Topology

1. | Topological spaces

Metric spaces

Definition 1. Let X be a set. A *distance* (or *metric*) in X is a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$ the following properties are satisfied:

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y) \quad$ (*triangular inequality*).

Definition 2. A *metric space* is a pair (X, d), where X is a set and d is a distance in X.

Proposition 3. Let $x, y \in \mathbb{R}^n$ such that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The following functions are metrics in $\mathbb{R}^n$.

1. *Euclidean metric*:
$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
2. *Taxicab metric*:
$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$
3. *Maximum metric*:
$$d(x, y) = \max\{|x_i - y_i| : i \in \{1, \ldots, n\}\}$$

Proposition 4. Let X be a set. Then, (X, d) is a metric space, where:
$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
This metric d is called *discrete metric*.

Definition 5. Let (X, d) be a metric space $a \in X$ and $r \in \mathbb{R}$. We define the *ball* $B_d(a, r)$ of center a and radius r with the metric (X, d) as:
$$B_d(a, r) = \{x \in X : d(x, a) < r\}$$

Definition 6. Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : (X, d_X) \to (Y, d_Y)$ be a function. We say that f is continuous if $\forall a \in X$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x, a) < \delta$ or, equivalently:
$$f(B_{d_X}(a, \delta)) \subseteq B_{d_Y}(f(a), \varepsilon)$$
which is equivalent to $B_{d_X}(a, \delta) \subseteq f^{-1}(B_{d_Y}(f(a), \varepsilon))$.

Definition 7. Let (X, d) be a metric space. We say that a subset $A \subseteq X$ is *open* if $\forall a \in A$, $\exists \varepsilon > 0$ such that $B_d(a, \varepsilon) \subseteq A$.

Proposition 8. Let (X, d) be a metric space. Then:

- $\varnothing$ and X are open sets.
- If I is an arbitrary index set and $\{U_i : U_i \subseteq X \; \forall i \in I\}$ is a collection of open sets, then $\bigcup_{i \in I} U_i$ is an open set.
- If $\{U_i : U_i \subseteq X \; \forall i \in \{1, \ldots, n\}\}$ is a finite collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is an open set.

Proposition 9. Let (X, d) be a metric space and $x \in X$. Then, the ball $B_d(x, r)$ is open $\forall r \in \mathbb{R}$.

Proposition 10. Let (X, d) be a metric space and $A \subseteq X$ be a subset of X. Then, A is open if and only if $A = \bigcup_{i \in I} B_d(a_i, \varepsilon_i)$, where I is an index set, $a_i \in A$ and $\varepsilon_i > 0$ for all $i \in I$.

Theorem 11. Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : (X, d_X) \to (Y, d_Y)$ be a function. The following statements are equivalent:

1. f is continuous.
2. If $A \subseteq Y$ is open, then $f^{-1}(A) \subseteq X$ is also open.

Proposition 12. Let (X, d) be a metric space with $|X| \geq 2$ and $x, y \in X$. Then, $\exists \delta > 0$ such that $x \in B_d(x, \delta)$, $y \in B_d(y, \delta)$ and $B_d(x, \delta) \cap B_d(y, \delta) = \varnothing$.

Topological spaces

Definition 13 (Topological space). Let X be a set. A *topology* τ on a set X is a collection of subsets of X (that is, $\tau \subseteq \mathcal{P}(X)$) satisfying the following properties:

1. $\varnothing, X \in \tau$.
2. The intersection of any finite subcollection of τ is in τ.
3. The union of any subcollection of τ is in τ.

The ordered pair (X, τ) is called a *topological space*[1]. The elements of X are called *points* and the elements of τ, *open sets*.

Definition 14. Let (X, τ) and (X, τ') be topological spaces. We say that τ is *finer* than τ' if $\tau' \subseteq \tau$.

Proposition 15. Let X be a set, $p \in X$ be a point of X and d be a metric defined on X. Then, we can construct some topologies on X as follows:

- *Topology induced from the metric*:
$$\tau := \{U \subseteq X : U \text{ is open with the metric } d\}$$

[1] Sometimes, in order to simplify the notation, we will write X instead of (X, τ) to denote the topological space (X, τ) as well as the set X.

- *Trivial topology*: $\tau_{\text{t}} := \{\varnothing, X\}$
- *Discrete topology*: $\tau_{\text{d}} := \mathcal{P}(X)$
- *Cofinite topology*:
$$\tau_{\text{f}} := \{U \subseteq X : U = \varnothing \vee X \setminus U \text{ is finite}\}$$
- *Cocountable topology*:
$$\tau := \{U \subseteq X : U = \varnothing \vee X \setminus U \text{ is finite or countable}\}$$
- *Particular point topology*:
$$\tau := \{U \subseteq X : U = \varnothing \vee p \in U\}$$
- *Excluded point topology*:
$$\tau := \{U \subseteq X : U = X \vee p \notin U\}$$
- *Sierpiński topology*: If $X = \{0, 1\}$,
$$\tau := \{\varnothing, \{1\}, \{0, 1\}\}$$

Definition 16. Let (X, τ) be a topological space and $C \subseteq X$. We say that C is *closed* if $X \setminus C \in \tau$, that is, if $X \setminus C$ is open.

Definition 17. Let (X, τ) be a topological space and $A \subseteq X$. We say that A is *clopen* if it is both open and closed.

Proposition 18. Let (X, τ) be a topological space. Then:

1. $\varnothing$ and X are closed.
2. The union of any finite subcollection of closed sets in (X, τ) is closed in (X, τ).
3. The intersection of any subcollection of closed sets in (X, τ) is closed in (X, τ).

Basis for a topology

Definition 19. Let (X, τ) be a topological space and $\mathcal{B} \subseteq \tau$ be a subset of open sets. We say that $\mathcal{B}$ is a *basis* of τ if $\forall U \in \tau$ and $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 20. Let (X, τ) be a topological space and $\mathcal{B}$ be a basis of τ. Then, for all $U \in \tau$ we have:
$$U = \bigcup_{x \in U} B_x$$
where $x \in B_x \subseteq U$ and $B_x \in \mathcal{B}\ \forall x \in U$.

Lemma 21. Let (X, τ) be a topological space, $\mathcal{B} \subseteq \tau$ be a basis of τ and $\{B_i \in \mathcal{B} : i = 1, \ldots, n\}$ be a collection of elements of $\mathcal{B}$. Then, $\forall x \in \bigcap_{i=1}^n B_i$, $\exists B' \in \mathcal{B}$ such that $x \in B' \subseteq \bigcap_{i=1}^n B_i$.

Proposition 22. Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of subsets of X such that:

a) $X = \bigcup_{B \in \mathcal{B}} B$
b) $\forall U, V \in \mathcal{B}$ and $\forall x \in U \cap V$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U \cap V$.

Then, there exists a unique topology τ of X such that:

1. $\mathcal{B}$ is a basis of τ.
2. τ is the least finer topology that contains $\mathcal{B}$.

Definition 23. Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The *topology generated* by $\mathcal{B}$ is:
$$\tau = \left\{ U \subseteq X : U = \bigcup_{i \in I} B_i, B_i \in \mathcal{B}\ \forall i \in I \right\}$$
Or equivalently:
$$\tau = \{U \subseteq X : \forall x \in U\ \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$$

Definition 24. Let
$$\mathcal{B} = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R}, a < b\}$$
We define the *lower limit topology* as the topology generated by $\mathcal{B}$.

Proposition 25. The lower limit topology is finer that the usual topology of $\mathbb{R}$.

Definition 26. Let
$$U_n = \begin{cases} \{n\} & \text{if } n \text{ is odd} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even} \end{cases}$$
and $\mathcal{B} = \{U_n \subset \mathbb{Z} : n \in \mathbb{Z}\}$. We define the *digital topology* as the topology generated by $\mathcal{B}$.

Definition 27. Let (X, τ) be a topological space and $\mathcal{S} \subseteq \tau$ be a subset. We say that $\mathcal{S}$ is a *subbasis* of τ if $\forall U \in \tau$, U can be written as a union of finite intersections of elements of $\mathcal{S}$.

Proposition 28. Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $X = \bigcup_{S \in \mathcal{S}} S$. Then, there exists a unique topology τ of X such that:

1. $\mathcal{S}$ is a subbasis of the topology τ.
2. τ is the least finer topology that contains $\mathcal{S}$.

Interior, closure and boundary of a set

Definition 29 (Interior). Let (X, τ) be a topological space and $A \subseteq X$ be a subset. The *interior* of A, $\operatorname{Int}_{(X,\tau)} A$ or simply $\operatorname{Int} A$, is the largest open subset of X contained in A.

Definition 30 (Closure). Let (X, τ) be a topological space and $A \subseteq X$ be a subset. The *closure* of A, $\operatorname{Cl}_{(X,\tau)} A$ or simply $\operatorname{Cl} A$, is the smallest closed subset of X containing A.

Proposition 31. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then:
$$\operatorname{Int} A = \bigcup_{\substack{U \subseteq A \\ U \text{ is open}}} U \qquad \operatorname{Cl} A = \bigcap_{\substack{C \supseteq A \\ C \text{ is closed}}} C$$
Hence, we have the inclusions:
$$\operatorname{Int} A \subseteq A \subseteq \operatorname{Cl} A$$
And, furthermore:

- $\operatorname{Int} A = A \iff A$ is open
- $\operatorname{Cl} A = A \iff A$ is closed

Definition 32. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. A is called *dense* in (X, τ) if $\forall U \in \tau$ with $U \neq \varnothing$ we have $U \cap A \neq \varnothing$.

