

Differential geometry

1. | Differentiable curves

Inner product of $\mathbb{R}^n$

Proposition 1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors and $\langle \mathbf{u}, \mathbf{v} \rangle$ be the usual inner product between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Then:

- Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

- Triangular inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Polarization identity:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2} (\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2)$$

Definition 2. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors. We define the angle between $\mathbf{u}$ and $\mathbf{v}$ as the unique value $\theta \in [0, \pi]$ such that:

$$\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Parametrized curves

Definition 3. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be a differentiable function. We say that $\mathbf{f}$ is a *local diffeomorphism* if $\forall p \in U$, there exists a neighbourhood $V \subseteq U$ of p such that $\mathbf{f}|_V : V \rightarrow \mathbf{f}(V)$ is a diffeomorphism.

Proposition 4. Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \rightarrow \mathbb{R}$ be a differentiable function. If $f'(x) \neq 0 \forall x \in I$, then $f(I)$ is an open set and f is a diffeomorphism.

Proposition 5. Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha, \beta : I \rightarrow \mathbb{R}^n$ be differentiable functions. Then:

1. $\langle \alpha(t), \beta(t) \rangle' = \langle \alpha'(t), \beta(t) \rangle + \langle \alpha(t), \beta'(t) \rangle$
2. If $t \mapsto \|\alpha(t)\|$ is a constant function, then $\alpha \perp \alpha'$.

Definition 6. Let $I \subseteq \mathbb{R}$ be an open interval and $C \subset \mathbb{R}^n$ be a curve. A *parametrization* of C of class $\mathcal{C}^k$ is a function $\alpha : I \rightarrow \mathbb{R}^n$ of class $\mathcal{C}^k$ such that $\alpha(I) = C$. The image of α , C , is called the *trace* of α , and it is sometimes denoted by $\alpha^* := \text{im}(\alpha)$ ¹.

Definition 7. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^n$ be a curve and $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization of C of class $\mathcal{C}^1$. We define the *tangent vector* of α at $t_0 \in \mathbb{R}$ as $\alpha'(t_0)$. We say that α is *regular* if $\alpha'(t) \neq 0 \forall t \in I$. In that last case, we define the *tangent line* of α at $\alpha(t_0)$ as the following parametrized line in $\mathbb{R}^n$:

$$s \mapsto \alpha(t_0) + s\alpha'(t_0)$$

Definition 8. Let $C \subset \mathbb{R}^n$ be a curve. We say that C is a *plane curve* if it is contained in a plane of $\mathbb{R}^n$.

Definition 9. Let $I, J \subseteq \mathbb{R}$ be open intervals, $C \subset \mathbb{R}^n$ be a curve, $\alpha : I \rightarrow \mathbb{R}^n$ be a regular parametrization of C of class $\mathcal{C}^1$ and $h : J \rightarrow I$ be a diffeomorphism. Then, $\beta := \alpha \circ h : J \rightarrow \mathbb{R}^n$ is a regular parametrization of C satisfying:

$$\beta'(s) = \alpha'(h(s))h'(s) \quad \forall s \in J$$

It is said that β is a *reparametrization* of α and h is a *change of parameter*. Moreover, the reparametrization is *positive* if $h'(s) > 0 \forall s \in J$, and it is *negative* if $h'(s) < 0 \forall s \in J$.

Length of curves

Definition 10. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^n$ be a curve, $\alpha : I \rightarrow \mathbb{R}^n$ be a continuous parametrization of C , $[a, b] \subset I$, $\mathfrak{P}([a, b])$ be the set of all partitions of $[a, b]$ and $\mathcal{P} = \{t_0, \dots, t_n\} \in \mathfrak{P}$. We define the *length of the polygonal* with vertices at $\alpha(t_i)$, $i = 1, \dots, n$ as:

$$L_{a,b}(\alpha, \mathcal{P}) = \sum_{i=1}^n \|\alpha(t_i) - \alpha(t_{i-1})\|$$

We define $L_{a,b}(\alpha)$ as:

$$L_{a,b}(\alpha) := \sup\{L_{a,b}(\alpha, \mathcal{P}) : \mathcal{P} \in \mathfrak{P}([a, b])\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

If $L_{a,b}(\alpha) < +\infty$, we say that C is *rectifiable* and that $L_{a,b}(\alpha)$ is its *length* in $[a, b]$.

Proposition 11. Let $I \subseteq \mathbb{R}$ be an open interval, $[a, b] \subset I$, $C \subset \mathbb{R}^n$ be a curve and $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization of C of class $\mathcal{C}^1$. Then, C is rectifiable and:

$$L_{a,b}(\alpha) = \int_a^b \|\alpha'(t)\| dt$$

Proposition 12. Let $I, J \subseteq \mathbb{R}$ be open intervals, $[a, b] \subset I$, $C \subset \mathbb{R}^n$ be a curve, $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization of C of class $\mathcal{C}^1$, $h : J \rightarrow I$ be a diffeomorphism, $\beta = \alpha \circ h$ be a reparametrization of α . Suppose $[c, d] = h^{-1}([a, b])$. Then:

$$\int_c^d \|\beta'(u)\| du = \int_a^b \|\alpha'(t)\| dt$$

That is, the length of a curve does not depend on its parametrization.

Definition 13. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^n$ be a curve and $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization of C of class $\mathcal{C}^1$. We say that α is a *unit-speed parametrization* (or that it is parametrized by *arc-length parameter*) if $\|\alpha'(t)\| = 1 \forall t \in I$.

¹Sometimes α is referred to the curve as well as to the parametrization of it.

Definition 14. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^n$ be a curve, $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization C of class $\mathcal{C}^1$ and $t_0 \in I$. We define the *arc-length function* of α with origin t_0 , the function $s_{t_0} : I \rightarrow \mathbb{R}$ defined as:

$$s_{t_0}(t) = \int_{t_0}^t \|\alpha'(u)\| du$$

Proposition 15. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^n$ be a curve and $\alpha : I \rightarrow \mathbb{R}^n$ be a parametrization C of class $\mathcal{C}^1$ and $t_0 \in I$. Then:

1. s_{t_0} is of class $\mathcal{C}^1$ and $\frac{ds_{t_0}}{dt}(t_0) = \|\alpha'(t_0)\| \geq 0$.
2. If α is regular, then $J := s_{t_0}(I) \subseteq \mathbb{R}$ is an open interval and $s_{t_0} : I \rightarrow J$ is a diffeomorphism.
3. If α is regular, then $\beta(s_{t_0}) := \alpha(t(s_{t_0}))^2$ is an arc-length reparametrization of α .

Proposition 16. Let $I, J \subseteq \mathbb{R}$ be open intervals, $C \subset \mathbb{R}^n$ be a curve, $\alpha : I \rightarrow \mathbb{R}^n$ be a regular parametrization of C of class $\mathcal{C}^1$, $h : J \rightarrow I$ be a diffeomorphism and $\beta = \alpha \circ h$ be a reparametrization of α . If α and β are arc-length parametrizations, then:

$$\beta(u) = \alpha(\pm u + u_0)$$

for some $u_0 \in \mathbb{R}$.

Proposition 17. All regular parametrization of curves of class $\mathcal{C}^1$ can be arc-length parametrized.

Orientability and cross product

Definition 18. Let V be a vector space and $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases of V . We say that $\mathcal{B}_1 \sim \mathcal{B}_2$ if $\det([\text{id}]_{\mathcal{B}_1, \mathcal{B}_2}) > 0$. This relation is an equivalence relation on the set of all bases of V which has exactly two connected components.

Definition 19. Let V be a vector space and $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases of V . We say that $\mathcal{B}_1 \sim \mathcal{B}_2$ have the *same orientation* if $\det([\text{id}]_{\mathcal{B}_1, \mathcal{B}_2}) > 0$. Otherwise, we say that they have *opposite orientations*. Note that the property of having the same orientation defines an equivalence relation on the set of all bases for V .

Definition 20. An *orientation* on a vector space is the choice of one of the two equivalence classes under $\sim$. A vector space with an orientation selected is called an *oriented vector space*, while one not having an orientation selected, is called an *unoriented vector space*. A basis of an oriented vector space which has the orientation chosen is called *positive basis*, while one with the other orientation is called *negative basis*.

Definition 21. Let V be an oriented vector space, $\mathcal{B}$ be a basis of V and $f : V \rightarrow V$ be a linear isomorphism. We say that f is *orientation-preserving* (or *positively oriented*) if $\det([f]_{\mathcal{B}}) > 0$. Analogously, if $\det([f]_{\mathcal{B}}) < 0$ we say that f is *negatively-oriented*.

²Here, $t(s_{t_0})$ represent the inverse function of $s_{t_0}(t)$.

³From now on, if we do not explicitly fix a basis it will mean that the standard basis of $\mathbb{R}^n$ is the chosen one.

⁴Another commonly used notation for the cross product is $\mathbf{u} \wedge \mathbf{v}$.

Definition 22. Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a basis of $\mathbb{R}^n$. Suppose for each $i \in \{1, \dots, n\}$ we have

$$\mathbf{v}_i = \sum_{j=1}^n \lambda_{ij} \mathbf{e}_j$$

where $\lambda_{ij} \in \mathbb{R}$ and $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the standard basis of $\mathbb{R}^n$. We define the *determinant* of $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ as:

$$\det(\mathbf{v}_1, \dots, \mathbf{v}_n) := \begin{vmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{vmatrix}^3$$

Proposition 23. Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a basis of $\mathbb{R}^n$ and $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$. Then:

$$\det(\mathbf{A}\mathbf{v}_1, \dots, \mathbf{A}\mathbf{v}_n) = \det \mathbf{A} \det(\mathbf{v}_1, \dots, \mathbf{v}_n)$$

Proposition 24. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors of $\mathbb{R}^n$ and P be the parallelepiped they generate. Then:

$$\text{vol } P = |\det(\mathbf{v}_1, \dots, \mathbf{v}_n)|$$

Definition 25. Let $\mathbf{u}, \mathbf{v}$ be vectors of $\mathbb{R}^3$. We define the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$ ⁴, as the unique vector $\mathbf{w}$ satisfying:

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

Proposition 26. Let $\mathbf{u}, \mathbf{v}$ be vectors of $\mathbb{R}^3$ such that $\mathbf{u} = \sum_{i=1}^3 u_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^3 v_i \mathbf{e}_i$. Then:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Proposition 27. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors of $\mathbb{R}^3$. Then:

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $\mathbf{u} \times \mathbf{v} = 0 \iff \mathbf{u} = \lambda \mathbf{v}$, for some $\lambda \in \mathbb{R}$.
3. $\mathbf{u} \times \mathbf{v} \in \langle \mathbf{u}, \mathbf{v} \rangle^\perp$
4. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$ is a positive basis of $\mathbb{R}^3$.
5. If $\mathbf{x}, \mathbf{y}$ are vectors of $\mathbb{R}^3$, then:

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{x} \times \mathbf{y} \rangle = \begin{vmatrix} \mathbf{u} \times \mathbf{x} & \mathbf{v} \times \mathbf{x} \\ \mathbf{u} \times \mathbf{y} & \mathbf{v} \times \mathbf{y} \end{vmatrix}$$

6. Let $\theta \in [0, \pi]$ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

7. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{w} \rangle \mathbf{u}$

8. *Jacobi identity:*

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$$

Proposition 28. Let $\alpha, \beta : I \rightarrow \mathbb{R}^3$ be parametrized curves of class $\mathcal{C}^\infty$. Then:

$$(\alpha(t) \times \beta(t))' = \alpha'(t) \times \beta(t) + \alpha(t) \times \beta'(t)$$

Frenet-Serret formulas

Definition 29. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^2$. We define the unit tangent vector of α at $s_0 \in I$ as:

$$\mathbf{T}_\alpha(s_0) := \alpha'(s_0)$$

Note that $\|\mathbf{T}_\alpha\| = 1$ and $\mathbf{T}_\alpha \perp \mathbf{T}_\alpha'$.

Definition 30. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^2$. We define the *curvature* of α at $s_0 \in I$ as:

$$k_\alpha(s_0) := \|\alpha''(s_0)\| = \|\mathbf{T}_\alpha'(s_0)\|$$

Definition 31. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^2$, $s_0 \in I$ and suppose that $k_\alpha(s_0) \neq 0$. We define the *unit normal vector* of α at s_0 as:

$$\mathbf{N}_\alpha(s_0) := \frac{\mathbf{T}_\alpha'(s_0)}{k_\alpha(s_0)} = \frac{\alpha''(s_0)}{\|\alpha''(s_0)\|}$$

Note that $\|\mathbf{N}_\alpha\| = 1$, $\mathbf{N}_\alpha \perp \mathbf{T}_\alpha$ and $\mathbf{T}_\alpha'(s) = k_\alpha(s)\mathbf{N}_\alpha(s) \forall s \in I$.

