t)$, in contradiction with Theorem 25. □

Corollary 27. Let $A_1, A_2, A_3 \subseteq S^2$ be closed sets such that $S^2 = A_1 \cup A_2 \cup A_3$. Then there exists $x \in S^2$ such that $x, -x \in A_i$ for some i.

Proof. Consider $f : S^2 \to \mathbb{R}^2$ such that $f(x) = (d_1(x), d_2(x))$, where $d_i(x)$ is the distance from x to A_i, and apply Theorem 26. □

Proposition 28. S^2 is contractible.

Seifert-Van Kampen theorem

Definition 29. The *free group* F_S over a set S is the set of all the finite words $s_1^{n_1} s_2^{n_2} \ldots s_r^{n_r}$ such that $s_i \in S$, $n_i \in \mathbb{Z} \setminus \{0\}$ and $s_i \neq s_{i+1}$, and the empty word, which will the identity element. The group operation in F_S is the juxtaposition, that is,

$$s_1^{n_1} \ldots s_r^{n_r} \cdot g_1^{m_1} \ldots g_r^{m_t} = \\ = \begin{cases} s_1^{n_1} \ldots s_r^{n_r} g_1^{m_1} \ldots g_t^{m_t} & \text{if } s_r \neq g_1 \\ s_1^{n_1} \ldots s_r^{n_r + m_1} g_2^{m_2} \ldots g_t^{m_t} & \text{if } s_r = g_1,\, n_r + m_1 \neq 0 \\ s_1^{n_1} \ldots s_{r-1}^{n_{r-1}} g_2^{m_2} \ldots g_t^{m_t} & \text{if } s_r = g_1,\, n_r + m_1 = 0 \end{cases}$$

Proposition 30. Let G be a group, and $S \subseteq G$ a generating set of G. Then, there exists an epimorphism from F_S to G.

Definition 31. Let G be a group, $S \subseteq G$ a generating set of G, and $\varphi : F_S \to G$ an epimorphism. An element of $\ker(\varphi)$ is called a relation. If $R = \{r_i\}_{i \in I}$ is a set of relations such that $\triangleleft R \triangleright = \ker(\varphi)$, then we say that G has a presentation $< S \,|\, R >$.

Definition 32. The free product $*_{i \in I} G_i$ of a set of groups $\{G_i\}_{i \in I}$ is the set of all finite words $g_1 g_2 \ldots g_n$ such that $g_i \in G_{g_i} \setminus \{e_{g_i}\}$, where e_{g_i} is the identity element of the group G_{g_i}. The group operation in $*_{i \in I} G_i$ is the juxtaposition, that is,

$$g_1 \ldots g_r \cdot h_1 \ldots h_t = \\ = \begin{cases} g_1 \ldots g_r h_1 \ldots h_t & \text{if } G_{g_r} \neq G_{h_1} \\ g_1 \ldots (g_r \cdot h_1) h_2 \ldots h_t & \text{if } G_{g_r} = G_{h_1},\, g_r \cdot h_1 \neq e_{g_r} \\ g_1 \ldots g_{r-1} h_2 \ldots h_t & \text{if } G_{g_r} = G_{h_1},\, g_r \cdot h_1 = e_{g_r} \end{cases}$$

Proposition 33. Let $\{G_i\}_{i\in I}$ be a set of groups, G a group, and $f_i : G_i \to G$ a group homomorphism for every $i \in I$. Then, there exists a unique group homomorphism $\varphi : *_{i\in I} G_i \to G$ such that $\varphi|_{G_i} = f_i$

Theorem 34 (Seifert–Van Kampen theorem). If X is the union of two path-connected open sets U, V containing the basepoint x_0 such that $U \cap V$ is path-connected, then the homomorphism $\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ defined by $\Phi|_{\pi_1(U,x_0)}([\alpha]_U) = [\alpha]_X$ and $\Phi|_{\pi_1(V,x_0)}([\alpha]_V) = [\alpha]_X$ is surjective, and $\ker(\Phi) = \triangleleft\{[\alpha]_U[\alpha]_V^{-1} : [\alpha]_{U\cap V} \in \pi_1(U \cap V, x_0)\}\triangleright$.

2. | Singular Homology

Definition 35. The standard *p-simplex* is the set

$$\Delta_p := \{(x_0, ..., x_p) \in \mathbb{R}^{p+1} \ : x_i \geq 0, \sum x_i = 1\}$$

Definition 36. A *singular p-simplex* (or simply *p-simplex*) in X is a continuous map $\sigma : \Delta_p \to X$.