Proposition 33. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then, A is dense in (X, τ) if and only if $\operatorname{Cl} A = X$.

Proposition 34. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then:

- If $U \subseteq A$ is open, then $U \subseteq \operatorname{Int} A$.
- If $C \supseteq A$ is closed, then $\operatorname{Cl} A \subseteq C$.

Definition 35 (Boundary). Let (X, τ) be a topological space and $A \subseteq X$ be a subset. The *boundary* of A, $\partial_{(X,\tau)} A$ or simply ∂A, is:

$$\partial A := \operatorname{Cl}(A) \cap \operatorname{Cl}(X \setminus A)$$

Proposition 36. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then:

$$X = \operatorname{Int} A \sqcup \partial A \sqcup \operatorname{Int}(X \setminus A)$$

Definition 37. Let (X, τ) be a topological space and $x \in X$. We say that $N \subseteq X$ is a *neighbourhood* of x if $\exists U \in \tau$ such that $x \in U \subseteq N$. We denote by $\mathcal{N}_x$ the set of all neighbourhoods in (X, τ) of x.

Definition 38. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. We say that $x \in X$ is an *interior point* of A if A is a neighbourhood of x.

Definition 39. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. We say that $x \in X$ is an *adherent point* of A if $\forall N \in \mathcal{N}_x$ we have that $N \cap A \neq \varnothing$.

Proposition 40. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then:

1. $\operatorname{Int} A$ is the set containing all the interior points of A.
2. $\operatorname{Cl} A$ is the set containing all the adherent points of A.

Proposition 41. Let (X, τ) be a topological space and $A, B \subseteq X$ be subsets.
Properties regarding the interior:

1.1. $\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int} A$

1.2. $A \subseteq B \implies \operatorname{Int} A \subseteq \operatorname{Int} B$

1.3. $\operatorname{Int}(X \setminus A) = X \setminus \operatorname{Cl} A$

1.4. $\operatorname{Int}(B \setminus A) = \operatorname{Int} B \setminus \operatorname{Cl} A$

1.5. $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$

1.6. $\operatorname{Int}(A \cup B) \supseteq \operatorname{Int} A \cup \operatorname{Int} B$

Properties regarding the closure:

2.1. $\operatorname{Cl}(\operatorname{Cl}(A)) = \operatorname{Cl} A$

2.2. $A \subseteq B \implies \operatorname{Cl} A \subseteq \operatorname{Cl} B$

2.3. $\operatorname{Cl}(X \setminus A) = X \setminus \operatorname{Int} A$

2.4. $\operatorname{Cl}(A \cap B) \subseteq \operatorname{Cl} A \cap \operatorname{Cl} B$

2.5. $\operatorname{Cl}(A \cup B) = \operatorname{Cl} A \cup \operatorname{Cl} B$

Properties regarding the boundary:

3.1. $\partial A \cap \operatorname{Int} A = \varnothing$

3.2. $\partial A = \operatorname{Cl} A \setminus \operatorname{Int} A$

3.3. $\partial A \cup \operatorname{Int} A = \operatorname{Cl} A$

3.4. $\partial(A \cup B) \subseteq \partial A \cup \partial B$

3.5. $\partial(\partial A) \subseteq \partial A$

3.6. $\partial A \subseteq A \iff A$ is closed

3.7. $\partial A \cap A = \varnothing \iff A$ is open

3.8. $\partial A = \varnothing \iff A$ is clopen

Proposition 42 (Kuratowski's problem). Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then:

$$\begin{aligned}\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int} A))) &= \operatorname{Cl}(\operatorname{Int} A)\\ \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} A))) &= \operatorname{Int}(\operatorname{Cl} A)\end{aligned}$$

Definition 43. Let (X, τ) be a topological space and $A, B \subseteq X$ be subsets. We say that A and B are *separated* if

$$\operatorname{Cl} A \cap B = A \cap \operatorname{Cl} B = \varnothing$$

Definition 44. Let (X, τ) be a topological space and $A, B \subseteq X$ be subsets. We say that A and B are *separated by closed neighbourhoods* if there are closed neighbourhoods C_A and C_B of A and B respectively, such that $C_A \cap C_B = \varnothing$.

Proposition 45. Let (X, τ) be a topological space and $A, B \subseteq X$ be subsets. Then, A and B are separated by closed neighbourhoods if and only if $\operatorname{Cl} A \cap \operatorname{Cl} B = \varnothing$.

2. | Functions between topological spaces

Definition 46 (Continuous function). Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. We say that f is continuous if for all $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$.

Proposition 47. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. We say that f is continuous if and only if for all closed sets $C \subseteq Y$, we have $f^{-1}(C) \subseteq X$ is closed.

Theorem 48. Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function and $\mathcal{B}_Y$ be a basis of τ_Y. Then:

$$f \text{ is continuous} \iff f^{-1}(B) \in \tau_X \ \forall B \in \mathcal{B}_Y$$

Theorem 49. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. Then, the following statements are equivalent:

1. f is continuous.
2. $f^{-1}(\mathrm{Int}(B)) \subseteq \mathrm{Int}(f^{-1}(B))$ for all subsets $B \subseteq Y$.
3. $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A))$ for all subsets $A \subseteq X$.

Theorem 50. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. Then, f is continuous if and only if $\forall x \in X$ and $\forall U \in \tau_Y$ such that $f(x) \in U$, there exists a neighbourhood N of x with $f(N) \subseteq U$.

Proposition 51. Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$, $g : (Y, \tau_Y) \to (Z, \tau_Z)$ be continuous functions. Then, $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is continuous.

Definition 52. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A *homeomorphism* between (X, τ_X) and (Y, τ_Y) is a bijective function that is continuous and whose inverse is also continuous. We say that (X, τ_X) and (Y, τ_Y) are *homeomorphic*, and we denote it by $(X, \tau_X) \cong (Y, \tau_Y)$, if there exists a homeomorphism between them.

Definition 53 (Open function). Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. We say that f is *open* if $\forall U \in \tau_X$, we have $f(U) \in \tau_Y$.

Definition 54 (Closed function). Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. We say that f is *closed* if for all closed subsets $C \subseteq X$, we have that $f(C)$ is closed.

Theorem 55. Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function and $\mathcal{B}_X$ be a basis of τ_X. Then:

$$f \text{ is open} \iff f(B) \in \tau_Y \ \forall B \in \mathcal{B}_X$$

Theorem 56. Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f : (X, \tau_X) \to (Y, \tau_Y)$ be a bijective function. Then:

$$f \text{ is open} \iff f \text{ is closed}$$

Proposition 57. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. Then, the following statements are equivalent:

1. f is open.
2. $f(\mathrm{Int}(A)) \subseteq \mathrm{Int}(f(A))$ for all subsets $A \subseteq X$.

Proposition 58. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous bijective function. Then, the following statements are equivalent:

1. f is a homeomorphism.
2. f is open.
3. f is closed.

Proposition 59. Being homeomorphic as topological spaces is an equivalence relation.

3. | Subspaces

Subspace topology

Definition 60. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. We define the following set:

$$\tau_A = \{U \subseteq A : \exists V \in \tau \text{ such that } V \cap A = U\}$$

Then, (A, τ_A) is a topological space (called *topological subspace* of (X, τ)) and τ_A is called the *subspace topology* on A. We will write $(A, \tau_A) \subseteq (X, \tau)$ to denote that (A, τ_A) is a topological subspace.

Proposition 61. Let (X, τ) be a topological space and $(A, \tau_A) \subseteq (X, \tau)$ be a topological subspace. Then, $C \subseteq A$ is closed on (A, τ_A) if and only if $C = K \cap A$, where $K \subseteq X$ is a closed subset on (X, τ).

Proposition 62. Let (X, τ) be a topological space and $(A, \tau_A) \subseteq (X, \tau)$ be a topological subspace. Then:

1. If A is open and $U \subseteq A$, then:
$$U \in \tau_A \iff U \in \tau$$
2. If A is closed and $C \subseteq A$, then:
$$C \text{ is closed on } (A, \tau_A) \iff C \text{ is closed on } (X, \tau)$$

Proposition 63. Let (X, τ) be a topological space and $(A, \tau_A) \subseteq (X, \tau)$ be a topological subspace. Then, the inclusion $\iota : (A, \tau_A) \hookrightarrow (X, \tau)$ is continuous and τ_A is the least finer topology where ι is continuous.

Corollary 64. Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function and $(A, \tau_A) \subseteq (X, \tau_X)$ be a topological subspace. Then, $f|_A$ is also continuous.

Proposition 65. Let (X, τ) be a topological space, $\mathcal{B}$ be a basis of τ and $(A, \tau_A) \subseteq (X, \tau)$ be a topological subspace. Then,

$$\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$$

is basis of τ_A.

Proposition 66. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $(B, \tau_B) \subseteq (Y, \tau_Y)$ be a topological subspace. Let $f : (X, \tau_X) \to (B, \tau_B)$ be a function. Then, f is continuous if and only if $\iota \circ f : (X, \tau_X) \to (B, \tau_B) \hookrightarrow (Y, \tau_Y)$ is continuous.

Corollary 67. Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Then, $g : (X, \tau_X) \to (f(X), \tau_{f(X)})$ is also continuous.