Definition 32. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular arc-length parametrization of C of class $\mathcal{C}^2$ such that $\alpha''(s) \neq 0 \forall s \in I$. We define the *binormal vector* of α at $s_0 \in I$ as:

$$\mathbf{B}_\alpha(s_0) = \mathbf{T}_\alpha(s_0) \times \mathbf{N}_\alpha(s_0)$$

Then, the triplet $(\mathbf{T}_\alpha(s_0), \mathbf{N}_\alpha(s_0), \mathbf{B}_\alpha(s_0))$ is an orthonormal positive basis⁵, and the affine frame $\{\alpha(s_0); (\mathbf{T}_\alpha(s_0), \mathbf{N}_\alpha(s_0), \mathbf{B}_\alpha(s_0))\}$ is called *Frenet-Serret frame* (or *TNB frame*).

Proposition 33. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ such that $\alpha''(s) \neq 0 \forall s \in I$. Then:

$$\mathbf{B}_\alpha'(s) = \tau_\alpha(s)\mathbf{N}_\alpha(s) \quad \forall s \in I$$

This coefficient $\tau_\alpha(s)$ is called *torsion* of α at $s \in I$.

Proposition 34. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ such that $\alpha''(s) \neq 0 \forall s \in I$. The following statements are equivalent:

1. α is a plane curve.
2. $\mathbf{B}_\alpha = \text{const.}$
3. $\tau_\alpha = 0$.

Theorem 35 (Frenet-Serret formulas). Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ such that $\alpha''(s) \neq 0 \forall s \in I$. Then⁶:

$$\begin{pmatrix} \mathbf{T}_\alpha \\ \mathbf{N}_\alpha \\ \mathbf{B}_\alpha \end{pmatrix}' = \begin{pmatrix} 0 & k_\alpha & 0 \\ -k_\alpha & 0 & -\tau_\alpha \\ 0 & \tau_\alpha & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_\alpha \\ \mathbf{N}_\alpha \\ \mathbf{B}_\alpha \end{pmatrix}$$

⁵Observe that the binormal vector (together with the tangent and normal vectors) is the unique vector that satisfies this property.

⁶Note that an inversion of the orientation of α would change the sign of $\mathbf{T}_\alpha$ and $\mathbf{B}_\alpha$, but it would preserve the sign of $\mathbf{N}_\alpha$, k_α and τ_α .

Definition 36. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ such that $\alpha''(s) \neq 0 \forall s \in I$ and $s_0 \in I$. We define the following planes of $\mathbb{R}^3$:

- *Osculating plane*: plane generated by $\mathbf{T}_\alpha(s_0)$ and $\mathbf{N}_\alpha(s_0)$ that contains $\alpha(s_0)$.
- *Normal plane*: plane generated by $\mathbf{N}_\alpha(s_0)$ and $\mathbf{B}_\alpha(s_0)$ that contains $\alpha(s_0)$.
- *Rectifying plane*: plane generated by $\mathbf{T}_\alpha(s_0)$ and $\mathbf{B}_\alpha(s_0)$ that contains $\alpha(s_0)$.

Proposition 37. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be a regular parametrization of C of class $\mathcal{C}^3$ and $h(t) = s(t)$ be the arc-length parameter. Let $\beta = (\alpha \circ h^{-1})(s)$, which is an arc-length parametrization of C . Then, assuming $\beta'' \neq 0$, we can define the TNB frame of α as:

$$\mathbf{T}_\alpha := \mathbf{T}_\beta \circ h \quad \mathbf{N}_\alpha := \mathbf{N}_\beta \circ h \quad \mathbf{B}_\alpha := \mathbf{B}_\beta \circ h$$

And the curvature and torsion of α as:

$$k_\alpha := k_\beta \circ h \quad \tau_\alpha := \tau_\beta \circ h$$

Lemma 38. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be a regular parametrization of C of class $\mathcal{C}^3$ and $h(t) = s(t)$ be the arc-length parameter. Let $\beta = (\alpha \circ h^{-1})(s)$. Then, $\beta'' = 0 \iff \alpha' \times \alpha'' = 0$.

Lemma 39. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be a regular parametrization of C of class $\mathcal{C}^3$ such that $\alpha' \times \alpha'' \neq 0$ and $v(t) := \|\alpha'(t)\|$. Then:

- $\alpha' = v\mathbf{T}_\alpha$
- $\alpha'' = v'\mathbf{T}_\alpha + k_\alpha v^2\mathbf{N}_\alpha$

Theorem 40 (General Frenet-Serret formulas). Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be a regular parametrization of C of class $\mathcal{C}^3$ such that $\alpha' \times \alpha'' \neq 0$ and $v(t) := \|\alpha'(t)\|$. Then:

$$\begin{pmatrix} \mathbf{T}_\alpha \\ \mathbf{N}_\alpha \\ \mathbf{B}_\alpha \end{pmatrix}' = \begin{pmatrix} 0 & k_\alpha v & 0 \\ -k_\alpha v & 0 & -\tau_\alpha v \\ 0 & \tau_\alpha v & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_\alpha \\ \mathbf{N}_\alpha \\ \mathbf{B}_\alpha \end{pmatrix}$$

Corollary 41. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be a regular parametrization of C of class $\mathcal{C}^3$ such that $\alpha' \times \alpha'' \neq 0$. Then:

$$\mathbf{T}_\alpha = \frac{\alpha'}{\|\alpha'\|} \quad \mathbf{N}_\alpha = \mathbf{B}_\alpha \times \mathbf{T}_\alpha \quad \mathbf{B}_\alpha = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

Moreover:

$$k_\alpha = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \quad \tau_\alpha = -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2}$$

Contact between curves and surfaces

Definition 42. Let $I \subseteq \mathbb{R}$ be an open interval, $\alpha, \beta : I \rightarrow \mathbb{R}^n$ be arc-length parametrizations of two curves of class $\mathcal{C}^\infty$ and $s_0 \in I$. We say that α and β have *contact* of order $\geq r$ at s_0 if

$$\lim_{s \rightarrow s_0} \frac{\alpha(s) - \beta(s)}{(s - s_0)^r} = \mathbf{0}$$

We say that α and β have *contact* of order r at s_0 if they have contact of order $\geq r$ but not contact of order $\geq r+1$.

Proposition 43. Let $I \subseteq \mathbb{R}$ be an open interval, $\alpha, \beta : I \rightarrow \mathbb{R}^n$ be arc-length parametrizations of two curves of class $\mathcal{C}^\infty$ and $s_0 \in I$. Then, α and β have contact of order $\geq r$ at s_0 if and only if:

$$\alpha^{(k)}(s_0) = \beta^{(k)}(s_0) \quad \text{for } k = 0, \dots, r$$

Proposition 44. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^2$ and $s_0 \in I$. Then, the *tangent line* at $\alpha(s_0)$ is the unique line that has contact of order ≥ 1 with α at this point. An arc-length parametrization of the tangent line is:

$$u \mapsto \alpha(s_0) + u\mathbf{T}_\alpha(s_0)$$

Proposition 45. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^2$, $s_0 \in I$ and suppose that $k_\alpha(s_0) \neq 0$. Then, there exists a unique circle of $\mathbb{R}^3$ that has contact of order ≥ 2 at $\alpha(s_0)$. This circle is called *osculating circle* and its radius (called *radius of curvature*) is $\rho_\alpha(s_0) := \frac{1}{k_\alpha(s_0)}$. Its center is $c(s_0) = \alpha(s_0) + \rho_\alpha(s_0)\mathbf{N}_\alpha(s_0)$ ⁷.

Proposition 46. Let $\alpha : I \rightarrow \mathbb{R}^2$ be a regular parametrization of C of class $\mathcal{C}^3$, $s_0 \in I$ and suppose that $k_\alpha(s_0) \neq 0$. If $\alpha(t) = (x(t), y(t))$, then the center of the osculating circle at $\alpha(s_0)$ has coordinates (X, Y) given by:

$$X = x + y' \frac{x'^2 + y'^2}{x''y' - x'y''} \quad Y = y - x' \frac{x'^2 + y'^2}{x''y' - x'y''}$$

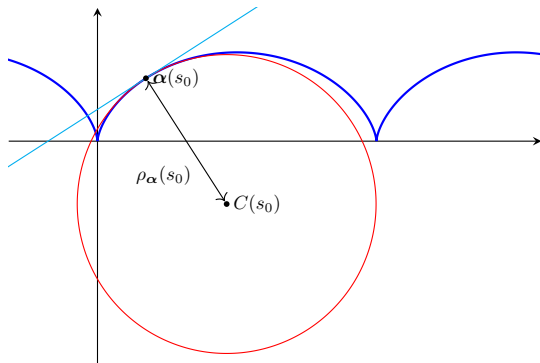


Figure 1: Osculating circle of a cycloid at a certain point

⁷An arc-length parametrization of the osculating circle is, for example:

$$u \mapsto c(s_0) + \rho_\alpha(s_0) \left(-\cos\left(\frac{u}{\rho_\alpha(s_0)}\right) \mathbf{N}_\alpha(s_0) + \sin\left(\frac{u}{\rho_\alpha(s_0)}\right) \mathbf{T}_\alpha(s_0) \right)$$

Definition 47. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^3$, $s_0 \in I$ and suppose that $k_\alpha(s_0) \neq 0$. Then, there exists a unique sphere of $\mathbb{R}^3$ that has contact of order ≥ 3 at $\alpha(s_0)$. This sphere is called *osculating sphere* of α at $\alpha(s_0)$ and its center $c(s_0)$ and radius $r(s_0)$ are given by:

$$c(s_0) = \alpha(s_0) + \rho_\alpha(s_0)\mathbf{N}_\alpha(s_0) - \frac{\rho_\alpha'(s_0)}{\tau_\alpha'(s_0)}\mathbf{B}_\alpha(s_0)$$

$$r(s_0)^2 = \rho_\alpha(s_0)^2 + \left(\frac{\rho_\alpha'(s_0)}{\tau_\alpha'(s_0)} \right)^2$$

Envelopes: evolute and involute

Definition 48. An *envelope* of a family of plane curves is a curve that is tangent to each of the members of the family at some point.

Definition 49. Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha, \beta : I \rightarrow \mathbb{R}^2$ be regular parametrizations of two curves of class $\mathcal{C}^3$ such that $k_\alpha(s), \tau_\alpha(s) \neq 0 \forall s \in I$. We say that β is the *evolute* of α if β is the envelope of all the normal lines to α .

Proposition 50. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^2$ be regular parametrization of C of class $\mathcal{C}^3$ such that $k_\alpha(s) \neq 0 \forall s \in I$. Then, a parametrization of the evolute of α is:

$$t \mapsto \alpha(t) + \rho_\alpha(t)\mathbf{N}_\alpha(t)$$

Definition 51. Let $I \subseteq \mathbb{R}$ be an open interval and $\alpha, \beta : I \rightarrow \mathbb{R}^2$ be regular parametrizations of two curves of class $\mathcal{C}^3$. We say that β is the *involute* of α if β intersects orthogonally all the tangent lines to α .

Proposition 52. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^2$ be regular parametrization of C of class $\mathcal{C}^3$ such that $k_\alpha(s) \neq 0 \forall t_0 \in I$ and $s_0 \in I$. Then, a parametrization of the involute of α is:

$$t \mapsto \alpha(t) - \mathbf{T}_\alpha(t) \int_{t_0}^t \|\alpha'(u)\| du$$

Proposition 53. The evolute of the involute of a curve $C \subset \mathbb{R}^3$ is the curve C itself.

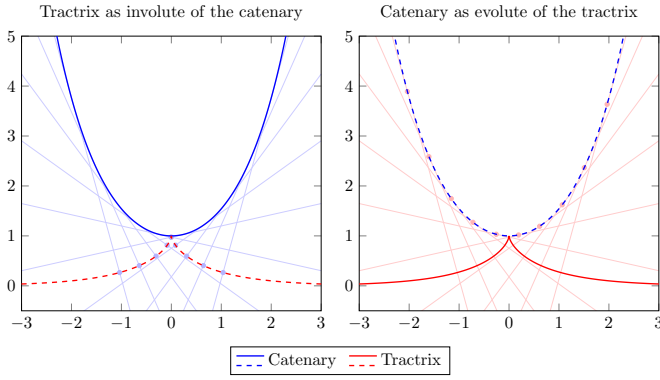


Figure 2: Construction of the evolute and involute of a curve

Curvature of plane curves

Lemma 54. Let $I \subseteq \mathbb{R}$ be an open interval, $a, b : I \rightarrow \mathbb{R}$ be differentiable functions such that $a^2 + b^2 = 1$, $t_0 \in I$ and $\theta_0 \in \mathbb{R}$ be such that $a(t_0) = \cos \theta_0$ and $b(t_0) = \sin \theta_0$. Then, the differentiable function $\theta(t)$ defined as:

$$\theta(t) = \theta_0 + \int_{t_0}^t (a(u)b'(u) - a'(u)b(u)) du$$

satisfies $a(t) = \cos \theta(t)$, $b(t) = \sin \theta(t)$ and $\theta(t_0) = \theta_0 \forall t \in I$.

Proposition 55. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^2$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$. Then, there is a unique vector $\hat{\mathbf{N}}_\alpha$ such that $(\mathbf{T}_\alpha, \hat{\mathbf{N}}_\alpha)$ is a positive orthonormal basis of $\mathbb{R}^2$. Thus, $\mathbf{T}_\alpha' \parallel \hat{\mathbf{N}}_\alpha$.