Definition 37. We denote by $C_p(X)$ the abelian free group generated by the set of all p-simplex in X. An element of $C_p(X)$ is called a *singular p-chain* (or simply a *p-chain*).

Definition 38. The i-th side of the standard p-simplex Δ_p is the $(p-1)$-simplex

$$\begin{aligned} F_i^p : \Delta_{p-1} &\longrightarrow \Delta_p \\ (x_0, ..., x_{p-1}) &\longmapsto (x_0, ..., x_{i-1}, 0, x_i, ..., x_{p-1}). \end{aligned}$$

Definition 39. The *boundary* of a p-simplex σ is the $(p-1)$-chain

$$\partial\sigma := \sum_{i=0}^{p} (-1)^i \sigma \circ F_i^p,$$

and the boundary of a p-chain $\sum a_\sigma \sigma$ is the $(p-1)$-chain

$$\partial \sum a_\sigma \sigma := \sum a_\sigma \partial\sigma.$$

Definition 40. Let c be a p-chain in X.

1. We say that a c is a *p-cycle* if $\partial c = 0$. The set of all p-cycles in X is denoted by $Z_p(X)$.
2. We say that c is a *p-boundary* if there exists $c' \in C_{p+1}(X)$ such that $\partial c' = c$. The set of all *p-boundarys* in X is denoted by $B_p(X)$.

Proposition 41. $\partial : C_p \to C_{p-1}$ is a group homomorphism, and, therefore, $Z_p(X)$ and $B_p(X)$ are subgroups of $C_p(X)$.

Lemma 42. Let c be a p-chain. Then, $\partial\partial c = 0$. As a result, $B_p(X) \subseteq Z_p(X)$.

Definition 43. The p-th homology group of X is

$$H_p(X) := {Z_p(X)}\big/{B_p(X)}$$

Morphisms of chains

Definition 44. Let $f : X \to Y$ be a continuous map. f induces a map $f_\#$ from $C_p(X)$ to $C_p(Y)$ as follows:

$$\begin{aligned} f_\# : C_p(X) &\longrightarrow C_p(Y) \\ \sum a_\sigma \sigma &\longmapsto \sum a_\sigma f \circ \sigma. \end{aligned}$$

Proposition 45. Let $f : X \to Y$ be a continuous map. Then:

1. $f_\#$ is a group homomorphism.
2. $f_\# \circ \partial = \partial \circ f_\#$.
3. $f_\#(Z_p(X)) \subseteq Z_p(Y)$, and $f_\#(B_p(X)) \subseteq B_p(Y)$. Therefore, $f_\#$ induces a group homeomorphism $f_* : H_p(X) \to H_p(Y)$.

Definition 46. Let $f : X \to Y$ be a continuous map. f induces a group homomorphism f_* from $H_p(X)$ to $H_p(Y)$ defined by $f_*([c]) = [f_\#(c)]$, $c \in C_p(X)$, which is well-defined by Theorem 45.

Proposition 47. Let $f : X \to Y$, $g : Y \to Z$ be two continuous maps. Then,

1. $(f \circ g)_* = g_* \circ f_*$.
2. $(id_X)_* = id_{H_p(X)}$.

Corollary 48. If $f : X \to Y$ is an homeomorphism, then $f_* : H_p(X) \to H_p(Y)$ is a group isomorphism.

Proposition 49. Let $X = \bigsqcup X_\alpha$ be the decomposition of X in path-connected components, and $i_\alpha : X_\alpha \to X$ the inclusion. Then $\bigoplus_\alpha i_\alpha : \bigoplus_\alpha H_p(X_\alpha) \to H_p(X)$ is a group isomorphism.

Proposition 50. If X is path-connected, then $H_0(X) \cong \mathbb{Z}$.

Proof.

$$\begin{aligned} \varepsilon : H_0(X) &\longrightarrow \mathbb{Z} \\ \left[\sum a_\sigma \sigma\right] &\longmapsto \sum a_\sigma \end{aligned}$$

is a well-defined group isomorphism. □

Relationship between homotopy and homology

Lemma 51. Let $f, g : X \to Y$ be two continuous maps such that $f \simeq g$. There exists a group homomorphism $P : C_p(X) \to C_{p+1}(Y)$ such that $\partial P + P\partial = g_\# - f_\#$.

Theorem 52. Let $f, g : X \to Y$ be two continuous maps. If $f \simeq g$, then $f_* = g_*$.

Corollary 53. If $X \simeq Y$, then $H_p(X) \cong H_p(Y)$.