Proposition 68. Let (X, τ_X) and (Y, τ_Y) be topological spaces such that $X = A \cup B$, for some sets A, B. Consider a function $f : (X, \tau_X) \to (Y, \tau_Y)$ satisfying that that $f|_A$ and $f|_B$ are continuous. Then:

1. If A, B are open, then f is continuous.
2. If A, B are closed, then f is continuous.

Cantor set

Definition 69. Let $C_0 = [0,1]$. Define $I_1 := \left(\frac{1}{3}, \frac{2}{3}\right)$ and $C_1 := C_0 \setminus I_1$. Then, define $I_2 := I_1 \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and $C_2 := C_0 \setminus I_2$. In general, define:

$$I_{n+1} = I_n \cup \left[\bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)\right]$$
$$C_{n+1} = C_0 \setminus I_{n+1}$$

We define the *Cantor set* $\mathcal{C}$ as:

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n$$

Proposition 70. The Cantor set $\mathcal{C}$ can be expressed as:

$$\mathcal{C} = \{x \in [0,1] : {x_3}^{2} \text{ does not contain the digit } 1\}$$

Proposition 71. The Cantor set $\mathcal{C}$ satisfies the following properties:

1. $\mathcal{C} \neq \varnothing$.
2. $\mathcal{C}$ is closed in $\mathbb{R}$.
3. $\mathcal{C}$ does not contain any interval of $\mathbb{R}$.
4. $\operatorname{Int} \mathcal{C} = \varnothing$.
5. $\mathcal{C}$ does not have the discrete topology.
6. $\mathcal{C}$ is not countable.

4. | Product topology

Finite product

Definition 72. Let (X, τ_X), (Y, τ_Y) be topological spaces. We define the *product topology* on $X \times Y$, denoted by $\tau_{X\times Y}$, as the topology generated by:

$$\mathcal{B} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$$

Proposition 73. Let (X, τ_X), (Y, τ_Y) be topological spaces. Then, the projections

$$\pi_X : (X \times Y, \tau_{X\times Y}) \longrightarrow (X, \tau_X)$$
$$\pi_Y : (X \times Y, \tau_{X\times Y}) \longrightarrow (Y, \tau_Y)$$

are continuous and open.

Proposition 74. Let (X, τ_X), (Y, τ_Y) be topological spaces. Then, $A \subseteq X \times Y$ is open on $(X \times Y, \tau_{X\times Y})$ if and only if $\forall a \in A$ there exist $U \in \tau_X$ and $V \in \tau_Y$ such that $a \in U \times V \subseteq A$.

Proposition 75. Let (X, τ_X), (Y, τ_Y) be topological spaces. If $\mathcal{B}_X$ is a basis for τ_X and $\mathcal{B}_Y$ is a basis for τ_Y, then

$$\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a basis for $\tau_{X\times Y}$.

[2] Here, x_3 mean the expression of x in base 3.

Proposition 76. Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces and $f : (Z, \tau_Z) \to (X \times Y, \tau_{X\times Y})$ be a function. Then, f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

Proposition 77. Let (X_i, τ_{X_i}), (Y_i, τ_{Y_i}) be topological spaces and $f_i : (X_i, \tau_{X_i}) \to (Y_i, \tau_{Y_i})$ be functions for $i = 1, 2$. Then,

$$\begin{aligned} f_1 \times f_2 : (X_1 \times X_2, \tau_{X_1\times X_2}) &\longrightarrow (Y_1 \times Y_2, \tau_{Y_1\times Y_2}) \\ (x_1, x_2) &\longmapsto (f_1(x_1), f_2(x_2)) \end{aligned}$$

is continuous if and only if f_1 and f_2 are both continuous.

Proposition 78. Let (X, τ_X), (Y, τ_Y) be topological spaces and $A \subseteq X$, $B \subseteq Y$ be closed subsets. Then, $A \times B$ is closed.

Proposition 79. Let (X, τ_X), (Y, τ_Y) be topological spaces and $(A, \tau_A) \subseteq (X, \tau_X)$, $(B, \tau_B) \subseteq (Y, \tau_Y)$ be topological subspaces. Then:

1. $\operatorname{Int}(A \times B) = \operatorname{Int} A \times \operatorname{Int} B$
2. $\operatorname{Cl}(A \times B) = \operatorname{Cl} A \times \operatorname{Cl} B$
3. $\partial(A \times B) = (\partial A \times \operatorname{Cl} B) \cup (\operatorname{Cl} A \times \partial B)$

Arbitrary product

Definition 80. Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ be a collection of topological spaces and $X := \prod_{i\in I} X_i$. We define the *box topology* on X as the topology generated by

$$\mathcal{B} = \left\{\prod_{i\in I} U_i : U_i \in \tau_{X_i}\ \forall i \in I\right\}$$

Definition 81. Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ be a collection of topological spaces and $X := \prod_{i\in I} X_i$. We define the *infinite product topology* on X, denoted by τ_X, as the topology generated by

$$\mathcal{B} = \left\{\prod_{i\in I} U_i : U_i \in \tau_{X_i}\ \forall i \in I \wedge U_i = X_i \text{ except for a finite number of indices}\right\}$$

Proposition 82. Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ be a collection of topological spaces and $X := \prod_{i\in I} X_i$. Then, the projection

$$\pi_{X_i} : (X, \tau_X) \longrightarrow (X_i, \tau_{X_i})$$

is continuous and open for all $i \in I$.

Proposition 83. Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ be a collection of topological spaces and $X := \prod_{i\in I} X_i$. If $\mathcal{B}_{X_i}$ is a basis of τ_{X_i} $\forall i \in I$, then

$$\mathcal{B} = \left\{\prod_{i\in I} U_i : U_i \in \mathcal{B}_{X_i}\ \forall i \in I\right\}$$

is a basis of τ_X.

Proposition 84. Let I be an index set, $\{(Y_i, \tau_{Y_i}) : i \in I\}$ and (X, τ_X) be topological spaces, $Y := \prod_{i\in I} Y_i$ and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. Then, f is continuous if and only if $\pi_{Y_i} \circ f$ is continuous for all $i \in I$.

Proposition 85. Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ and $\{(Y_i, \tau_{Y_i}) : i \in I\}$ be two collections of topological spaces, $X := \prod_{i\in I} X_i$, $Y := \prod_{i\in I} Y_i$ and $f_i : (X_i, \tau_{X_i}) \to (Y_i, \tau_{Y_i})$ be a function for all $i \in I$. Then,

$$\begin{aligned}\prod_{i\in I} f_i : (X, \tau_X) &\longrightarrow (Y, \tau_Y)\\ (x_i)_{i\in I} &\longmapsto (f_i(x_i))_{i\in I}\end{aligned}$$

is continuous if and only if f_i is continuous $\forall i \in I$.

Proposition 86. Let (X_i, τ_{X_i}) be topological spaces and $(A_i, \tau_{A_i}) \subseteq (X_i, \tau_{X_i})$ be topological subspaces for $i = 1, \dots, n$. Consider the following topological spaces:

1. The topological space created from the product of subspaces A_i.
2. The topological space created from the subspace $\prod_{i=1}^n A_i$ of the product $\prod_{i=1}^n X_i$.

Then, these topological spaces are the same.

Proposition 87. Let I be and index set, (X_i, τ_{X_i}) be topological spaces $\forall i \in I$ and $A_i \subseteq X_i$ be subsets $\forall i \in I$. Let $X := \prod_{i\in I} X_i$. Then, $\prod_{i\in I} A_i$ is dense in (X, τ_X) if and only if A_i is dense in (X_i, τ_{X_i}) $\forall i \in I$.

Theorem 88. The function

$$\begin{aligned}\varphi : \prod_{i=1}^{\infty}\{0, 2\} &\longrightarrow \mathcal{C}\\ (a_i) &\longmapsto \sum_{i=1}^{\infty} \frac{a_i}{3^i}\end{aligned}$$

is a homeomorphism.

Definition 89. We define the *$n-1$-th sphere* $S^{n-1} \subset \mathbb{R}^n$ as:

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

We define the *n-th ball* $B^n \subset \mathbb{R}^n$ as:

$$B^n := \{x \in \mathbb{R}^n : \|x\|^2 < 1\}$$

Definition 90 (Torus). We define the *torus* $T^2 \subset \mathbb{R}^3$ of major radius R and minor radius r (see Fig. 1) as:

$$T^2 := \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2\right\}$$

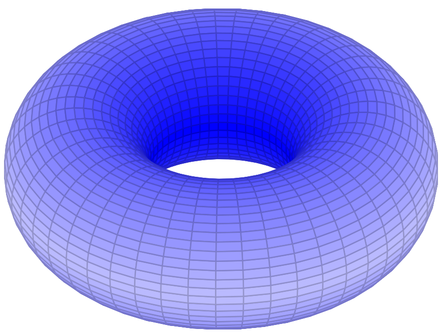

Figure 1: Torus T^2

Proposition 91. With the ordinary topology of $\mathbb{R}^n$ we have:

- $S^n \setminus (0, \overset{(n)}{\dots}, 0, 1) \cong \mathbb{R}^n$
- $S^1 \times S^1 \cong T^2$

5. | Quotient topology

Quotient topology

Definition 92. Let (X, τ_X) be a topological space, Y be a set and $f : X \to Y$ be a function. We define the *quotient topology* on Y defined by f as:

$$\tau_f := \{U \subseteq Y : f^{-1}(U) \in \tau_X\}$$

Proposition 93. Let (X, τ_X), (Y, τ_f) be topological spaces, where $f : X \to Y$ is a function. Then, f (thought as a function between topological spaces) is continuous and τ_f is the finest topology for which f is continuous.

Proposition 94. Let (X, τ_X), (Y, τ_f) be topological spaces, where $f : X \to Y$ is a function. Then, $C \subseteq Y$ is closed on (Y, τ_f) if and only if $f^{-1}(C) \subseteq X$ is closed on (X, τ_X).