Definition 56. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^2$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ and $s_0 \in I$. We define the *signed curvature* of α at $\alpha(s_0)$ as the value $\kappa_\alpha(s_0)$ satisfying $\mathbf{T}_\alpha'(s_0) = \kappa_\alpha(s_0)\hat{\mathbf{N}}_\alpha(s_0)$ ⁸. Moreover:

$$\kappa_\alpha = \det(\mathbf{T}_\alpha, \mathbf{T}_\alpha')$$

Proposition 57. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve and $\alpha : I \rightarrow \mathbb{R}^2$ be a regular parametrization of C of class $\mathcal{C}^3$. Then, the signed curvature of α is:

$$\kappa_\alpha = \frac{\det(\alpha', \alpha'')}{\|\alpha'\|^3}$$

Local form of a curve

Definition 58. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^2$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ and $s_0 \in I$. Consider the affine frame of reference $\mathcal{R} = \{\alpha(s_0); (\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha)\}$ and sup-

⁸Using the notation of the last proposition, note that $\hat{\mathbf{N}}_\alpha = \pm \mathbf{N}_\alpha$ and therefore $\kappa_\alpha = \pm \kappa_\alpha$.

⁹Recall that an Euclidean motion is a function that preserves the distance, that is, if $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an Euclidean motion, then $\|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\| = \|\mathbf{u} - \mathbf{v}\| \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

pose $\alpha(s)_\mathcal{R} = (x(s), y(s), z(s))$. Then:

$$\begin{cases} x(s) \simeq s - \frac{k_\alpha(0)^2}{6}s^3 \\ y(s) \simeq \frac{k_\alpha(0)^2}{2}s^2 - \frac{k'_\alpha(0)}{6}s^3 \\ z(s) \simeq -\frac{k_\alpha(0)\tau_\alpha(0)}{6}s^3 \end{cases}$$

This expression of $\alpha(s)_\mathcal{R}$ is called *local canonical form* of α in a neighbourhood of s_0 .

Corollary 59. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^2$ be a regular arc-length parametrization of C of class $\mathcal{C}^3$ and $s_0 \in I$. Then, in the reference $\mathcal{R} = \{\alpha(s_0); (\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha)\}$ we have:

- If $\tau < 0$, at $s = 0$ the curve cross the osculating plane towards the direction that points $\mathbf{B}_\alpha$ (*dextro-rotation*).
- If $\tau > 0$, at $s = 0$ the curve cross the osculating plane towards the opposite direction that points $\mathbf{B}_\alpha$ (*levorotation*).

Orthogonal group

Definition 60. We define that *orthogonal group* as the group of all linear transformations that preserve the inner product. That is:

$$\mathbf{O}(n) := \{\mathbf{A} \in \mathcal{M}_n(\mathbb{R}) : \langle \mathbf{A}\mathbf{u}, \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n\}$$

Proposition 61. Let $\mathbf{A} \in \mathbf{O}(n)$. Then, $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$ and $\det \mathbf{A} = \pm 1$.

Definition 62. We define that *special orthogonal group* as:

$$\mathbf{SO}(n) := \{\mathbf{A} \in \mathbf{O}(n) : \det \mathbf{A} = 1\}$$

Lemma 63. Let $\mathbf{A} \in \mathbf{O}(n)$ and $\lambda \in \sigma(\mathbf{A})$. Then, $\lambda \in \mathbb{R} \implies \lambda = \pm 1$.

Proposition 64. Let $\mathbf{A} \in \mathbf{O}(2)$. Then:

$$\mathbf{A} = \begin{cases} \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} & \text{if } \det \mathbf{A} = 1 \\ \begin{pmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{pmatrix} & \text{if } \det \mathbf{A} = -1 \end{cases}$$

for some $\omega \in \mathbb{R}$.

Proposition 65. Let $\mathbf{A} \in \mathbf{O}(3)$. Then, there exists an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ such that

$$[\text{id}]_{\mathcal{B}, \text{Can}(\mathbb{R}^3)}^{-1} \mathbf{A} [\text{id}]_{\mathcal{B}, \text{Can}(\mathbb{R}^3)} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mathbf{A}' & \\ 0 & & \end{pmatrix}$$

where $\mathbf{A}' \in \mathbf{O}(2)$.

Proposition 66. Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an Euclidean motion⁹. Then, $\exists \mathbf{A} \in \mathbf{O}(n)$ and $\mathbf{u} \in \mathbb{R}^n$ such that:

$$\mathbf{f}(\mathbf{v}) = \mathbf{A}\mathbf{v} + \mathbf{u}$$

Proposition 67. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrization of C of class $\mathcal{C}^3$ and $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$. Then:

$$(\mathbf{A}\alpha)'(t) = \mathbf{A}\alpha'(t)$$

Proposition 68. Let $\mathbf{A} \in \text{SO}(3)$. Then, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ we have:

$$\mathbf{A}(\mathbf{u} \times \mathbf{v}) = (\mathbf{A}\mathbf{u}) \times (\mathbf{A}\mathbf{v})$$

Corollary 69. Let $I \subseteq \mathbb{R}$ be an open interval, $C \subset \mathbb{R}^3$ be a curve, $\alpha : I \rightarrow \mathbb{R}^3$ be an arc-length parametrization of C of class $\mathcal{C}^3$ and $\beta := \mathbf{A}\alpha + \mathbf{u}$, where $\mathbf{A} \in \text{SO}(3)$ and $\mathbf{u} \in \mathbb{R}^3$. Then, β is arc-length parametrized and the TNB frame of β is:

$$\mathbf{T}_\beta = \mathbf{A}\mathbf{T}_\alpha \quad \mathbf{N}_\beta = \mathbf{A}\mathbf{N}_\alpha \quad \mathbf{B}_\beta = \mathbf{A}\mathbf{B}_\alpha$$

And the curvature and torsion of β are:

$$k_\beta = k_\alpha \quad \tau_\beta = \tau_\alpha$$

Fundamental theorem of curves

Theorem 70 (Fundamental theorem of curves). Let $I \subseteq \mathbb{R}$ be an open interval and $k, \tau : I \rightarrow \mathbb{R}$ be functions of class $\mathcal{C}^3$ with $k(s) > 0 \forall s \in I$. Then, there is a curve C , arc-length parametrized by $\alpha : I \rightarrow \mathbb{R}^3$ of class $\mathcal{C}^3$, whose curvature and torsion are k and τ , respectively. Moreover, if $\tilde{C}$ is another curve arc-length parametrized by $\tilde{\alpha} : I \rightarrow \mathbb{R}^3$ satisfying these restrictions, then there exists an Euclidean motion that carries $\tilde{C}$ into C .

2. | Submanifolds of $\mathbb{R}^n$

Planar functions

Definition 71. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}$ be a function. We define the *support* of f as:

$$\text{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}}$$

Lemma 72. Let $x_0 \in \mathbb{R}^n$ and $a, b \in \mathbb{R}_{>0}$ with $a < b$. Then, there exists a function $\rho : \mathbb{R}^n \rightarrow [0, 1]$ of class $\mathcal{C}^\infty$ such that $\text{supp}(\rho) \subseteq \overline{B(x_0, b)}$ and $\rho|_{B(x_0, a)} = 1$.

Proposition 73. Let $U \subseteq \mathbb{R}^n$ be an open set and $K \subset U$ be a compact set. Then, there exists a function $\rho : \mathbb{R}^n \rightarrow [0, 1]$ of class $\mathcal{C}^\infty$ such that $\text{supp}(\rho) \subseteq U$ and $\rho|_K = 1$.

Corollary 74. Let $U \subseteq \mathbb{R}^n$ be an open set, $K \subset U$ be a compact set and $f : U \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^\infty$. Then, there exists a function $\tilde{f} : U \rightarrow \mathbb{R}$ such that $\tilde{f}|_K = f|_K$ and $\tilde{f}|_{\mathbb{R}^n \setminus U} = 0$.

Immersion and submersion

Definition 75 (Immersion). Let $n, m \in \mathbb{N}$ with $n \leq m$, $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a function of class $\mathcal{C}^\infty$. We say that $\mathbf{f}$ is an *immersion* at $x_0 \in U$ if $d\mathbf{f}_{x_0}$ is injective. We say that $\mathbf{f}$ is an *immersion* (on U) if it is an immersion at each point $x \in U$.

Definition 76 (Submersion). Let $n, m \in \mathbb{N}$ with $n \geq m$, $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a function of class $\mathcal{C}^\infty$. We say that $\mathbf{f}$ is a *submersion* at $x_0 \in U$ if $d\mathbf{f}_{x_0}$ is surjective. We say that $\mathbf{f}$ is a *submersion* (on U) if it is a submersion at each point $x \in U$.

Proposition 77. Let $n, m \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a function of class $\mathcal{C}^\infty$. Then:

- If $n \leq m$, then:

$$\mathbf{f} \text{ is immersion} \iff \text{rank } d\mathbf{f}_p = n \quad \forall p \in U$$

- If $n \geq m$, then:

$$\mathbf{f} \text{ is submersion} \iff \text{rank } d\mathbf{f}_p = m \quad \forall p \in U$$

Theorem 78 (Local structure of immersions). Let $n, m \in \mathbb{N}$ with $n \leq m$, $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be an immersion at $x_0 \in U$ and $\iota : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the inclusion map. Then, there exist neighbourhoods $V \subseteq U$ of x_0 and $W \subseteq \mathbb{R}^m$ of $\iota(x_0)$ and a diffeomorphism $\mathbf{g} : W \rightarrow \mathbf{g}(W)$ such that the following diagram is commutative, that is, $\mathbf{f} = \mathbf{g} \circ \iota$.

$$\begin{array}{ccc} V & \xrightarrow{\mathbf{f}} & \mathbf{g}(W) \\ \downarrow \iota & \nearrow \mathbf{g} & \\ W & & \end{array}$$

Figure 3

Theorem 79 (Local structure of submersions). Let $n, m \in \mathbb{N}$ with $n \geq m$, $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a submersion at $x_0 \in U$ and $\pi_1 : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$ be the projection map into the first coordinate. Then, there exists a neighbourhood $V \subseteq U$ of x_0 and a diffeomorphism $\mathbf{g} : V \rightarrow \mathbf{g}(V)$ such that the following diagram is commutative, that is, $\mathbf{f} = \pi_1 \circ \mathbf{g}$.

$$\begin{array}{ccc} V & \xrightarrow{\mathbf{f}} & \mathbb{R}^m \\ \downarrow \mathbf{g} & \nearrow \pi_1 & \\ \mathbf{g}(V) & & \end{array}$$

Figure 4

Submanifolds of $\mathbb{R}^n$

Definition 80. Let $M \subseteq \mathbb{R}^n$ be a set. We say that M is a *submanifold* of $\mathbb{R}^n$ of dimension p (and codimension $q := n - p$) if $\forall z \in M$ there exists a neighbourhood $U \subseteq \mathbb{R}^n$ of z and a diffeomorphism $\mathbf{g} : U \rightarrow \mathbf{g}(U)$ such that:

$$\mathbf{g}(U \cap M) = \mathbf{g}(U) \cap (\mathbb{R}^p \times \{0\})$$

Theorem 81. Let $M \subseteq \mathbb{R}^n$ be a set. The following statements are equivalent:

1. M is a submanifold of $\mathbb{R}^n$ of dimension p and codimension q .
2. $\forall z \in M$ there exists a neighbourhood $U \subseteq \mathbb{R}^n$ of z and a submersion $\phi : U \rightarrow \mathbb{R}^q$ such that $U \cap M = \phi^{-1}(0)$.
3. $\forall z \in M$ there exists a neighbourhood $V \subseteq \mathbb{R}^p$ of z and an immersion $\varphi : V \rightarrow \mathbb{R}^n$ such that $z \in \varphi(V) \subseteq M$ and $\varphi : V \rightarrow \varphi(V)$ is a homeomorphism.

Definition 82. Let $M \subseteq \mathbb{R}^n$ be a submanifold, $V \subseteq \mathbb{R}^p$ and $\varphi : V \rightarrow \varphi(V) \subseteq M$ be an immersion and a homeomorphism. We say that the pair (V, φ) is a *parametrization* of M and the pair $(\varphi(V), \varphi^{-1})$, a *coordinate chart* of M .

Proposition 83. Let $(V_1, \varphi_1), (V_2, \varphi_2)$ be two parametrizations of a submanifold $M \subseteq \mathbb{R}^n$. Then, the composition $\varphi_2^{-1} \circ \varphi_1$ is differentiable on its domain.

Proposition 84. Let $M \subseteq \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ of dimension p , $V \subseteq \mathbb{R}^p$ be an open set and $\varphi : V \rightarrow M$ be a differentiable injective immersion. Then, $\varphi(V) \subseteq M$ is an open set and $\varphi : V \rightarrow \varphi(V)$ is a homeomorphism. Hence, (V, φ) is a parametrization of M .

Surfaces of $\mathbb{R}^3$

Definition 85. A submanifold of $\mathbb{R}^3$ of dimension 2 is called a *regular surface* (or simply *surface*) of $\mathbb{R}^3$.

Proposition 86. Let $U \subseteq \mathbb{R}^2$ be an open set and $h : U \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$. Then, $\text{graph}(h)$ is a surface.

Proposition 87. Let $S \subseteq \mathbb{R}^3$ be a set. Then, S is a surface if and only if $\forall z \in S$ there exists an open neighbourhood $U \subseteq \mathbb{R}^3$ of z , a change of the order of the variables σ , a neighbourhood $V \subseteq \mathbb{R}^2$ of $\pi_1(\sigma(z))$ (where $\pi_1 : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is the projection map) and a differentiable function $h : V \rightarrow \mathbb{R}$ such that:

$$\sigma(S \cap U) = \text{graph}(h)$$

Proposition 88. Let $U \subseteq \mathbb{R}^3$, $f : U \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$ and $a \in \mathbb{R}$ such that $df_p \neq \mathbf{0} \ \forall p \in f^{-1}(a)$. Then, $f^{-1}(a)$ is a surface.

Definition 89. Let $(a(u), v(u))$, $u \in I$, be a parametrization of class $\mathcal{C}^1$ of a planar curve C . We define the *surface of revolution* created by rotating C around an axis of rotation. The curve C is called *generatrix*. In particular, if we choose the y -axis as the axis of rotation,

$$\varphi(u, v) = (a(u) \cos v, a(u) \sin v, b(u)) \quad (u, v) \in I \times (0, 2\pi)$$

is a parametrization of the induced surface of revolution.

Differentiable functions

Definition 90. Let $S \subseteq \mathbb{R}^3$ be a surface. We say that a function $\mathbf{f} : S \rightarrow \mathbb{R}^n$ is *differentiable* at a point $p \in S$ if there is a local parametrization (V, φ) of S with $p \in \varphi(V)$ such that $\mathbf{f} \circ \varphi$ is differentiable at $\varphi^{-1}(p)$. We say that $\mathbf{f}$ is *differentiable* on S if it is differentiable at each point $p \in S$.

Proposition 91. Let $S \subseteq \mathbb{R}^3$ be a surface, $\mathbf{f} : S \rightarrow \mathbb{R}^n$ be a differentiable function and $p \in S$. Then, there exists an open neighbourhood $U \subseteq \mathbb{R}^3$ of p and a function $\tilde{\mathbf{f}} : S \rightarrow \mathbb{R}^n$ such that $\tilde{\mathbf{f}}|_{U \cap S} = \mathbf{f}|_{U \cap S}$.

Corollary 92. Let $S \subseteq \mathbb{R}^3$ be a surface, $U \subseteq \mathbb{R}^n$ and $\mathbf{f} : U \rightarrow \mathbb{R}^3$ be a differentiable function such that $\mathbf{f}(U) \subseteq S$. If (V, φ) is a local parametrization of S , then $\varphi^{-1} \circ \mathbf{f}$ is also a differentiable function on its domain.

Corollary 93. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $\alpha : I \rightarrow \mathbb{R}^3$ be a curve of class $\mathcal{C}^\infty$ such that $\alpha(I) \subseteq \varphi(V)$. Then, α can be written as $\alpha(t) = \varphi(u(t), v(t))$, where $u(t), v(t)$ are differentiable functions.

Definition 94. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces. We say that a function $\mathbf{f} : S_1 \rightarrow S_2$ is differentiable if $\forall p \in S_1$, there exist parametrizations (V_1, φ_1) and (V_2, φ_2) of S_1 and S_2 respectively with $p \in \varphi_1$ and $\mathbf{f}(p) \in \varphi_2$ and such that $\varphi_2^{-1} \circ \mathbf{f} \circ \varphi_1$ is differentiable on its domain¹⁰.

Proposition 95. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces and $\mathbf{f} : S_1 \rightarrow S_2$ be a function. Then, $\mathbf{f}$ is differentiable if $\mathbf{f} \circ \iota : S_1 \hookrightarrow \mathbb{R}^3$ is differentiable.

Tangent space

Definition 96. Let $S \subseteq \mathbb{R}^3$ be a surface and $p \in S$. If $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$ is a parametrization of a curve of class $\mathcal{C}^\infty$ such that $\alpha(0) = p$, we say that $\alpha'(0)$ is a *tangent vector* to S at p . The set of all such vectors is called *tangent space* (or *tangent plane*) and it is denoted as $T_p S$.

Proposition 97. Let $S \subseteq \mathbb{R}^3$ be a surface, $p \in S$, (V, φ) be a local parametrization of S with $p \in \varphi(V)$ and $f : U \rightarrow \mathbb{R}$ be a submersion with $S \cap U = f^{-1}(0)$. Then:

$$\text{im } d\varphi_{\varphi^{-1}(p)} = T_p S = \ker df_p$$

Therefore, $T_p S$ is a vector space and $\dim T_p S = 2$.

Proposition 98. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $p = \varphi(u_0, v_0) \in S$. Then, the tangent vectors

$$\left(\frac{\partial \varphi}{\partial u}(u_0, v_0), \frac{\partial \varphi}{\partial v}(u_0, v_0) \right)^{11}$$

form a basis of the tangent plane $T_p S$.

Lemma 99. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a differentiable function and $\alpha : (-\varepsilon, \varepsilon) \rightarrow U$ be a parametrization of a curve of class $\mathcal{C}^\infty$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. Then:

$$d\mathbf{f}_p(\mathbf{v}) = (\mathbf{f} \circ \alpha)'(0)$$

¹⁰In particular, note that if $S_1 = S_2$ and $\mathbf{f} = \text{id}$, then $\varphi_2^{-1} \circ \text{id} \circ \varphi_1 = \varphi_2^{-1} \circ \varphi_1$ is a change of coordinates on the surface.

¹¹Usually we will denote these partial derivatives by $\varphi_u = \frac{\partial \varphi}{\partial u}(u_0, v_0)$ and $\varphi_v = \frac{\partial \varphi}{\partial v}(u_0, v_0)$, respectively.

Definition 100. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces, $p \in S_1$ and $\mathbf{f} : S_1 \rightarrow S_2$ be a differentiable function. We define the *tangent function* (or *differential*) of $\mathbf{f}$ at p as the function:

$$\begin{aligned} \mathbf{df}_p : T_p S_1 &\longrightarrow T_{\mathbf{f}(p)} S_2 \\ \mathbf{v} &\longmapsto (\mathbf{f} \circ \alpha)'(0) \end{aligned}$$

where $\alpha : (-\varepsilon, \varepsilon) \rightarrow S_1$ is a parametrization of a curve of class $\mathcal{C}^\infty$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.

Proposition 101. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces, $p \in S_1$ and $\mathbf{f} : S_1 \rightarrow S_2$ be a differentiable function. Then, $\mathbf{df}_p$ is linear. Moreover, if $(V_1, \varphi_1(u, v))$ and $(V_2, \varphi_2(\tilde{u}, \tilde{v}))$ are parametrizations of S_1 and S_2 respectively, $\tilde{u} = f_1(u, v)$, $\tilde{v} = f_2(u, v)$ ¹² and $\mathcal{B}_1 = \left(\frac{\partial \varphi_1}{\partial u}, \frac{\partial \varphi_1}{\partial v} \right)$, $\mathcal{B}_2 = \left(\frac{\partial \varphi_2}{\partial \tilde{u}}, \frac{\partial \varphi_2}{\partial \tilde{v}} \right)$, we have that:

$$[\mathbf{df}_p]_{\mathcal{B}_1, \mathcal{B}_2} = \begin{pmatrix} \frac{\partial f_1}{\partial u}(\varphi^{-1}(p)) & \frac{\partial f_1}{\partial v}(\varphi^{-1}(p)) \\ \frac{\partial f_2}{\partial u}(\varphi^{-1}(p)) & \frac{\partial f_2}{\partial v}(\varphi^{-1}(p)) \end{pmatrix}$$

Theorem 102 (Inverse function theorem for surfaces). Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces, $p \in S_1$ and $\mathbf{f} : S_1 \rightarrow S_2$ be a differentiable function. Suppose $\mathbf{df}_p$ is an isomorphism. Then, $\mathbf{f}$ is a diffeomorphism between a neighbourhood $U_1 \subseteq S_1$ of p and a neighbourhood $U_2 \subseteq S_2$ of $\mathbf{f}(p)$.

3. | First fundamental form

First fundamental form

Definition 103. Let $S \subseteq \mathbb{R}^3$ be a surface and $p \in S$. We define the *first fundamental form* of S at p as the quadratic form $I_p : T_p S \times T_p S \rightarrow \mathbb{R}$ defined by:

$$I_p(\mathbf{v}) := \langle \mathbf{v}, \mathbf{v} \rangle_p := \|\mathbf{v}\|^2$$
¹³

Proposition 104. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $p \in S$. Then, in the basis (φ_u, φ_v) we have:

$$I_p = \begin{pmatrix} E_\varphi & F_\varphi \\ F_\varphi & G_\varphi \end{pmatrix} := \begin{pmatrix} \langle \varphi_u, \varphi_u \rangle_p & \langle \varphi_u, \varphi_v \rangle_p \\ \langle \varphi_v, \varphi_u \rangle_p & \langle \varphi_v, \varphi_v \rangle_p \end{pmatrix}$$
¹⁴

That is, if $\mathbf{u} = a\varphi_u + b\varphi_v$ and $\mathbf{v} = c\varphi_u + d\varphi_v$, then

$$\langle \mathbf{u}, \mathbf{v} \rangle_p = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E_\varphi & F_\varphi \\ F_\varphi & G_\varphi \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

and

$$I_p(\mathbf{u}) = a^2 E_\varphi + 2ab F_\varphi + b^2 G_\varphi$$

Definition 105. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $p \in S$. We say that the parametrization $(V, \varphi(u, v))$ is *orthogonal* (or that u and v are *orthogonal coordinates*) if $F_\varphi = 0$.

Proposition 106. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $\alpha : I \rightarrow \varphi(V)$ be a parametrization of a curve of class $\mathcal{C}^\infty$. We can write α as $\alpha(t) = \varphi(u(t), v(t))$. Then:

$$\|\alpha'(t)\| = \sqrt{u'(t)^2 E_\varphi + 2u'(t)v'(t)F_\varphi + v'(t)^2 G_\varphi}$$

where $E_\varphi = E_\varphi(u(t), v(t))$, $F_\varphi = F_\varphi(u(t), v(t))$, $G_\varphi = G_\varphi(u(t), v(t))$. The arc-length parameter is thus:

$$s(t) = \int_{t_0}^t \sqrt{u'^2 E_\varphi + 2u'v'F_\varphi + v'^2 G_\varphi} d\xi$$

Proposition 107. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $p \in S$. Then, the angle β between the coordinates lines of the parametrization $(V, \varphi(u, v))$ is:

$$\cos \beta = \frac{\langle \varphi_u, \varphi_v \rangle}{\|\varphi_u\| \|\varphi_v\|} = \frac{F_\varphi}{\sqrt{E_\varphi G_\varphi}}$$

Area

Definition 108. Let $S \subseteq \mathbb{R}^3$ be a surface and $D \subseteq S$ be a subset. We say that D is a *regular domain* (or simply *domain*) if D is open, connected and $\partial D \subset S$ is the image of a piecewise curve of class $\mathcal{C}^1$. A *region* $R \subseteq S$ is the union of a domain D with its boundary, $R = D \cup \partial D$.

Definition 109. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S and $R \subset \varphi(V)$ be a compact region. Let $Q = \varphi^{-1}(R) \subseteq \mathbb{R}^2$. We define the area of R as:

$$\begin{aligned} \text{area}(R) &= \int_Q \|\varphi_u \times \varphi_v\| du dv \\ &= \int_Q \sqrt{E_\varphi G_\varphi - F_\varphi^2} du dv \end{aligned}$$
¹⁵

Definition 110. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S , $R \subset \varphi(V)$ be a compact region and $f : S \rightarrow \mathbb{R}$ be a function. Let $Q = \varphi^{-1}(R) \subseteq \mathbb{R}^2$. We define the *integral of f over the region R* as:

$$\int_R f dS := \int_Q (f \circ \varphi) \sqrt{E_\varphi G_\varphi - F_\varphi^2} du dv$$
¹⁶

Isometries

Definition 111. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces and $\mathbf{f} : S_1 \rightarrow S_2$ be a differentiable function. We say that $\mathbf{f}$ is a *local isometry* if the differential function $\mathbf{df}_p$ is an isometry $\forall p \in S_1$. That is, for each $p \in S_1$ we have:

$$\langle \mathbf{v}, \mathbf{w} \rangle_1 = \langle \mathbf{df}_p(\mathbf{v}), \mathbf{df}_p(\mathbf{w}) \rangle_2 \quad \forall \mathbf{v}, \mathbf{w} \in T_p S_1$$
¹⁷

We say that $\mathbf{f}$ is an *isometry* if it is a local isometry, and it is invertible.

¹²That is, f_1 and f_2 are the component functions of $\varphi_2^{-1} \circ \mathbf{f} \circ \varphi_1$.

¹³Abusing notation, we will denote the bilinear associated function to I_p also as I_p . That is, $I_p(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle_p$.

¹⁴Sometimes we will omit writing the subindex of E_φ , F_φ and G_φ .

¹⁵One can check that this definition does not depend on the parametrization $(V, \varphi(u, v))$ of S .

¹⁶Here, $\langle \cdot, \cdot \rangle_i$ represents the first fundamental form of S_i , $i = 1, 2$.