Proposition 95. Let (X, τ_X), (Y, τ_f) and (Z, τ_Z) be topological spaces, where $f : X \to Y$ is a function, and $h : (Y, \tau_f) \to (Z, \tau_Z)$ be a function. Then, h is continuous if and only if $h \circ f$ is continuous.

Definition 96. Let (X, τ_X) be a topological space and $f : X \to Y$ be a function. We say that f is a *quotient map* if it is surjective and Y is equipped with the topology τ_f.

Definition 97. Let (X, τ_X) be a topological space and $\sim$ be an equivalence relation on X. Consider the canonical function $f : X \twoheadrightarrow X/\sim$. We define the *quotient space* as $(X/\sim, \tau_f)$.

Definition 98. Let (X, τ_X) be a topological space and $A \subseteq X$ be a subset. Consider the partition of X:

$$X = A \sqcup \bigsqcup_{x\in X\setminus A} \{x\}$$

and define the equivalence relation $\sim_A$ as follows:

$$x \sim_A y \iff x, y \in A \vee x = y \in X \setminus A$$

We define the *quotient space of collapsing a set to a point* as $X/A := X/\sim_A$ together with the quotient topology. We will write $[A] := [a]$ $\forall a \in A$, which is well-defined.

Proposition 99. Let (X, τ_X) be a topological space, $A \subseteq X$ be a subset and $\pi : X \twoheadrightarrow X/A$ be the projection. Then, for all $U \subseteq X/A$ we have:

$$\pi^{-1}(U) = \begin{cases} \displaystyle\bigcup_{[x]\in U} \{x\} & \text{if } [A] \notin U \\ \displaystyle A \cup \bigcup_{\substack{[x]\in U \\ [x]\neq[A]}} \{x\} & \text{if } [A] \in U \end{cases}$$

Group actions on topological spaces

Definition 100. Let (X,τ) be a topological space and $(G,\cdot)$ be a group. An *action* of $(G,\cdot)$ on (X,τ) is a function:

$$\begin{aligned} f:(G,\cdot)\times(X,\tau) &\longrightarrow (X,\tau)\\ (g,x) &\longmapsto f_g(x)\end{aligned}$$

where $f_g:(X,\tau)\to(X,\tau)$ is a homeomorphism for all $g\in G$ such that:

1. $f_e=\mathrm{id}$.

2. $f_{g\cdot h}=f_g\circ f_h,\ \forall g,h\in G$.

The pair $((X,\tau),f)$ is called a $(G,\cdot)$-*space*.

Proposition 101. Let $((X,\tau),f)$ be a G-space. Consider the equivalence relation: $x\sim y \iff \exists g\in G$ such that $y=f_g(x)$[3]. Then, the set of orbits under f, $X/G:=X/\sim$, is a topological space together with the quotient topology. Furthermore, the projection $\pi:X\twoheadrightarrow X/G$ is open.

Homeomorphisms of quotient spaces

Proposition 102. With the ordinary topology of $\mathbb{R}^n$ we have:

$$[0,1]\big/\{0,1\}\cong S^1$$

Proposition 103. Consider $X=[0,1]^2$. Define an equivalence relation $\sim$ in X in the following ways:

1. $(0,t)\sim(1,t)\ \forall t\in[0,1]$. Then, $X\cong S^1\times[0,1]$, which is a cylinder.

2. $(0,t)\sim(1,t)$ and $(s,0)\sim(s,1)\ \forall s,t\in[0,1]$. Then, $X\cong T^2$.

3. $(0,t)\sim(1,1-t)\ \forall t\in[0,1]$. In that case, $X\cong\mathcal{M}$, where $\mathcal{M}$ is the *Möbius band* (see Fig. 3).

4. $(0,t)\sim(1,t)$ and $(s,0)\sim(1-s,1)\ \forall s,t\in[0,1]$. In that case, $X\cong\mathcal{K}$, where $\mathcal{K}$ is the *Klein bottle* (see Fig. 4).

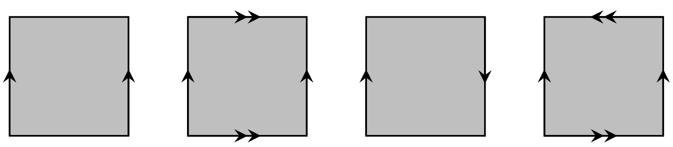

Figure 2: Representation of the quotient spaces $[0,1]^2/\sim$, where $\sim$ is the equivalence relation defined on Theorem 103

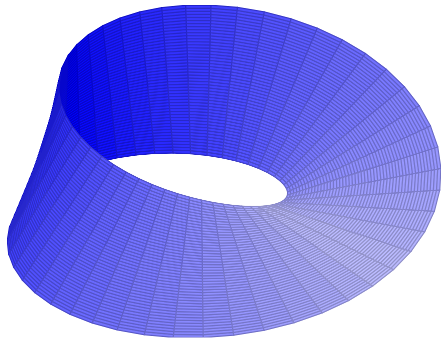

Figure 3: Möbius band

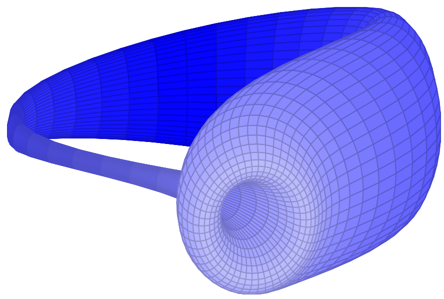

Figure 4: Klein bottle

Proposition 104. Let $\mathcal{P}_n(\mathbb{R})$ be the projective space of $\mathbb{R}^{n+1}$. Consider the relation $\sim$ on $\mathbb{R}^{n+1}$ such that $\mathbf{v}\sim-\mathbf{v}$ $\forall\mathbf{v}\in\mathbb{R}^{n+1}$. Then:

$$\mathcal{P}_n(\mathbb{R})\cong S^n\big/\sim$$

Proposition 105. Consider the ball B^2 and the equivalence relation $\sim$ in $\partial B^2=S^1$ such that for all $(x,y)\in\partial B^2$, $(x,y)\sim(x,-y)$. Then:

$$B^2\big/\sim\ \cong S^2$$

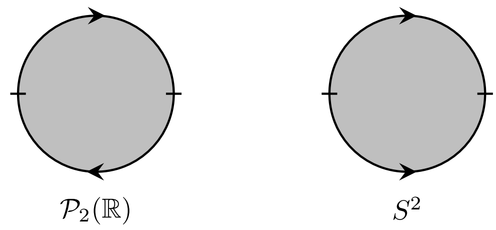

Figure 5: Representation of the quotient space $S^2/\sim$ (left), where $\sim$ is the equivalence relation defined on Theorem 104 and the quotient space $B^2/\sim$ (right), where $\sim$ is the equivalence relation defined on Theorem 105

6. | Separation axioms

Definition 106 **(T_0 space).** Let (X,τ) be a topological space. We say that (X,τ) is T_0[4] (or *Kolmogorov*) if for any two distinct points of X, there exists an open set that contains one of them but not the other.

Definition 107 **(T_1 space).** Let (X,τ) be a topological space. We say that (X,τ) is T_1 (or *Fréchet*) if for any two distinct points $x,y\in X$, there exists an open set $U\in\tau$ such that $x\in U$ and $y\notin U$.

[3]Note that this relation creates a partition of X in terms of the orbits under f.

[4]The letter T comes from the German word "Trennungsaxiom" which means "separation axioms".

Theorem 108. Let (X, τ) be a topological space. The statements following are equivalent:

1. (X, τ) is T_1.
2. For all $x \in X$, $\{x\} = \bigcap_{N \in \mathcal{N}_x} N$.
3. For all $x \in X$, $\{x\}$ is closed.

Definition 109 (T_2 space). Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or *Hausdorff*) if for any two distinct points $x, y \in X$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Theorem 110. Let (X, τ) be a topological space. The statements following are equivalent:

1. (X, τ) is T_2.
2. For all $x \in X$, $\{x\} = \bigcap_{N \in \mathcal{N}_x} \mathrm{Cl}(N)$.
3. The diagonal $\Delta(X) := \{(x, x) \in X \times X\} \subset X \times X$ is closed.

Proposition 111. Let (X, τ_X), (Y, τ_Y) be Hausdorff topological spaces. Then, $(X \times Y, \tau_{X \times Y})$ is Hausdorff.

Definition 112 ($T_{2\frac{1}{2}}$ space). Let (X, τ) be a topological space. We say that (X, τ) is $T_{2\frac{1}{2}}$ if for any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ separated by closed neighbourhoods ($\mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \varnothing$) such that $x \in U$ and $y \in V$.

Definition 113. Let (X, τ) be a topological space. We say that (X, τ) is *regular* if for all $x \in X$ and for all a closed sets $C \subseteq X$ such that $x \notin C$, there exist $U, V \in \tau$ such that $x \in U$, $C \subseteq V$ and $U \cap V = \varnothing$.

Definition 114 (T_3 space). Let (X, τ) be a topological space. We say that (X, τ) is T_3 if it is T_1 and regular.

Theorem 115. Let (X, τ) be a topological space. The statements following are equivalent:

1. (X, τ) is T_3.
2. For all $x \in X$ and for all $U \in \tau$ such that $x \in U$, $\exists V \in \tau$ such that $x \in V \subseteq \mathrm{Cl}(V) \subseteq U$.

Theorem 116. A subspace of a topological space T_3 is T_3.

Theorem 117. The product of topological spaces T_3 is T_3.

Definition 118. Let (X, τ) be a topological space. We say that (X, τ) is *normal* if for all closed sets $C, K \subseteq X$ such that $C \cap K = \varnothing$, there exist $U, V \in \tau$ such that $C \subseteq U$, $K \subseteq V$ and $U \cap V = \varnothing$.