Proposition 112. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces and $\mathbf{f} : S_1 \rightarrow S_2$ be a local isometry. Then, $\mathbf{d}\mathbf{f}_p$ is an isomorphism.

Proposition 113. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces and $\mathbf{f} : S_1 \rightarrow S_2$ be a function of class $\mathcal{C}^1$. Then, $\mathbf{f}$ is a local isometry if and only if $\mathbf{f}$ preserves lengths, that is, for any curve $\alpha : I \rightarrow S_1$, we have $L(\alpha) = L(\mathbf{f} \circ \alpha)$.

Proposition 114. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces, (V, φ) be a local parametrization of S_1 , $\mathbf{f} : S_1 \rightarrow S_2$ be a function of class $\mathcal{C}^1$. Then, $(V, \psi = \mathbf{f} \circ \varphi)$ is a local parametrization of S_2 and moreover:

$$\mathbf{f} \text{ is an isometry} \iff E_\varphi = E_\psi, F_\varphi = F_\psi, G_\varphi = G_\psi$$

Corollary 115. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces and $\mathbf{f} : S_1 \rightarrow S_2$ be an isometry. Then, $\mathbf{f}$ preserves areas.

Conformal maps

Definition 116. Let $U, V \subseteq \mathbb{R}^n$ be open sets, $\mathbf{f} : U \rightarrow V$ be a function and $p \in U$. We say that $\mathbf{f}$ is *conformal* (or *angle-preserving*) at p if it preserves angles between directed curves through p , as well as preserving orientation. We say that $\mathbf{f}$ is *conformal* (on U) if it is conformal at each $p \in U$.

Theorem 117. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be surfaces, (V, φ) be a parametrization of S_1 and $\mathbf{f} : S_1 \rightarrow S_2$ be a conformal map. Consider the parametrization $(V, \psi = \mathbf{f} \circ \varphi)$ be a parametrization of S_2 . Then, there exists a function $\rho : U \rightarrow \mathbb{R}$ such that:

$$E_\varphi = \rho^2 E_\psi \quad F_\varphi = \rho^2 F_\psi \quad G_\varphi = \rho^2 G_\psi$$

4. | Second fundamental form

Orientation of surfaces and Gauß map

Definition 118. Let $S \subseteq \mathbb{R}^3$ be a surface. We say that S is *orientable* if it admits a *normal unit field*, that is, a differentiable function $\nu_S : S \rightarrow S^2 \subseteq \mathbb{R}^3$ such that $\nu_S(p) \in T_p S^\perp \forall p \in S$. This function ν_S ¹⁸ is known as *Gauß map*.

Definition 119. Let $S \subseteq \mathbb{R}^3$ be an orientable and connected surface. An *orientation* of S is the choice of one of the two $(\nu_S$ or $-\nu_S)$ unit normal fields.

Definition 120. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be a local parametrization of S . We say that $(V, \varphi(u, v))$ is *compatible* with the orientation of S if

$$\nu_S = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}$$

Proposition 121. Let $S \subseteq \mathbb{R}^3$ be a surface. S is orientable if and only if S can be covered by the images $\varphi_i(V_i)$ of a collection of parametrizations $\{(V_i, \varphi_i) : i \in I\}$ of S such that

$$\det \mathbf{d}(\varphi_j^{-1} \circ \varphi_i) > 0 \quad \forall i, j \in I$$

¹⁸Unless necessary, we will omit writing the subindex S .

Weingarten endomorphism

Definition 122. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν . We define the *Weingarten endomorphism* of S at the point $p \in S$ as the endomorphism:

$$\begin{aligned} \mathbf{W}_p : T_p S &\longrightarrow T_p S \\ \mathbf{v} &\longmapsto -\mathbf{d}\nu_p(\mathbf{v}) \end{aligned}$$

Lemma 123. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ be a local parametrization of S , $\alpha(t) = \varphi(u(t), v(t))$ be a curve on S and $p = \alpha(0)$. We denote $\nu(t) = (\nu \circ \alpha)(t) = \nu(u(t), v(t))$. Then:

$$\begin{aligned} \mathbf{d}\nu_p(\alpha'(0)) &= \mathbf{d}\nu_p(u'(0)\varphi_u + v'(0)\varphi_v) \\ &= u'(0)\nu_u + v'(0)\nu_v \end{aligned}$$

In particular, $\mathbf{d}\nu_p(\varphi_u) = \nu_u$ and $\mathbf{d}\nu_p(\varphi_v) = \nu_v$.

Proposition 124. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν and $p \in S$. Then, the Weingarten endomorphism is auto-adjoint with respect to the first fundamental form. That is:

$$\langle \mathbf{W}_p(\mathbf{u}), \mathbf{v} \rangle_p = \langle \mathbf{u}, \mathbf{W}_p(\mathbf{v}) \rangle_p \quad \forall \mathbf{u}, \mathbf{v} \in T_p S$$

Proposition 125. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $p \in S$. Then, the Weingarten endomorphism has real eigenvalues, and it diagonalizes in an orthonormal basis of $T_p S$.

Definition 126. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $p \in S$. We define the *principal directions* of S at p as the eigenspaces of $\mathbf{W}_p$. We define the *principal curvatures* of S at p as the eigenvalues of $\mathbf{W}_p$.

Definition 127. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $p \in S$. We say that the point p is an *umbilic point* if $\mathbf{W}_p = \lambda \text{id}$, for some $\lambda \in \mathbb{R}$.

Definition 128. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and k_1, k_2 be the principal curvatures of S at p . We define the *Gauß curvature* of S at p as:

$$K(p) := \det \mathbf{W}_p = k_1 k_2$$

We define the *mean curvature* of S at p as:

$$H(p) := \frac{\text{tr } \mathbf{W}_p}{2} = \frac{k_1 + k_2}{2}$$

Definition 129. Let $S \subseteq \mathbb{R}^3$ be an orientable surface. We say that S is a *minimal surface* if $H = 0$.

Second fundamental form

Definition 130. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field $\boldsymbol{\nu}$ and $p \in S$. We define the *second fundamental form* of S at p as the quadratic form $\mathbb{I}_p : T_p S \times T_p S \rightarrow \mathbb{R}$ defined by:

$$\mathbb{I}_p(\mathbf{v}) = \mathbb{I}_p(\mathbf{W}_p(\mathbf{v}), \mathbf{v}) = \langle \mathbf{W}_p(\mathbf{v}), \mathbf{v} \rangle_p^{19}$$

Definition 131. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field $\boldsymbol{\nu}$, $p \in S$ and $\boldsymbol{\alpha} : I \rightarrow S$ be a regular curve. Suppose $\cos \theta = \langle \mathbf{N}_{\boldsymbol{\alpha}}(p), \boldsymbol{\nu}(p) \rangle$. We define the *normal curvature* of $\boldsymbol{\alpha}$ at p as:

$$k_n(p) := k_{\boldsymbol{\alpha}} \cos \theta$$

Proposition 132 (Meusnier's theorem). Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and $\boldsymbol{\alpha} : I \rightarrow S$ be an arc-length parametrization of a curve C of class $\mathcal{C}^\infty$ such that $\boldsymbol{\alpha}(0) = p$. Then:

$$k_n(p) = \mathbb{I}_p(\boldsymbol{\alpha}'(0))$$

In particular, $k_n(p)$ depends only on the tangent line to $\boldsymbol{\alpha}$ at p .

Definition 133. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and $\mathbf{v} \in T_p S$ with $\|\mathbf{v}\| = 1$. We define the *normal curvature* at p in the direction of $\mathbf{v}$ as:

$$k_n(\mathbf{v}) := \mathbb{I}_p(\mathbf{v})$$

Proposition 134. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and $(\mathbf{v}_1, \mathbf{v}_2)$ be an orthonormal basis of $T_p S$, where $\mathbf{v}_i$ is an eigenvector of eigenvalue k_i of $\mathbf{W}_p$ for $i = 1, 2$. Then, for $i = 1, 2$ we have:

$$k_i = k_n(\mathbf{v}_i)$$

Definition 135. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and $\mathbf{v} \in T_p S$ with $\|\mathbf{v}\| = 1$. We say that the direction of $\mathbf{v}$ in $T_p S$ is an *asymptotic direction* if $k_n(\mathbf{v}) = \mathbb{I}_p(\mathbf{v}) = 0$.

Definition 136. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $C \subset S$ be a curve. We say that C is a *line of curvature* of S if the tangent line at C is a principal direction at each point $p \in C$. We say that C is an *asymptotic line* of S if the tangent line at C is an asymptotic direction at each point $p \in C$.

Proposition 137 (Olinde Rodrigues' theorem). Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\boldsymbol{\alpha} : I \rightarrow S$ be a regular parametrization of a curve C of class $\mathcal{C}^\infty$. Let $\boldsymbol{\nu}(t) := (\boldsymbol{\nu} \circ \boldsymbol{\alpha})(t)$. Then, C is a line of curvature of S if and only if

$$\boldsymbol{\nu}'(t) = \lambda(t)\boldsymbol{\alpha}'(t)$$

where $\lambda(t)$ is a differentiable function. In this case, $-\lambda(t)$ is the principal curvature of S in the direction of $\boldsymbol{\alpha}'(t)$.

¹⁹Abusing notation, we will denote the bilinear associated function to $\mathbb{I}_p$ also as $\mathbb{I}_p$. That is, $\mathbb{I}_p(\mathbf{u}, \mathbf{v}) = \langle \mathbf{W}_p(\mathbf{u}), \mathbf{v} \rangle_p$.

²⁰Sometimes we will omit writing the subindex of e_φ , f_φ and g_φ .

Proposition 138 (Euler's formula). Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ and $(\mathbf{v}_1, \mathbf{v}_2)$ be an orthonormal basis of $T_p S$, where $\mathbf{v}_i$ is an eigenvector of eigenvalue k_i of $\mathbf{W}_p$ for $i = 1, 2$. Then:

$$k_n(\cos \theta \mathbf{v}_1 + \sin \theta \mathbf{v}_2) = k_1(\cos \theta)^2 + k_2(\sin \theta)^2$$

Hence, we will denote $k_n(\theta) := k_1(\cos \theta)^2 + k_2(\sin \theta)^2$.

Corollary 139. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$. Then, the extrema of $k_n(p)$ are precisely the principal curvatures k_1 and k_2 at p .

Proposition 140. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field $\boldsymbol{\nu}$ and $p \in S$. Then, if we invert the orientation of $\boldsymbol{\nu}$, the curvatures k_1 , k_2 , H and k_n change their sign but K remains invariant.

Proposition 141. Let $S \subseteq \mathbb{R}^3$ be an orientable surface. Then:

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} k_n(\theta) d\theta$$

Proposition 142. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field $\boldsymbol{\nu}$ and $\boldsymbol{\alpha} : I \rightarrow S$ be a regular curve of class $\mathcal{C}^\infty$. Then:

$$k_n = \frac{\langle \boldsymbol{\alpha}'', \boldsymbol{\nu} \circ \boldsymbol{\alpha} \rangle}{\|\boldsymbol{\alpha}'\|^2}$$

Definition 143. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $p \in S$. We say that p is

- an *elliptic point* if $K(p) > 0$.
- a *hyperbolic point* if $K(p) < 0$.
- a *parabolic point* if $K(p) = 0$ but $\mathbf{W}_p \neq 0$.
- a *plane point* if $K(p) = 0$ and $\mathbf{W}_p = 0$.

Proposition 144. Let $S \subseteq \mathbb{R}^3$ be an orientable and connected surface such that all of its points are umbilic. Then, S is contained in a sphere or in a plane.

Gauß map in coordinates

Proposition 145. Let $(V, \varphi(u, v))$ be a local parametrization of a surface $S \subseteq \mathbb{R}^3$ oriented with a normal unit field $\boldsymbol{\nu}$ and $p \in S$. Suppose

$$\begin{aligned} \boldsymbol{\nu}_u &= a_{11}\boldsymbol{\varphi}_u + a_{21}\boldsymbol{\varphi}_v \\ \boldsymbol{\nu}_v &= a_{12}\boldsymbol{\varphi}_u + a_{22}\boldsymbol{\varphi}_v \end{aligned}$$

Then:

$$d\boldsymbol{\nu}_p = -\mathbf{W}_p = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Proposition 146. Let $(V, \varphi(u, v))$ be a local parametrization of a surface $S \subseteq \mathbb{R}^3$ oriented with a normal unit field $\boldsymbol{\nu}$ and $p \in S$. Then, we have:

$$e_\varphi := -\langle \boldsymbol{\nu}_u, \boldsymbol{\varphi}_u \rangle = \langle \boldsymbol{\nu}, \boldsymbol{\varphi}_{uu} \rangle$$

$$\begin{aligned} f_\varphi &:= -\langle \nu_v, \varphi_u \rangle = \langle \nu, \varphi_{uv} \rangle = \langle \nu, \varphi_{vu} \rangle = -\langle \nu_u, \varphi_v \rangle \\ g_\varphi &:= -\langle \nu_v, \varphi_v \rangle = \langle \nu, \varphi_{vv} \rangle \end{aligned}$$

Moreover, in the basis (φ_u, φ_v) we have:

$$\mathbb{I}_p = \begin{pmatrix} e_\varphi & f_\varphi \\ f_\varphi & g_\varphi \end{pmatrix}^{20}$$

Proposition 147. Let $(V, \varphi(u, v))$ be a local parametrization of a surface $S \subseteq \mathbb{R}^3$ oriented with a normal unit field ν and $p \in S$. Then:

$$\mathbf{W}_p = \mathbf{I}_p^{-1} \mathbb{I}_p$$

Hence:

$$\mathbf{W}_p = \frac{1}{E_\varphi G_\varphi - F_\varphi^2} \begin{pmatrix} e_\varphi G_\varphi - f_\varphi F_\varphi & f_\varphi G_\varphi - g_\varphi F_\varphi \\ -e_\varphi F_\varphi + f_\varphi E_\varphi & -f_\varphi F_\varphi + g_\varphi E_\varphi \end{pmatrix}$$

Corollary 148. Let $(V, \varphi(u, v))$ be a local parametrization of a surface $S \subseteq \mathbb{R}^3$ oriented with a normal unit field ν and $p \in S$. Then:

$$\begin{aligned} K &= \frac{e_\varphi g_\varphi - f_\varphi^2}{E_\varphi G_\varphi - F_\varphi^2} \\ H &= \frac{1}{2} \frac{e_\varphi G_\varphi - 2f_\varphi F_\varphi + g_\varphi E_\varphi}{E_\varphi G_\varphi - F_\varphi^2} \end{aligned}$$

Moreover the principal curvatures are given by:

$$k_1, k_2 = H \pm \sqrt{H^2 - K}$$

Proposition 149. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $(V, \varphi(u, v))$ be a parametrization of S and $\alpha : I \rightarrow S$ be a regular parametrization of a curve C of class $\mathcal{C}^\infty$ such that $\alpha(t) = \varphi(u(t), v(t))$. Then:

1. C is an asymptotic line if and only if:

$$e_\varphi u'^2 + 2f_\varphi u'v' + g_\varphi v'^2 = 0$$

2. C is a line of curvature if and only if:

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E_\varphi & F_\varphi & G_\varphi \\ e_\varphi & f_\varphi & g_\varphi \end{vmatrix} = 0$$

Geometric interpretation of the Gauß curvature

Lemma 150. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$, $(\mathbf{v}_1, \mathbf{v}_2)$ be a basis of $T_p S$ and $\mathbf{A} : T_p S \rightarrow T_p S$ be a linear isomorphism. Then:

$$\mathbf{A}\mathbf{v}_1 \times \mathbf{A}\mathbf{v}_2 = \det \mathbf{A} (\mathbf{v}_1 \times \mathbf{v}_2)$$

In particular, if ν is a normal unit field of S and $\mathbf{w}_1, \mathbf{w}_2 \in T_p S$, then:

$$d\nu_p(\mathbf{w}_1) \times d\nu_p(\mathbf{w}_2) = K(p) (\mathbf{w}_1 \times \mathbf{w}_2)$$

If $\mathbf{w}_1 = \varphi_u$ and $\mathbf{w}_2 = \varphi_v$ we have:

$$\nu_u \times \nu_v = K (\varphi_u \times \varphi_v)$$

²⁰Observe that $\Gamma_{ij}^k = \Gamma_{ji}^k \forall i, j, k \in \{1, 2\}$.

Definition 151. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν and $R \subseteq S$ be a region on S where the curvature K doesn't vanish. We define the signed area of $\nu(R)$ as:

$$\text{area}_s(\nu(R)) = \text{sgn}(K) \text{area}(\nu(R))$$

Proposition 152. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $p \in S$ such that $K(p) \neq 0$ and $V \subseteq S$ be a connected neighbourhood of p where K has constant sign. Let $(B_n) \subseteq V$ be a sequence of regions that converge to p . Then:

$$K(p) = \lim_{n \rightarrow \infty} \frac{\text{area}_s(\nu(B_n))}{\text{area}(B_n)}$$

Ruled surfaces

Definition 153. Let $S \subseteq \mathbb{R}^3$ be a surface is called *ruled surface* if it has a parametrization of the form

$$\varphi(u, v) = \alpha(u) + v\beta(u)$$

where α and β are curves of $\mathbb{R}^3$ such that $|\beta| = 1$.

Proposition 154. Let $S \subseteq \mathbb{R}^3$ be a ruled surface. Then, $K \leq 0$.

Definition 155. Let $S \subseteq \mathbb{R}^3$ be a surface. We say that S is *developable* if it is ruled and $K = 0$.

Proposition 156. Let $S \subseteq \mathbb{R}^3$ be a developable surface and $(V, \varphi(u, v))$ be a parametrization of S . Then, there exists a curve $v = h(u)$ where φ stops being regular. This curve is called *regression axis*.

5. | Intrinsic geometry of surfaces

Gauß' Theorema Egregium

Definition 157. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be a parametrization of S . Then

$$\begin{aligned} \varphi_{uu} &= \Gamma_{11}^1 \varphi_u + \Gamma_{11}^2 \varphi_v + e\nu \\ \varphi_{uv} &= \Gamma_{12}^1 \varphi_u + \Gamma_{12}^2 \varphi_v + f\nu \\ \varphi_{vu} &= \Gamma_{21}^1 \varphi_u + \Gamma_{21}^2 \varphi_v + f\nu \\ \varphi_{vv} &= \Gamma_{22}^1 \varphi_u + \Gamma_{22}^2 \varphi_v + g\nu \end{aligned}$$

for some coefficients Γ_{ij}^k , $i, j, k \in \{1, 2\}$. These coefficients are called *Christoffel symbols*²¹.

Proposition 158. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be a parametrization of S . Then:

$$\begin{aligned} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix} \\ \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{pmatrix} \\ \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} &= \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix} \end{aligned}$$

That is, the Christoffel symbols only depend on the coefficients of the first fundamental form.

Proposition 159. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be a parametrization of S . Then:

1. $(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 = -EK$
2. $(\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^1 = FK$
3. $(\Gamma_{22}^2)_u - (\Gamma_{12}^2)_v + \Gamma_{22}^1 \Gamma_{11}^2 - \Gamma_{12}^1 \Gamma_{12}^2 = -FK$
4. $(\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{12}^1 - \Gamma_{12}^1 \Gamma_{12}^1 - \Gamma_{12}^2 \Gamma_{22}^1 = GK$

These equations are called *Gauß equations*. Moreover, we have:

5. $e_v - f_u = e \Gamma_{12}^1 + f (\Gamma_{12}^2 - \Gamma_{11}^1) - g \Gamma_{11}^2$
6. $f_v - g_u = e \Gamma_{22}^1 + f (\Gamma_{22}^2 - \Gamma_{12}^1) - g \Gamma_{12}^2$

These equations are called *Codazzi-Mainardi equations*.

Corollary 160. Let $S \subseteq \mathbb{R}^3$ be an orientable surface. Then, its Gauß curvature depends only on the coefficients of the first fundamental form.

Theorem 161 (Gauß' Theorema Egregium). The Gauß curvature is invariant under local isometries between surfaces.

Theorem 162 (Bonnet's theorem). Let $V \subseteq \mathbb{R}^2$ be an open set and $E, F, G, e, f, g : V \rightarrow \mathbb{R}$ be functions of class $\mathcal{C}^\infty$ such that $E, G, EG - F^2 > 0$ and such that they satisfy Gauß and Codazzi-Mainardi equations. Then $\forall p \in V$, there exists a neighbourhood $U \subseteq V$ of p and an immersion $\varphi : U \rightarrow \mathbb{R}^3$ such that $S := \varphi(U)$ is a regular surface whose first and second fundamental forms coefficients are E, F, G and e, f, g , respectively. Moreover, if $\psi : U \rightarrow \mathbb{R}^3$ satisfy the same conditions, then there exist $\mathbf{A} \in \text{O}(3)$ and $\mathbf{c} \in \mathbb{R}^3$ such that $\psi = \mathbf{A}\varphi + \mathbf{c}$.

Proposition 163. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be an orthogonal parametrization of S . Then:

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right]$$

Parallel transport

Definition 164. Let $S \subseteq \mathbb{R}^3$ be a surface and $U \subseteq S$ be an open set. A *vector field* tangent to S defined on U is a correspondence $\mathbf{X}$ that at each point $p \in U$ it assigns a tangent vector $\mathbf{X}(p) =: \mathbf{X}_p \in T_p S$. We say that $\mathbf{X}$ is *differentiable* at $p \in U$ if there is a parametrization $\varphi(u, v)$ of S whose image contains p such that

$$\mathbf{X} = a\varphi_u + b\varphi_v$$

for some differentiable functions $a(u, v)$, $b(u, v)$ at p . We say that $\mathbf{X}$ is *differentiable* if it is differentiable at each point $p \in U$ ²².

Definition 165. Let $S \subseteq \mathbb{R}^3$ be a surface, $\mathbf{X}$ be a differentiable vector field tangent to S , $p \in S$ and $\mathbf{w} \in T_p S$. Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ a parametrized curve of class $\mathcal{C}^\infty$ with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{w}$. We denote $\mathbf{X}(t) := (\mathbf{X} \circ \alpha)(t)$. We define the *covariant derivative* of $\mathbf{X}$ at the point p in the direction of $\mathbf{w}$, denoted as $\frac{D\mathbf{X}}{dt}(0)$, as the orthogonal projection $\pi^\perp$ of $\mathbf{X}'(0)$ over the vector field $T_p S$. That is:

$$\frac{D\mathbf{X}}{dt}(0) = \pi^\perp(\mathbf{X}'(0))$$

Proposition 166. Let $S \subseteq \mathbb{R}^3$ be a surface, $\mathbf{X}$ be a differentiable vector field tangent to S , $p \in S$ and $\mathbf{w} \in T_p S$. Let $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ a parametrized curve of class $\mathcal{C}^\infty$ with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{w}$. Suppose $(V, \varphi(u, v))$ is a parametrization of S whose image contains p . Suppose $\mathbf{X}(t) = a(u(t), v(t))\varphi_u + b(u(t), v(t))\varphi_v = a(t)\varphi_u + b(t)\varphi_v$. Then:

$$\begin{aligned} \frac{D\mathbf{X}}{dt} &= (a' + \Gamma_{11}^1 a u' + \Gamma_{12}^1 a v' + \Gamma_{21}^1 b u' + \Gamma_{22}^1 b v') \varphi_u \\ &\quad + (b' + \Gamma_{11}^2 a u' + \Gamma_{12}^2 a v' + \Gamma_{21}^2 b u' + \Gamma_{22}^2 b v') \varphi_v \end{aligned} \quad (1)$$

Definition 167. Let $S \subseteq \mathbb{R}^3$ be a surface and $\alpha : I \rightarrow S$ be a curve of class $\mathcal{C}^\infty$. A *vector field* tangent to S along α is a correspondence $\mathbf{X}$ that at each $t \in U$ it assigns a tangent vector $\mathbf{X}(t) =: \mathbf{X}_{\alpha(t)} \in T_{\alpha(t)} S$. We say that $\mathbf{X}$ is *differentiable* if at each local chart $(V, \varphi(u, v))$ we have:

$$\mathbf{X}(t) = a(t)\varphi_u + b(t)\varphi_v$$

for some differentiable functions $a(t)$, $b(t)$. We define its covariant derivative (along α) as the vector field $\frac{D\mathbf{X}}{dt}$ defined by Eq. (1), which is differentiable along α .

Definition 168. Let $S \subseteq \mathbb{R}^3$ be a surface and $\mathbf{X}$ be a vector field tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. We say that $\mathbf{X}$ is *parallel* if:

$$\frac{D\mathbf{X}}{dt} = 0$$

Proposition 169. Let $S \subseteq \mathbb{R}^3$ be a surface and $\mathbf{X}, \mathbf{Y}$ be vector fields tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$ such that they are parallel. Then, $t \mapsto \langle \mathbf{X}(t), \mathbf{Y}(t) \rangle$ is constant. In particular, the norms $\|\mathbf{X}(t)\|$, $\|\mathbf{Y}(t)\|$ as well as the angle between $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are constant.

Proposition 170. Let $S \subseteq \mathbb{R}^3$ be a surface, $(V, \varphi(u, v))$ is a parametrization of S and $\alpha : I \rightarrow S$ be a parametrized curve of class $\mathcal{C}^\infty$ such that $\alpha = \varphi(u(t), v(t))$. Then, given $t_0 \in I$ and $\mathbf{w} \in T_{\alpha(t_0)} S$ there exists a unique parallel vector field $\mathbf{X} = a\varphi_u + b\varphi_v$ along α such that $\mathbf{X}(t_0) = \mathbf{w}$. This vector field is called *parallel transport* of the vector $\mathbf{w}$ along α , and it is defined on the entire interval I . It can be found by solving this system of ODEs:

$$\begin{cases} a' + \Gamma_{11}^1 a u' + \Gamma_{12}^1 a v' + \Gamma_{21}^1 b u' + \Gamma_{22}^1 b v' = 0 \\ b' + \Gamma_{11}^2 a u' + \Gamma_{12}^2 a v' + \Gamma_{21}^2 b u' + \Gamma_{22}^2 b v' = 0 \end{cases}$$

with initial conditions $a(t_0) = a_0$, $b(t_0) = b_0$ and $\mathbf{w} = a_0\varphi_u + b_0\varphi_v$.