Definition 119 (T_4 space). Let (X, τ) be a topological space. We say that (X, τ) is T_4 if it is T_1 and normal.

Theorem 120. Let (X, τ) be a topological space. The statements following are equivalent:

1. (X, τ) is T_4.
2. For all closed set $C \subseteq X$ and for all $U \in \tau$ such that $C \subseteq U$, $\exists V \in \tau$ such that $C \subseteq V \subseteq \mathrm{Cl}(V) \subseteq U$.

Theorem 121. A closed subspace of a topological space T_4 is T_4.

Theorem 122. If we also denote by T_i the set of all topological spaces which are T_i, for $i \in \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$, we have that:

$$T_4 \subset T_3 \subset T_{2\frac{1}{2}} \subset T_2 \subset T_1 \subset T_0$$

Lemma 123 (Urysohn's lemma). Let (X, τ) be a topological space T_4 and $A, B \subseteq X$ be closed sets. Then, there exists a continuous function $f : (X, \tau) \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

Theorem 124 (Tietze extension theorem). Let (X, τ) be a topological space T_4, $C \subseteq X$ be a closed set and $f : C \to [0, 1]$ be a continuous function. Then, there exists a continuous function $F : (X, \tau) \to [0, 1]$ such that $F(x) = f(x)\ \forall x \in C$.

Definition 125. A topological space (X, τ) is said to be *metrizable* if there is a metric d such that the topology induced by d is τ.

Theorem 126 (Urysohn's metrization theorem). Let (X, τ) be a topological space T_3 such that it admits a countable basis of open sets. Then, (X, τ) is metrizable.

7. | Compactness

Definition 127. We say that a property P of a topological space is a *topological property* if it is preserved under homeomorphisms. That is, if (X, τ_X), (Y, τ_Y) are homeomorphic topological spaces such that (X, τ_X) has the property P, then (Y, τ_Y) has the property P too.

Proposition 128. The properties T_i are topological properties for $i \in \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$.

Covers

Definition 129 (Cover). Let (X, τ) be a topological space. A *cover* of X is a collection $\{U_i : i \in I\}$ with $U_i \subseteq X\ \forall i \in I$ such that $X = \bigcup_{i \in I} U_i$.

Definition 130. Let (X, τ) be a topological space and $U = \{U_i : i \in I\}$ be a cover of X.

- We say that U is *finite* if I is finite.
- We say that U is *countable* if I is countable.
- We say that U is an *open cover* if $U_i \in \tau\ \forall i \in I$.

Definition 131. Let (X, τ) be a topological space, $A \subseteq X$ be a subset and $U = \{U_i : i \in I\}$ with $U_i \subseteq X\ \forall i \in I$. We say that U is a *cover* of A if $A \subseteq \bigcup_{i \in I} U_i$.

Definition 132. Let (X, τ) be a topological space and $U = \{U_i : i \in I\}$ be a cover of X. A *subcover* of U is a collection $\{U_j : j \in J\}$ with $J \subseteq I$.

Compactness

Definition 133 (Compact space). Let (X, τ) be a topological space. We say that (X, τ) is *compact* if each of its open covers has a finite subcover.

Proposition 134. Let (X, τ) be a topological space. Then, (X, τ) is compact if and only if for any collection $C = \{C_i : i \in I\}$ of closed sets such that $\bigcap_{i \in I} C_i = \varnothing$, there exists a finite subcollection $\{C_{i_1}, \ldots, C_{i_n}\}$ of C such that $\bigcap_{j=1}^{n} C_{i_j} = \varnothing$

Proposition 135. The *compactness* is a topological property.

Proposition 136. Let (X, τ_X), (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a surjective continuous function. If (X, τ_X) is compact, then (Y, τ_Y) is also compact.

Corollary 137. The quotient space of a compact space is compact.

Definition 138. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. We say that A is a *compact subset* of X if (A, τ_A) is a compact space.

Lemma 139. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. Then, A is compact if and only if each open cover of A in (X, τ) admits a finite subcover.

Proposition 140. Let (X, τ) be a topological space and $\{K_i \subseteq X : i = 1, \ldots, n\}$ be a collection of compact sets. Then, $\bigcup_{i=1}^{n} K_i$ is also compact.

Theorem 141. Let (X, τ_X), (Y, τ_Y) be topological spaces, $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function and $A \subseteq X$ be a subset. If A is compact, then $f(A)$ is also compact.

Theorem 142. Let (X, τ) be a compact topological space and $C \subseteq X$ be a closed subset. Then, C is compact.

Theorem 143. Let (X, τ) be a Hausdorff topological space and $K \subseteq X$ be a compact subset. Then, K is closed.

Corollary 144. Let (X, τ) be a Hausdorff topological space and $\{K_i : i = 1, \ldots, n\}$ be a collection of compact sets $K_i \subseteq X$, $i = 1, \ldots, n$. Then, $\bigcap_{i \in I} K_i$ is compact.

Corollary 145. Let (X, τ_X) be a compact topological space, (Y, τ_Y) be Hausdorff topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Then, f is closed. Furthermore, if f is bijective, then f is a homeomorphism.

Corollary 146. Let (X, τ) be a compact Hausdorff topological space and $\tau_1 \subseteq \tau \subseteq \tau_2$. Then:

- If (X, τ_1) is Hausdorff, then $\mathrm{id} : (X, \tau) \to (X, \tau_1)$ is a homeomorphism and $\tau_1 = \tau$.
- If (X, τ_2) is compact, then $\mathrm{id} : (X, \tau_2) \to (X, \tau)$ is a homeomorphism and $\tau_2 = \tau$.

Proposition 147. Let (X, τ) be a compact Hausdorff topological space and $C \subseteq X$ be a closed subset. Then, X/C, together with the quotient topology, is compact Hausdorff.

Proposition 148. Let (X, τ) be a Hausdorff topological space and $C, K \subseteq X$ be compact subsets. Then, there exist $U, V \in \tau$ such that $C \subseteq U$, $K \subseteq V$ and $U \cap V = \varnothing$.

Corollary 149. Let (X, τ) be a compact Hausdorff topological space. Then, (X, τ) is T_4.

Compactness of the product

Lemma 150. Let (X, τ_X), (Y, τ_Y) be topological spaces such that (Y, τ_Y) is compact and $U \in \tau_{X \times Y}$ be an open set such that $\{x\} \times Y \subseteq U \subseteq X \times Y$. Then, $\exists V \in \tau_X$ such that $x \in V$ and $\{x\} \times Y \subseteq V \times Y \subseteq U$.

Corollary 151. Let (X, τ_X), (Y, τ_Y) be topological spaces such that (Y, τ_Y) is compact. Then, the projection $\pi_X : (X \times Y, \tau_{X \times Y}) \to (X, \tau_X)$ is closed.

Theorem 152 (Tychonoff's theorem). Let I be an index set, $\{(X_i, \tau_{X_i}) : i \in I\}$ be a collection of topological spaces and $X := \prod_{i \in I} X_i$. Then, (X, τ_X) is compact if and only if (X_i, τ_{X_i}) is compact $\forall i \in I$.

Axiom 153 (Axiom of choice). The Cartesian product of a collection of non-empty sets is non-empty.

Theorem 154 (Kelley's theorem). Tychonoff's theorem implies the axiom of choice.

Alexandroff extension

Definition 155. Let (X, τ_X), (Y, τ_Y) be topological spaces and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. We say that f is a *topological embedding* if f yields a homeomorphism between (X, τ_X) and $f(X)$ together with the subspace topology inherited from (Y, τ_Y).

Definition 156. Let (X, τ) be topological space and $X^* := X \cup \{\infty\}$. We define the following set:

$$\tau^* := \{U \subseteq X^* : U \in \tau \vee (\infty \in U \wedge X^* \setminus U \text{ is compact})\}$$

Theorem 157 (One-point compactification). Let (X, τ) be Hausdorff topological space. Then, (X^*, τ^*) is a compact topological space, called *one-point compactification* of (X, τ).

Proposition 158. Let (X, τ) be Hausdorff topological space. Then, the inclusion $\iota : (X, \tau) \to (X^*, \tau^*)$ is a topological embedding.

Definition 159. Let (X, τ) be topological space and P be a property. We say that (X, τ) satisfies *locally* P if $\forall x \in X$ and $\forall U \in \tau$ such that $x \in U$, there exists a neighbourhood $N \in \mathcal{N}_x$, with $x \in N \subseteq U$, that satisfies P.

Definition 160. Let (X, τ) be topological space. We say that (X, τ) is *locally compact* if $\forall x \in X$ and $\forall U \in \tau$ such that $x \in U$, there exists a compact neighbourhood $N \in \mathcal{N}_x$ such that $x \in N \subseteq U$.

Proposition 161. The local compactness is a topological property.

Definition 162. Let (X, τ) be topological space. (X, τ) is locally compact Hausdorff if and only if (X^*, τ^*) is compact Hausdorff.

Proposition 163. We the usual topology, $\mathbb{Q} \subset \mathbb{R}$ is not locally compact.

Theorem 164. Let (X, τ) be a locally compact Hausdorff topological space. Then, (X, τ) is T_3.

Compactness of $\mathbb{R}^n$

Theorem 165 (Heine-Borel theorem). Let $a, b \in \mathbb{R}$ with $a < b$. Then, $[a, b] \subset \mathbb{R}$ is compact.

Theorem 166 (Heine-Borel theorem). Consider $\mathbb{R}^n$ with the usual topology and $A \subseteq \mathbb{R}^n$. Then, A is compact if and only if A is closed and bounded.