²²From now on, all the vector fields considered will be differentiable, so sometimes we will omit to say it explicitly.

Geodesics

Definition 171. Let $S \subseteq \mathbb{R}^3$ be a surface and $\alpha I \rightarrow S$ be a parametrized curve of class $\mathcal{C}^\infty$. We say that α is a *geodesic* of S if the tangent vector α' is parallel along α . That is, if it satisfies:

$$\frac{D\alpha'}{dt} = 0$$

Proposition 172. Let α be a geodesic of a surface $S \subseteq \mathbb{R}^3$ be a surface. Then, $\|\alpha'\|$ is constant.

Proposition 173. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $(V, \varphi(u, v))$ be a local parametrization of S and $\alpha I \rightarrow S$ be a parametrized curve of class $\mathcal{C}^\infty$ with $\alpha^* \subset \varphi(V)$. Suppose $\alpha(t) = \varphi(u(t), v(t))$, for some differentiable functions $u, v : I \rightarrow \mathbb{R}$. Then, α is a geodesic of S if and only if:

$$\begin{cases} u'' + \Gamma_{11}^1(u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1(v')^2 = 0 \\ v'' + \Gamma_{11}^2(u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2(v')^2 = 0 \end{cases}$$

Proposition 174. Let $S \subseteq \mathbb{R}^3$ be a surface, $p \in S$ and $\mathbf{v} \in T_p S$. Then, there exists $\varepsilon > 0$ and a curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ of class $\mathcal{C}^\infty$ such that it is a geodesic, $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.

Geodesic curvature

Definition 175. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν and $\mathbf{X}$ be a unit vector field tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. We define the by $\bar{\mathbf{X}}$ the unique unit vector field along α such that $(\mathbf{X}, \bar{\mathbf{X}}, \nu)$ is a positive orthonormal basis of $\mathbb{R}^3$ ²³.

Definition 176. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν and $\mathbf{X}$ be a unit vector field tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. Since $\mathbf{X}$ is a unit field, we have:

$$\frac{D\mathbf{X}}{dt} = \lambda \bar{\mathbf{X}} = \lambda \nu \times \mathbf{X}$$

We define the *algebraic value* of the covariant derivative of $\mathbf{X}$ at time t as:

$$\left[\frac{D\mathbf{X}}{dt}(t) \right] := \lambda(t)$$
²⁴

Definition 177. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\alpha : I \rightarrow S$ be a regular arc-length parametrization of a curve C of class $\mathcal{C}^\infty$. The algebraic value of the covariant derivative of α' at $\alpha(s)$ is:

$$k_g(s) := \left[\frac{D\alpha'}{ds}(s) \right]$$

This value k_g is called *geodesic curvature* of C at $\alpha(s)$.

Proposition 178. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\alpha : I \rightarrow S$ be a regular arc-length parametrization of a curve C of class $\mathcal{C}^\infty$. Then:

$$C \text{ is a geodesic} \iff k_g = 0$$

Proposition 179. Let $S \subseteq \mathbb{R}^3$ be a surface oriented with a normal unit field ν and $\mathbf{X}$ be a unit vector field tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. Then:

$$\left[\frac{D\mathbf{X}}{dt}(t) \right] = \left\langle \frac{d\mathbf{X}}{dt}, \bar{\mathbf{X}} \right\rangle = \left\langle \frac{d\mathbf{X}}{dt}, \nu \times \mathbf{X} \right\rangle$$

In particular if α is arc-length parametrized, then:

$$k_g = \langle \alpha'', \nu \times \alpha' \rangle$$

Or more generally:

$$k_g = \frac{\langle \alpha'', \nu \times \alpha' \rangle}{\|\alpha'\|^3}$$

Proposition 180. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\alpha : I \rightarrow S$ be a regular arc-length parametrization of a curve C of class $\mathcal{C}^\infty$. Then:

$$k^2 = k_g^2 + k_n^2$$

Proposition 181. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\mathbf{X}, \mathbf{Y}$ be two unit vector fields tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. Then:

$$\left[\frac{D\mathbf{Y}}{dt} \right] - \left[\frac{D\mathbf{X}}{dt} \right] = \frac{d\theta}{dt}$$

where θ is a differentiable determination of the angle between $\mathbf{X}$ and $\mathbf{Y}$.

Corollary 182. Let $S \subseteq \mathbb{R}^3$ be an orientable surface, $\mathbf{X}$ be a parallel unit vector field along an arc-length parametrized curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$ and θ is a differentiable determination of the angle between $\mathbf{X}$ and α' . Then:

$$k_g(s) = \left[\frac{D\mathbf{X}}{ds}(s) \right]$$

Proposition 183. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be an orthogonal parametrization of S compatible with the orientation. Let $\mathbf{X}$ be a unit vector field tangent to S along a curve $\alpha : I \rightarrow S$ of class $\mathcal{C}^\infty$. Suppose $\alpha(t) = \varphi(u(t), v(t))$. Then:

$$\left[\frac{D\mathbf{X}}{dt} \right] = \frac{1}{2\sqrt{EG}} (G_u v' - E_v u') + \frac{d\theta}{dt}$$

where θ is the angle from φ_u to $\mathbf{X}$. In particular if the curve α is arc-length parametrized, then:

$$k_g = \frac{1}{2\sqrt{EG}} (G_u v' - E_v u') + \frac{d\theta}{ds}$$

Theorem 184 (Liouville's formula). Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $(V, \varphi(u, v))$ be an orthogonal parametrization of S compatible with the orientation. Let $\alpha : I \rightarrow S$ be an arc-length parametrized curve of class $\mathcal{C}^\infty$ such that $\alpha(t) = \varphi(u(t), v(t))$ and let $\theta = \theta(s)$ be the angle between φ_u and $\alpha'(s)$. Then:

$$k_g = (k_g)_1 \cos \theta + (k_g)_2 \sin \theta + \frac{d\theta}{ds}$$

where $(k_g)_1$ and $(k_g)_2$ denote the geodesic curvature of the coordinate lines $v = \text{const.}$ and $u = \text{const.}$, respectively.

²³Or equivalently such that $(\mathbf{X}, \bar{\mathbf{X}})$ is a positive orthonormal basis of $T_{\alpha(t)} S$.

²⁴Note that the sign of $\left[\frac{D\mathbf{X}}{dt}(t) \right]$ does depend on the orientation of ν .

Proposition 185. Let $S \subseteq \mathbb{R}^3$ be an orientable surface and $\alpha : I \rightarrow S$ be a parametrized curve of class $\mathcal{C}^\infty$. Then, α is a geodesic in a convenient parametrization if and only if:

$$\frac{D\alpha'}{dt} = \lambda(t)\alpha'(t)$$

6. | Differential forms

Vector fields of $\mathbb{R}^n$

Definition 186. Let $p \in \mathbb{R}^n$. We denote by $T_p\mathbb{R}^n$ the vector space defined by:

$$T_p\mathbb{R}^n := \{(p, \mathbf{v}) : \mathbf{v} \in \mathbb{R}^n\}$$

Definition 187. Let $U \subseteq \mathbb{R}^n$ be an open set. A *vector field* defined on U is a correspondence $\mathbf{X}$ that at each point $p \in U$ it assigns a vector $\mathbf{X}(p) =: \mathbf{X}_p \in T_p\mathbb{R}^n$.

Definition 188. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the standard basis of $\mathbb{R}^n$. For $i = 1, \dots, n$, we define the following vector fields:

$$\mathbf{E}_i := \frac{\partial}{\partial x^i} : \mathbb{R}^n \longrightarrow T_p\mathbb{R}^n \\ p \longmapsto (p, \mathbf{e}_i)$$

Proposition 189. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, for each $p \in \mathbb{R}^n$, $(\mathbf{E}_1, \dots, \mathbf{E}_n)$ is a basis of $T_p\mathbb{R}^n$. Consequently, given a vector field $\mathbf{X}$ defined on U , it can be uniquely written as:

$$\mathbf{X} = \sum_{i=1}^n X^i \mathbf{E}_i = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \quad ^{25}$$

Definition 190. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{X} = \sum X^i \frac{\partial}{\partial x^i}$ be a vector field defined on U . We say that $\mathbf{X}$ is *differentiable* at $p \in U$ if the components X^i are differentiable at p . We say that $\mathbf{X}$ is *differentiable* on U if it is differentiable at each point $p \in U$. We denote by $\mathcal{X}(U)$ the set of all differentiable vector fields defined on U ²⁶.

Definition 191. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{X}, \mathbf{Y} \in \mathcal{X}(U)$. We define the inner product of $\mathbf{X}$ and $\mathbf{Y}$ as:

$$\langle \mathbf{X}, \mathbf{Y} \rangle : U \longrightarrow \mathbb{R} \\ p \longmapsto \langle \mathbf{X}_p, \mathbf{Y}_p \rangle$$

Definition 192. Let $U \subseteq \mathbb{R}^n$ be an open set, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X} = \sum X^i \frac{\partial}{\partial x^i} \in \mathcal{X}(U)$. We denote by $\mathbf{X}f$ the differentiable function defined as:

$$\mathbf{X}f(p) := \mathbf{X}_p f := \sum_{i=1}^n X^i(p) \frac{\partial f}{\partial x^i} \quad ^{27}$$

Lemma 193. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{X} \in \mathcal{X}(U)$. Then, the function

$$\mathbf{X} : \mathcal{C}^\infty(U) \longrightarrow \mathcal{C}^\infty(U) \\ f \longmapsto \mathbf{X}f$$

satisfies the following properties:

²⁵The superscript notation is used due to historical reasons (Einstein notation). See https://en.wikipedia.org/wiki/Einstein_notation for further information.

²⁶Note that $\mathcal{X}(U)$ is a $\mathbb{R}$ -vector space.

²⁷Observe that $\mathbf{X}_p f$ is the partial derivative of f at p in the direction of $\mathbf{X}_p$.

1. It is $\mathbb{R}$ -linear.

2. $\mathbf{X}(fg) = (\mathbf{X}f)g + f(\mathbf{X}g)$ for all $f, g \in \mathcal{C}^\infty(U)$.

Lemma 194. Let $U \subseteq \mathbb{R}^n$ be an open set, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$\mathbf{X}f = \langle \mathbf{X}, \nabla f \rangle$$

Definition 195. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{X} = \sum X^i \frac{\partial}{\partial x^i} \in \mathcal{X}(U)$. We say that a parametrized curve $\gamma : I \rightarrow \mathbb{R}^n$ is an *integral curve* of $\mathbf{X}$ if:

$$\gamma'(t) = \mathbf{X}(\gamma(t)) \quad \forall t \in I$$

That is, the integral curve $\gamma(t) = (x^1(t), \dots, x^n(t))$ of $\mathbf{X}$ satisfies the following system of ODEs:

$$\begin{cases} (x^1)' = X^1(\gamma(t)) \\ \vdots \\ (x^n)' = X^n(\gamma(t)) \end{cases}$$

Proposition 196. Let $U \subseteq \mathbb{R}^n$ be an open set, $F \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. We say that F is a *first integral* of $\mathbf{X}$ if:

1. $dF_p \neq 0 \quad \forall p \in U$.

2. F is constant over the integral curves of $\mathbf{X}$. That is, $\mathbf{X}F = 0$.

Proposition 197. Let $n \geq 2$, $U \subseteq \mathbb{R}^n$ be an open set, $p \in U$ and $\mathbf{X} \in \mathcal{X}(U)$. Suppose that $\mathbf{X}_p \neq 0$. Then, there exists a neighbourhood $V \subseteq U$ of p and a differential function $F : V \rightarrow \mathbb{R}$ such that F is a first integral of $\mathbf{X}$.

Definition 198. Let $S \subseteq \mathbb{R}^3$ be a regular surface. A *vector field* defined on S is a correspondence $\mathbf{X}$ that at each point $p \in S$ it assigns a vector $\mathbf{X}(p) =: \mathbf{X}_p \in T_p\mathbb{R}^3$. If there is a parametrization $\varphi(u, v)$ of S , we can write

$$\mathbf{X} = \mathbf{X}(u, v) = \sum_{i=1}^3 X^i(u, v) \frac{\partial}{\partial x^i}$$

where $X^i(u, v) := (X^i \circ \varphi)(u, v)$. We say that $\mathbf{X}$ is *differentiable* if the functions $X^i(u, v)$ are differentiable. We say that $\mathbf{X}$ is *tangent* to S if $\mathbf{X}_p \in T_p S \quad \forall p \in S$. In this case we can write:

$$\mathbf{X} = \tilde{X}^1 \varphi_u + \tilde{X}^2 \varphi_v$$

Proposition 199. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $\mathbf{X}, \mathbf{Y}$ be tangent differential vector fields to S such that at some point $p \in S$, the vectors $\mathbf{X}_p, \mathbf{Y}_p$ are linearly independent. Then, there exists a local parametrization $(V, \varphi(u, v))$ of S such that $p \in \varphi(V)$ and

$$\mathbf{X} = \lambda \varphi_u \quad \mathbf{Y} = \mu \varphi_v$$

for some differentiable functions λ, μ .