Lemma 167. Let $K \subset \mathbb{R}$ be a compact subset. Then, $\exists m, M \in K$ such that $m \leq k \leq M$ $\forall k \in K$.

Theorem 168 (Weierstraß' theorem). Let (X, τ) be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then, f attains a maximum and a minimum.

Proposition 169. S^n, T^2, $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{P}_n(\mathbb{R})$ are compact.

8. | Connectedness

Connectedness

Definition 170 (Connected space). Let (X, τ) be a topological space. We say that (X, τ) is *connected* if do not exist non-empty open sets $U, V \in \tau$ such that $X = U \sqcup V$. Otherwise, that is, if there are non-empty open sets $U, V \in \tau$ such that $X = U \sqcup V$, we say that (X, τ) is *disconnected*.

Proposition 171. Let (X, τ) be a topological space. The following statements are equivalent:

1. (X, τ) is connected.
2. There are no non-empty closed sets $C, D \subset X$ such that $X = C \sqcup D$.
3. There isn't a non-empty clopen set $U \subset X$.

Definition 172. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. We say that A is a *connected subset* of X if (A, τ_A) is a connected space.

Proposition 173. Let (X, τ) be a topological space and $A \subseteq X$ be a subset. A is connected if and only if there are no open sets $U, V \in \tau$ such that $A \subseteq U \cup V$, $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$ and $A \cap U \cap V = \varnothing$.

Proposition 174. The connectedness is a topological property.

Theorem 175. Let (X, τ_X), (Y, τ_Y) be a topological spaces such that (X, τ_X) is connected and $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Then, $f(X) \subset Y$ is connected.

Corollary 176. The quotient space of a connected space is connected.

Lemma 177. Let (X, τ) be a topological space and $C, D \subseteq X$ be subsets such that $C \subseteq D$ and C is connected. Suppose that D is disconnected and so that there exist non-empty open sets $U, V \in \tau$ such that $D \subseteq U \cup V$, $D \cap U \neq \varnothing$, $D \cap V \neq \varnothing$ and $D \cap U \cap V = \varnothing$. Then, either $C \subseteq U$ or $C \subseteq V$.

Proposition 178. Let (X, τ) be a topological space and $\{C_i : i \in I\}$ be a collection of connected subsets of X such that $\bigcap_{i \in I} C_i \neq \varnothing$. Then, $\bigcup_{i \in I} C_i$ is connected.

Theorem 179. Let (X_i, τ_{X_i}) be connected topological spaces for $i = 1, \ldots, n$. Then, $\prod_{i=1}^n (X_i, \tau_{X_i})$ is connected.

Theorem 180. Let (X, τ) be a topological space and $C \subseteq X$ be a connected subset. If $A \subseteq X$ is such that $C \subseteq A \subseteq \operatorname{Cl} C$, then A is connected.

Proposition 181. Let (X, τ) be a connected topological space and $A \subset X$ be a non-empty subset. Then, $\partial A \neq \varnothing$.

Proposition 182. Let (X, τ) be a topological space and $C \subset X$ be a connected non-empty subset. If $A \subsetneq X$ is such that $C \cap A \neq \varnothing$ and $C \cap (X \setminus A) \neq \varnothing$, then $C \cap \partial A \neq \varnothing$.

Definition 183. Let (X, τ) be a topological space with $|X| > 1$. We say that (X, τ) is *totally disconnected* if all subsets with cardinal greater than 1 are disconnected.

Proposition 184. Let (X, τ) be a topological space with $|X| > 1$. Then, (X, τ) is totally disconnected if and only if $\forall x, y \in X$, $x \neq y$, $\exists U, V \in \tau$ such that $x \in U$, $y \in V$ and $X = U \sqcup V$.

Proposition 185. Let (X, τ_{d}) be a topological space with $|X| > 1$. Then, (X, τ_{d}) is totally disconnected.

Proposition 186. $\mathbb{Q} \subseteq \mathbb{R}$ with the usual topology is totally disconnected.

Connectedness of $\mathbb{R}^n$

Theorem 187. Consider $\mathbb{R}$ together with the usual topology and let $a, b \in \mathbb{R}$. Then, $[a, b]$ is connected.

Theorem 188. Consider $\mathbb{R}$ together with the usual topology. Then, $\mathbb{R}$ is connected.

Theorem 189. Consider $\mathbb{R}$ together with the usual topology and let $A \subseteq \mathbb{R}$ be a subset. Then:

$$A \text{ is connected} \iff A \text{ is an interval}$$

Theorem 190 (Intermediate value theorem). Let (X, τ) be a connected topological space, $f : (X, \tau) \to \mathbb{R}$ be a continuous function. Let $p, q \in \operatorname{im} f$ and $r \in \mathbb{R}$ be such that $p \leq r \leq q$. Then, $r \in \operatorname{im} f$.

Corollary 191 (Bolzano's theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)f(b) \leq 0$. Then, $\exists r \in [a, b]$ such that $f(r) = 0$.

Theorem 192 (Brouwer's fixed-point theorem). Let $\overline{B}^n \subset \mathbb{R}$ be a closed n-th ball and $\mathbf{f} : \overline{B}^n \to \overline{B}^n$ be a continuous function. Then, $\mathbf{f}$ has a fixed point.

Theorem 193 (Borsuk-Ulam theorem). Let $\mathbf{f} : S^n \to \mathbb{R}^n$ be a continuous function. Then, $\exists x \in S^n$ such that $\mathbf{f}(x) = \mathbf{f}(-x)$.

Connected components

Definition 194. Let (X,τ) be a topological space. We define the relation $\sim$ in (X,τ) as $\forall x,y\in X$, $x\sim y$ if and only if there exists a connected subset $C\subseteq X$ such that $x,y\in C$.

Proposition 195. Let (X,τ) be a topological space. The relation $\sim$ is an equivalence relation.

Definition 196. Let (X,τ) be a topological space with the relation $\sim$. We define the *connected components* of (X,τ) as the equivalence classes under $\sim$.

Proposition 197. Let (X,τ) be a topological space with the relation $\sim$. Then:

1. Each connected component $C\subseteq X$ is connected. Moreover, if $p\in C$, C is the maximal connected subset that contains p.
2. The connected components are pairwise disjoint.
3. If $A\subseteq X$ is a connected subspace, then $A\subseteq C$ for some connected component C of (X,τ).
4. The connected components are closed.
5. If there is a finite number of connected components, then they are open.

Theorem 198. Let (X,τ_X), (Y,τ_Y) be topological spaces, $f:(X,\tau_X)\to(Y,\tau_Y)$ be a continuous function and $C\subseteq X$ be a connected component. Then, $f(C)\subseteq D$, where D is a connected component of (Y,τ_Y). Furthermore, if f is a homeomorphism, $f(C)$ is a connected component of (Y,τ_Y).

Corollary 199. Let (X,τ_X), (Y,τ_Y) be homeomorphic topological spaces. Then, they have the same number of connected components.

Definition 200. Let (X,τ) be a topological space. We say that (X,τ) is *locally connected* if $\forall x\in X$ and $\forall U\in\tau$ such that $x\in U$, there exists a connected neighbourhood $N\in\mathcal{N}_x$ such that $x\in N\subseteq U$.

Proposition 201. Let (X,τ) be a locally connected topological space. Then, the connected components are open.

Path connectedness

Definition 202. Let (X,τ) be a topological space. A *path* in (X,τ) is a continuous function $\gamma:[0,1]\to(X,\tau)$. $\gamma(0)$ is called *initial point* of the path and $\gamma(1)$, *terminal point*.

Definition 203. Let (X,τ) be a topological space and $x,y\in X$. A *path from x to y* is a path whose initial point is x and whose terminal point is y. If $x=y$, we say that the path is a *loop*.

Proposition 204. Let (X,τ) be a topological space and γ be a path in (X,τ) and $x,y\in X$. Then:

1. $\operatorname{im}(\gamma)$ is a connected subspace of (X,τ). Therefore, the initial and terminal points of γ are in the same connected component.
2. If $A\subseteq X$ is a subset satisfying $\gamma(0)\in A$ and $\gamma(1)\notin A$, then $\exists r\in[0,1]$ such that $\gamma(r)\in\partial A$.
3. If $\gamma(t)$ is a path from x to y, then $\gamma(1-t)$ is a path from y to x

Proposition 205. Let (X,τ) be a topological space $x,y,z\in X$, γ_1 be a path from x to y and γ_2 be a path from y to z. Then,

$$(\gamma_1+\gamma_2)(t):=\begin{cases}\gamma_1(2t) & \text{if } 0\leq t\leq 1/2\\ \gamma_2(2t-1) & \text{if } 1/2<t\leq 1\end{cases}$$

is a path from x to z.

Definition 206. Let (X,τ) be a topological space. We say that (X,τ) is *path-connected* if for all $x,y\in X$, there exists a path in (X,τ) from x to y.

Proposition 207. The path-connectedness is a topological property.

Proposition 208. Let (X,τ_X), (Y,τ_Y) be topological space such that (X,τ_X) is path-connected, and $f:(X,\tau_X)\to(Y,\tau_Y)$ be a continuous function. Then $f(X)\subseteq Y$ is path-connected.

Corollary 209. The quotient space of a path-connected space is path-connected.

Definition 210. Let (X,τ) be a topological space and $A\subseteq X$. We say that A is *path-connected* if A, together with the subspace topology, is path-connected.

Proposition 211. Let (X,τ) be a topological space and $\{C_i : i\in I\}$ be a collection of path-connected subsets of X such that $\bigcap_{i\in I}C_i\neq\varnothing$. Then, $\bigcup_{i\in I}C_i$ is path-connected.

Theorem 212. Let (X_i,τ_{X_i}) be path-connected topological spaces for $i=1,\ldots,n$. Then, $\prod_{i=1}^n(X_i,\tau_{X_i})$ is path-connected.

Theorem 213. Let (X,τ) be a path-connected topological space. Then, (X,τ) is connected.

Definition 214. Let (X,τ) be a topological space. We define the relation $\sim_{\mathrm{p}}$ in (X,τ) as $\forall x,y\in X$, $x\sim_{\mathrm{p}}y$ if and only if there exists a path from x to y.

Proposition 215. Let (X,τ) be a topological space. The relation $\sim_{\mathrm{p}}$ is an equivalence relation.

Definition 216. Let (X,τ) be a topological space with the relation $\sim_{\mathrm{p}}$. We define the *path-connected components* of (X,τ) as the equivalence classes under $\sim_{\mathrm{p}}$.

Proposition 217. Let (X,τ) be a topological space with the relation $\sim_{\mathrm{p}}$. Then:

1. Each path-connected component $C\subseteq X$ is path-connected. Moreover, if $p\in C$, C is the maximal path-connected subset that contains p.
2. The path-connected components are pairwise disjoint.

Definition 218. Let (X,τ) be a topological space. We say that (X,τ) is *locally path-connected* if $\forall x \in X$ and $\forall U \in \tau$ such that $x \in U$, there exists a path-connected neighbourhood $N \in \mathcal{N}_x$ such that $x \in N \subseteq U$.

Theorem 219. Let (X,τ) be a locally path-connected topological space. Then, the connected components and the path-connected components of (X,τ) are the same.

Proposition 220. $\mathbb{R}^n$, S^n, T^2, $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{P}_n(\mathbb{R})$ are connected and path-connected.

Definition 221. Let (X,τ) be a topological space. We say that (X,τ) is *simply connected* if every path between two points can be continuously transformed into any other such path while preserving the two endpoints in question[5].

9. | Topological manifolds

Topological manifolds

Definition 222 (Topological manifold). A *topological manifold* of dimension m, abbreviated as m-manifold, is a topological space (M,τ) such that:

1. (M,τ) is Hausdorff.
2. τ admits a countable basis.
3. $\forall x \in M\ \exists N \in \mathcal{N}_x$ such that $N \cong \mathbb{R}^n$.

Proposition 223. Let (M,τ) be a m-manifold. Then:

1. (M,τ) is locally connected.
2. (M,τ) is locally path-connected.
3. (M,τ) is locally compact.

Corollary 224. Let (M,τ) be a m-manifold, I be a finite or countable set and $\{M_i : i \in I\}$ be the connected components of (M,τ). Then, $M_i \subseteq M$ are clopen and $M \cong \bigsqcup_{i\in I} M_i$. Moreover for all $i \in I$, M_i is a manifold itself of dimension m. Finally, if I is finite and (M,τ) is compact, M_i are compact connected manifolds $\forall i \in I$.

Proposition 225. Being a topological manifold is a topological property.

Definition 226. Let (M,τ) be a m-manifold. A *coordinate chart* is a pair (U,φ), where $U \in \tau$ and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism. A collection $\{(U_i,\varphi_i) : i \in I\}$ of coordinate charts is called an *atlas* if $M = \bigcup_{i\in I} U_i$.

Definition 227. Let (M,τ) be a m-manifold and $\{(U_i,\varphi_i) : i \in I\}$ be an atlas. For all $i,j \in I$ such that $U_i \cap U_j \neq \varnothing$, we define the following homeomorphism:

$$\begin{aligned} \phi_{ij} : \varphi_j(U_i \cap U_j) &\longrightarrow U_i \cap U_j \longrightarrow \varphi_i(U_i \cap U_j) \\ x &\longmapsto {\varphi_j}^{-1}(x) \longmapsto \varphi_i\left({\varphi_j}^{-1}(x)\right) \end{aligned}$$

That is, $\phi_{ij} = \varphi_i \circ {\varphi_j}^{-1}|_{U_i\cap U_j}$. These functions are called *transition functions*.

- If ϕ_{ij} is a piecewise linear function $\forall i,j \in I$, we say that M is a *piecewise linear manifold*.
- If ϕ_{ij} is a differentiable function $\forall i,j \in I$, we say that M is a *differentiable manifold*.

Proposition 228. $\mathbb{R}^n$, S^n and $\mathcal{P}_n(\mathbb{R})$ are n-manifolds.

Proposition 229. T^2, $\mathcal{M}$ and $\mathcal{K}$ are compact 2-manifolds.

Proposition 230. Let (M,τ_M) be a m-manifold and (N,τ_N) be a n-manifold. Then, $(M\times N,\tau_{M\times N})$ is a $m+n$-manifold.

Definition 231 (Connected sum). Let (M_1,τ_{M_1}) and (M_2,τ_{M_2}) be two m-manifolds, $p_1 \in M_1$, $p_2 \in M_2$ and (U_1,φ_1), (U_2,φ_2) be coordinate charts such that $p_i \in U_i$ and $\varphi_i(p_i) = 0$ for $i = 1,2$. Let $\varepsilon > 0$ be such that $B(0,2\varepsilon) \subseteq \varphi(U_1) \cap \varphi(U_2)$ and $R := B(0,2\varepsilon) \setminus \mathrm{Cl}(B(0,\varepsilon))$. Now consider the following homeomorphism:

$$\begin{aligned} \psi : \quad R \quad &\longrightarrow \quad R \\ (x_1,\dots,x_n) &\longmapsto \frac{2\varepsilon^2}{{x_1}^2+\cdots+{x_n}^2}(x_1,\dots,x_n) \end{aligned}$$

Then, note that for $i = 1,2$, ${M_i}' := M_i \setminus \mathrm{Cl}(\psi^{-1}(B(0,\varepsilon)))$ is a m-manifold. We define the *connected sum* of (M_1,τ_{M_1}) and (M_2,τ_{M_2}) as:

$$M_1 \# M_2 := {M_1}' \sqcup {M_2}' \big/ \sim$$

where $x \sim (\psi^{-1} \circ \phi_{21} \circ \psi)(x)\ \forall x \in \psi^{-1}(B(0,\varepsilon))$[6].

Orientability

Definition 232. Let V be a vector space and $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases of V. The bases $\mathcal{B}_1$ and $\mathcal{B}_2$ have the *same orientation* if $\det\left([\mathrm{id}]_{\mathcal{B}_1,\mathcal{B}_2}\right) > 0$. Otherwise, we say that they have *opposite orientations*. Note that the property of having the same orientation defines an equivalence relation on the set of all bases for V.

Definition 233. An *orientation* on a vector space is an assignment of $+1$ to one equivalence class and -1 to the other. A vector space with an orientation selected is called an *oriented vector space*, while one not having an orientation selected, is called an *unoriented vector space*.

Definition 234. Let V, W be oriented vector spaces and $f : V \to W$ be a linear isomorphism. We say that f is *orientation-preserving* if $\det\left([f]_{\mathcal{B}_1,\mathcal{B}_2}\right) > 0$ for some bases $\mathcal{B}_1$ of V and $\mathcal{B}_2$ of W according to the orientation chosen. Analogously, if $\det\left([f]_{\mathcal{B}_1,\mathcal{B}_2}\right) < 0$ we say that f is *not orientation-preserving*.

Definition 235. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable homeomorphism. We say that $\mathbf{f}$ is *orientation-preserving* if $\det\left(\mathbf{Df}(x)\right) > 0\ \forall x \in \mathbb{R}^n$. Otherwise, we say that f is *not orientation-preserving*.

[5] Roughly speaking this definition says that a sumply connected topological space doesn't have *holes*.

[6] Roughly speaking, a connected sum of two m-manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

Definition 236. We say that a manifold (M, τ) is *orientable* if it admits an atlas such that all the transition functions are orientation-preserving.

Proposition 237. Let (M, τ_M) and (N, τ_N) be orientable manifolds. Then, $(M \times N, \tau_{M\times N})$ is orientable.

Proposition 238. $\mathbb{R}^n$, S^n and T^2 are orientable, but $\mathcal{P}_n(\mathbb{R})$, $\mathcal{M}$ and $\mathcal{K}$ are not.

1-manifolds

Theorem 239 (Classification of connected 1-manifolds). Let (M, τ) be a connected 1-manifold. Then, M is homeomorphic to exactly one of the following manifolds:

- $\mathbb{R}$
- S^1

10. | Compact surfaces

Connected sum of surfaces

Definition 240. Let (M, τ) be a m-manifold. We say that (M, τ) is a *surface* if $m = 2$.

Proposition 241 (Connected sum of surfaces). Let (S_1, τ_{S_1}) and (S_2, τ_{S_2}) be two surfaces, $p_1 \in M_1$, $p_2 \in M_2$ and (U_1, φ_1), (U_2, φ_2) be coordinate charts such that $p_i \in U_i$ and $\varphi_i(p_i) = 0$ for $i = 1, 2$. Let $D_i := {\varphi_i}^{-1}(B(0, 1))$ for $i = 1, 2$. Then, note that for $i = 1, 2$, ${S_i}' := S_i \setminus D_i$ is a surface and $\partial {S_i}' = {\varphi_i}^{-1}(\partial B(0, 1)) \cong S^1$. Then, the connected sum of (S_1, τ_{S_1}) and (S_2, τ_{S_2}) is:

$$S_1 \# S_2 = {S_1}' \sqcup {S_2}' \big/ \partial {S_1}' \sim \partial {S_2}'$$

Proposition 242. Let (S_1, τ_{S_1}), (S_2, τ_{S_2}), (S_3, τ_{S_3}) be compact connected surfaces. Then:

1. $S_1 \# S_2 \cong S_2 \# S_1$
2. $(S_1 \# S_2) \# S_3 \cong S_1 \# (S_2 \# S_3)$
3. $S_1 \# S^2 \cong S_1$
4. $S_1 \# S_2$ is orientable $\iff$ S_1 and S_2 are both orientable.

Proposition 243. Let (M, τ) be a compact connected surface. Then:

1. $M \# T^2$ is a handle attached to M.
2. $M \# \mathcal{P}_n(\mathbb{R})$ is attaching a Möbius band to M.

Definition 244. Let $g \in \mathbb{N} \cup \{0\}$. We define the *genus g orientable surface* as:

$$S_g := S^2 \# T^2 \# \overset{(g)}{\cdots} \# T^2$$

Definition 245. Let $h \in \mathbb{N}$. We define the *genus h non-orientable surface* as:

$$N_h := \mathcal{P}_2(\mathbb{R}) \# \overset{(h)}{\cdots} \# \mathcal{P}_2(\mathbb{R})$$

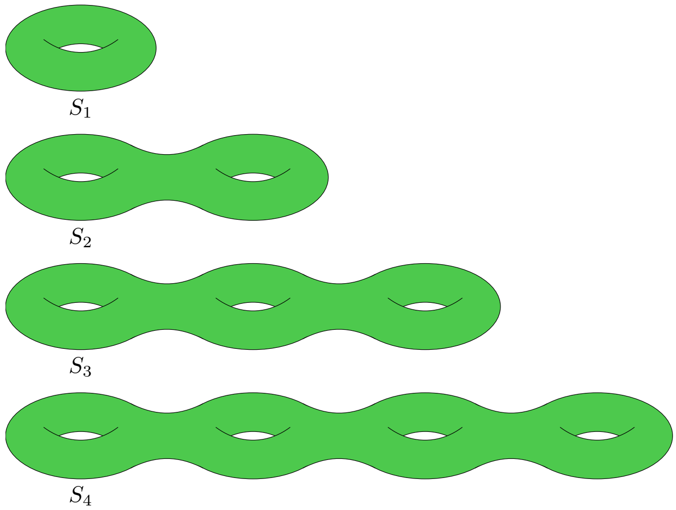

Figure 6: Genus g orientable surfaces

Triangularization

Definition 246. The *standard n-simplex* is the set

$$\Delta^n := \{(x_0, \dots, x_n) \in {\mathbb{R}_{\geq 0}}^{n+1} : x_0 + \cdots + x_n = 1\}$$

Definition 247. Let (S, τ) be a compact connected surface. A *triangularization* of S is a finite collection $\{T_1, \dots, T_n\}$ such that $S = \bigcup_{i=1}^n T_i$, $T_i \cong \Delta^2$ for $i = 1, \dots, n$ and if $T_i \cap T_j \neq \varnothing$ for $i \neq j$, then $T_i \cap T_j$ is an edge of D_i and D_j or a vertex of D_i and D_j.

Definition 248. A *simple curve* is a non-self-intersecting continuous loop in the plane.

Theorem 249 (Jordan curve theorem). All simple curves divides the plane in 2 connected components. One of these components is bounded and the other is unbounded.

Theorem 250 (Radó theorem). Each compact surfaces has a triangularization.

Theorem 251. Every compact surface can be constructed from a polygon with an even number of sides, called a *fundamental polygon of the surface*, by pairwise identification of its edges[7]. Reciprocally, every polygon whose edges are pairwise identified produces a surface.

Proposition 252. We have the following representations[8] of the most common surfaces [9]:

- S^2: $aa^{-1} =: 1$
- T^2: $aba^{-1}b^{-1}$
- $\mathcal{P}_2(\mathbb{R})$: aa
- $\mathcal{K}$: $aba^{-1}b$

[7] Any fundamental polygon can be written symbolically as follows. Begin at any vertex, and proceed around the perimeter of the polygon in either direction until returning to the starting vertex. During this traversal, record the label on each edge in order, with an exponent of -1 if the edge points opposite to the direction of traversal.

[8] Note that these representations are not unique.

[9] See Figs. 2 and 5 for a better understanding.

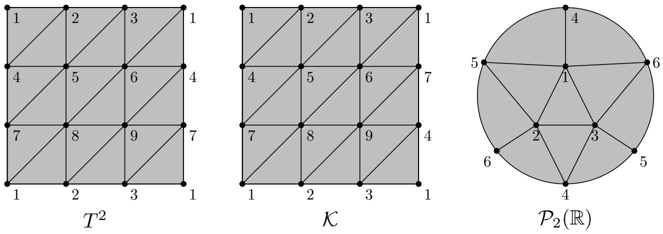

Figure 7: A triangularization of the torus T^2, the Klein bottle $\mathcal{K}$ and the projective plane $\mathcal{P}_2(\mathbb{R})$

Definition 253. We say that a representation of a surface is *normalized* if it is of one of the following forms:

$$a_1 b_1 {a_1}^{-1} {b_1}^{-1} \cdots a_n b_n {a_n}^{-1} {b_n}^{-1}$$
$$a_1 b_1 {a_1}^{-1} {b_1}^{-1} \cdots a_r b_r {a_r}^{-1} {b_r}^{-1} c_1 c_1 \cdots c_s c_s$$

Proposition 254. Let (S_1, τ_{S_1}) and (S_2, τ_{S_2}) be two surfaces whose respect representations are:

$$a_1 \cdots a_n \quad b_1 \cdots b_m$$

Then, $S_1 \# S_2$ is represented by

$$a_1 \cdots a_n b_1 \cdots b_m$$

Proposition 255. On the representations of polygons, we have:

- $aba^{-1}b^{-1} \equiv cdc^{-1}d^{-1}$
- $aba^{-1}b^{-1} \equiv a_1 a_2 b {a_2}^{-1} {a_1}^{-1} b^{-1}$
- $aba^{-1}b^{-1} \equiv ba^{-1}b^{-1}a \equiv a^{-1}b^{-1}ab \equiv b^{-1}aba^{-1}$
- $abab^{-1} \equiv abc, ab^{-1}e^{-1}$
- $abab^{-1} \equiv abcc^{-1}ab^{-1}$
- $abab^{-1} \equiv a^{-1}b^{-1}a^{-1}b$

Corollary 256. Let $g \in \mathbb{N} \cup \{0\}$ and $h \in \mathbb{N}$. Then, the representations of S_g and N_h are:

- S_g: $a_1 b_1 {a_1}^{-1} {b_1}^{-1} \cdots a_g b_g {a_g}^{-1} {b_g}^{-1}$ (a polygon with $4g$ sides)
- N_h: $a_1 a_1 \cdots a_h a_h$ (a polygon with $2n$ sides)

Proposition 257.

- $\mathcal{K} \cong \mathcal{P}_2(\mathbb{R}) \# \mathcal{P}_2(\mathbb{R})$
- $T^2 \# \mathcal{P}_2(\mathbb{R}) \cong \mathcal{K} \# \mathcal{P}_2(\mathbb{R}) \cong \mathcal{P}_2(\mathbb{R}) \# \mathcal{P}_2(\mathbb{R}) \# \mathcal{P}_2(\mathbb{R})$

Corollary 258. Let $g \in \mathbb{N} \cup \{0\}$ and $h \in \mathbb{N}$. Then:

$$S_g \# N_h \cong N_{h+2g}$$

Proposition 259. Let (S, τ) be a compact connected surface. Then, S is non-orientable if and only if it contains a Möbius strip, which can be identified by a representation of S of the form

$$aWaW'$$

where W and W' may contain more than one edge.

Euler characteristic

Definition 260. Let (S, τ) be a compact connected surface and $T = \{T_i : i = 1, \ldots, n\}$ be a triangularization of S. We define the *Euler characteristic* of S with the triangularization T as:

$$\chi_T(S) := V - E + F$$

where V, E, and F are respectively the numbers of vertices, edges and faces in the polygonal decomposition obtained from T taking into account the identifications between vertices and edges[10].

Proposition 261. Let (S, τ) be a compact connected surface and T, T' be triangularizations of S. Then:

$$\chi_T(S) = \chi_{T'}(S)$$

Therefore, from now on we will denote the Euler characteristic of S as $\chi(S)$.

Proposition 262. We have the following Euler characteristics of the most common surfaces:

- $\chi(S^2) = 2$
- $\chi(T^2) = 0$
- $\chi(\mathcal{P}_2(\mathbb{R})) = 1$
- $\chi(\mathcal{K}) = 0$

Proposition 263. Let (S_1, τ_{S_1}) and (S_2, τ_{S_2}) be two surfaces. Then:

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

Corollary 264. Let $g \in \mathbb{N} \cup \{0\}$ and $h \in \mathbb{N}$. Then:

- $\chi(S_g) = 2 - 2g$
- $\chi(N_h) = 2 - h$

Theorem 265. The Euler characteristic is a topological property.

Theorem 266 (Classification of compact connected surfaces). Every compact connected surface is homeomorphic to exactly one of the following surfaces:

- S_g for some $g \in \mathbb{N} \cup \{0\}$.
- N_h for some $h \in \mathbb{N}$.

Corollary 267. Two compact connected surfaces are homeomorphic if and only if they have the same orientability and the same Euler characteristic.

[10] That is, two identified vertices (or edges) count as one.