Corollary 200. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. Then, there exists a local orthogonal parametrization $(V, \varphi(u, v))$ of S such that $p \in \varphi(V)$.

Multilinear algebra

Definition 201. Let $V_1, \dots, V_k, W$ be a vector spaces over a field K . A *multilinear map* (or *k-linear map*) is a function

$$f : V_1 \times \dots \times V_k \longrightarrow W$$

that is linear separately in each variable. The value k is called *degree* of the multilinear map.

Definition 202. Let V be a vector space of dimension n and $\omega : V \times \dots \times V \longrightarrow \mathbb{R}$ be a k -linear map. We say that ω is *alternating* if

$$\omega(\mathbf{u}_{\sigma(1)}, \dots, \mathbf{u}_{\sigma(k)}) = \text{sgn}(\sigma) \omega(\mathbf{u}_1, \dots, \mathbf{u}_k) \quad \forall \sigma \in S_k$$

We denote by $\Lambda^k V^*$ the vector space of the alternating k -linear maps. The elements of $\Lambda^k V^*$ are called *multilinear forms*²⁸. By agreement, we denote $\Lambda^0 V^* := \mathbb{R}$ and:

$$\Lambda^* V^* := \bigoplus_{k=0}^n \Lambda^k V^*$$

Definition 203. Let V be a vector space, $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$. We define its *exterior product* as the multilinear map $\alpha \wedge \beta$ defined as:

$$\alpha \wedge \beta(\mathbf{u}_1, \dots, \mathbf{u}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \cdot \alpha(\mathbf{u}_{\sigma(1)}, \dots, \mathbf{u}_{\sigma(p)}) \beta(\mathbf{u}_{\sigma(p+1)}, \dots, \mathbf{u}_{\sigma(p+q)})$$

Proposition 204. Let V be a vector space and $\alpha \in \Lambda^p V^*$, $\beta \in \Lambda^q V^*$ and $\gamma \in \Lambda^r V^*$. Then:

1. $\alpha \wedge \beta \in \Lambda^{p+q} V^*$ (that is, $\alpha \wedge \beta$ is alternating)
2. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$
3. $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

Proposition 205. Let V be a vector space and $\omega^1, \dots, \omega^k \in V^* = \Lambda^1 V^*$. Then:

$$\omega^1 \wedge \dots \wedge \omega^k(\mathbf{u}_1, \dots, \mathbf{u}_k) = \begin{vmatrix} \omega^1(\mathbf{u}_1) & \dots & \omega^1(\mathbf{u}_k) \\ \vdots & \ddots & \vdots \\ \omega^k(\mathbf{u}_1) & \dots & \omega^k(\mathbf{u}_k) \end{vmatrix}$$

Corollary 206. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the standard basis of $\mathbb{R}^n$ and $(\mathbf{e}_1^*, \dots, \mathbf{e}_n^*)$ be its associated dual basis. Then:

$$\mathbf{e}_1^* \wedge \dots \wedge \mathbf{e}_n^*(\mathbf{u}_1, \dots, \mathbf{u}_n) = \det(\mathbf{u}_1, \dots, \mathbf{u}_n)$$

Proposition 207. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the standard basis of $E := \mathbb{R}^n$ and $(\mathbf{e}_1^*, \dots, \mathbf{e}_n^*)$ be its associated dual basis. Then, given $\omega \in \Lambda^k E^*$ we have that:

$$\omega(\mathbf{u}_1, \dots, \mathbf{u}_k) = \sum_{j_1 < \dots < j_k} A_{j_1, \dots, j_k} \omega(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_k})$$

where $A_{j_1, \dots, j_k} := \mathbf{e}_{j_1}^* \wedge \dots \wedge \mathbf{e}_{j_k}^*(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Thus:

$$\omega = \sum_{j_1 < \dots < j_k} \omega(\mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_k}) \mathbf{e}_{j_1}^* \wedge \dots \wedge \mathbf{e}_{j_k}^*$$

²⁸Here V^* denotes the dual space of V (see ??).

²⁹An element $\omega(p) =: \omega_p$ can be thought as an element of $\Lambda^k(T_p \mathbb{R}^n)^*$.

³⁰In particular, note that $dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{ij}$.

Corollary 208. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the standard basis of $E := \mathbb{R}^n$ and $(\mathbf{e}_1^*, \dots, \mathbf{e}_n^*)$ be its associated dual basis. Then, the set

$$\{\mathbf{e}_{j_1}^* \wedge \dots \wedge \mathbf{e}_{j_k}^* : j_1 < \dots < j_k, j_i \in \mathbb{N} \forall i = 1, \dots, k\}$$

is a basis of $\Lambda^k E^*$. In particular, $\dim \Lambda^k E^* = \binom{n}{k}$.

Corollary 209. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ and $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be the standard basis and an arbitrary basis of $E := \mathbb{R}^n$ with associated dual basis $(\mathbf{e}_1^*, \dots, \mathbf{e}_n^*)$ and $(\mathbf{v}_1^*, \dots, \mathbf{v}_n^*)$, respectively. Suppose $\mathbf{v}_i = \sum_{j=1}^n a_{ij} \mathbf{e}_j$ for $i = 1, \dots, n$ and define $\mathbf{A} := (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$. Then:

$$\mathbf{v}_1^* \wedge \dots \wedge \mathbf{v}_n^* = \frac{1}{\det \mathbf{A}} \mathbf{e}_1^* \wedge \dots \wedge \mathbf{e}_n^*$$

Proposition 210. Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ and P be the parallelepiped they generate. Then:

$$\text{vol } P = |\det(\mathbf{v}_1, \dots, \mathbf{v}_n)|$$

Differential forms

Definition 211. Let $U \subseteq \mathbb{R}^n$ be an open set. A *differential k-form* on U is a differentiable function $\omega : U \rightarrow \Lambda^k(\mathbb{R}^n)^* \cong \mathbb{R}^{\binom{n}{k}}$ ²⁹. We will denote the dual basis of $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right)$ by $(dx^1, \dots, dx^n)$. Thus, a differential k -form can be written as:

$$\omega = \sum_{j_1 < \dots < j_k} \omega_{j_1, \dots, j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

We denote by $\Omega^k(U)$ the set of all differential k -forms defined on U with the agreement that $\Omega^0(U) := \mathcal{C}^\infty(U)$.

Proposition 212. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, $\Omega^k(U)$ is a $\mathbb{R}$ -vector space.

Definition 213. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega \in \Omega^k(U)$. This form ω defines a $\mathcal{C}^\infty(U)$ -multilinear alternating function, which we'll denote also by ω , given by:

$$\omega : \mathcal{X}(U) \times \dots \times \mathcal{X}(U) \longrightarrow \mathcal{C}^\infty(U) \\ (\mathbf{X}_1, \dots, \mathbf{X}_k) \longmapsto \omega(\mathbf{X}_1, \dots, \mathbf{X}_k)$$

where $\omega(\mathbf{X}_1, \dots, \mathbf{X}_k)$ is the function defined by:

$$\omega(\mathbf{X}_1, \dots, \mathbf{X}_k)(p) = \omega_p((\mathbf{X}_1)_p, \dots, (\mathbf{X}_k)_p)$$

Definition 214. Let $U \subseteq \mathbb{R}^n$ be an open set and $h \in \mathcal{C}^\infty$. We define its *differential* as the differential 1-form dh given by:

$$dh = \sum_{i=1}^n \frac{\partial h}{\partial x^i} dx^i$$

Proposition 215. Let $U \subseteq \mathbb{R}^n$ be an open set, $h \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$dh(\mathbf{X}) = \mathbf{X}h^{30}$$

Definition 216. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets, $\mathbf{f} : U \rightarrow V$ be a differentiable function and $k \geq 1$. We define the *pull-back* by $\mathbf{f}$ as the linear function

$$\mathbf{f}^* : \Omega^k(V) \rightarrow \Omega^k(U)$$

defined by:

$$(\mathbf{f}^*\omega)(\mathbf{X}_1, \dots, \mathbf{X}_k)(p) = \omega_{f(p)}(\mathbf{df}_p(\mathbf{X}_1), \dots, \mathbf{df}_p(\mathbf{X}_k))$$

If $k = 0$, we define *pull-back* by $\mathbf{f}$ as $\mathbf{f}^*h = h \circ \mathbf{f}$.

Proposition 217. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^r$ be open sets, $\mathbf{f} : U \rightarrow V$, $\mathbf{g} : V \rightarrow W$ be differentiable functions, $\omega, \eta \in \Omega^k(V)$, $h \in \Omega^0(V)$ and $a, b \in \mathbb{R}$. Then:

1. $\mathbf{f}^*(a\omega + b\eta) = a\mathbf{f}^*\omega + b\mathbf{f}^*\eta$
2. $\mathbf{f}^*(\omega \wedge \eta) = \mathbf{f}^*\omega \wedge \mathbf{f}^*\eta$
3. $(\mathbf{g} \circ \mathbf{f})^* = \mathbf{f}^* \circ \mathbf{g}^*$
4. $\mathbf{f}^*dh = d(h \circ \mathbf{f}) = d(\mathbf{f}^*h)$

Corollary 218. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets, $\mathbf{f} = (f^1, \dots, f^m) : U \rightarrow V$ be a differentiable function, $\omega \in \Omega^k(V)$ and $h \in \Omega^0(V)$. Then:

1. $\mathbf{f}^*(h\omega) = (h \circ \mathbf{f})\mathbf{f}^*\omega$
2. $\mathbf{f}^*dx^j = d(x^j \circ \mathbf{f}) = df^j \forall j \in \{1, \dots, m\}$
3. If $\omega = \sum_{j_1 < \dots < j_k} \omega_{j_1, \dots, j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$, then:

$$\mathbf{f}^*\omega = \sum_{j_1 < \dots < j_k} (\omega_{j_1, \dots, j_k} \circ \mathbf{f}) df^{j_1} \wedge \dots \wedge df^{j_k}$$

Definition 219. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{X} \in \mathcal{X}(U)$ and $\omega \in \Omega^k(U)$, where $k \geq 1$. We define the *interior product* of ω by $\mathbf{X}$ as the differential $(k-1)$ -form $\iota_{\mathbf{X}}\omega \in \Omega^{k-1}(U)$ defined by:

$$\iota_{\mathbf{X}}\omega(\mathbf{Y}_2, \dots, \mathbf{Y}_k) = \omega(\mathbf{X}, \mathbf{Y}_2, \dots, \mathbf{Y}_k)$$

By agreement, we define $\iota_{\mathbf{X}}h = 0$ if $h \in \Omega^0(U)$.

Proposition 220. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{X} \in \mathcal{X}(U)$, $\alpha \in \Omega^k(U)$, and $\beta \in \Omega^m(U)$. Then:

$$\iota_{\mathbf{X}}(\alpha \wedge \beta) = (\iota_{\mathbf{X}}\alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_{\mathbf{X}}\beta)$$

Definition 221. The differential n -form $\eta \in \Omega^n(\mathbb{R}^n)$ defined by

$$\eta = dx^1 \wedge \dots \wedge dx^n$$

is called *volume element* of $\mathbb{R}^n$.

Lemma 222. Let $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ be an orthonormal positive basis of $T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$, and let η be the volume element. Then:

$$\eta_p(\mathbf{u}_1, \dots, \mathbf{u}_n) = 1$$

Proposition 223. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \rightarrow V$ be a differentiable function. Then:

$$\mathbf{f}^*\eta = J\mathbf{f} \cdot \eta$$

where $J\mathbf{f}$ is the Jacobian of $\mathbf{f}$.

Definition 224. Let $U \subseteq \mathbb{R}^n$ be an open set and $k \geq 0$. We define the *exterior differential* as the linear function $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ defined as follows: If

$$\omega = \sum_{j_1 < \dots < j_k} \omega_{j_1, \dots, j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

then:

$$d\omega := \sum_{m, j_1 < \dots < j_k} \frac{\partial \omega_{j_1, \dots, j_k}}{\partial x^m} dx^m \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

Proposition 225. Let $U, V \subseteq \mathbb{R}^n$ be open sets, $\mathbf{f} : U \rightarrow V$ be a differentiable function and $\alpha \in \Omega^k(U)$, $\beta \in \Omega^m(U)$. Then:

1. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
2. $d^2 = 0$
3. $\mathbf{f}^* \circ d = d \circ \mathbf{f}^*$

Definition 226. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega \in \Omega^k(U)$. We say that ω is *closed* if $d\omega = 0$.

Definition 227. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega \in \Omega^k(U)$. We say that ω is *exact* if $\exists \eta \in \Omega^{k-1}(U)$ such that $d\eta = \omega$.

7. | Integration

Submanifolds of $\mathbb{R}^n$

Definition 228. Let $M \subseteq \mathbb{R}^n$ be a submanifold and $p \in M$. If $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$ is a parametrization of a curve of class $\mathcal{C}^\infty$ such that $\alpha(0) = p$, we say that $\alpha'(0)$ is a *tangent vector* to M at p . The set of all such vectors is called *tangent space* to M at p , and it is denoted as T_pM . Moreover, T_pM is a vector space of dimension $\dim M$.

Definition 229. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k and $U \subseteq S$ be an open set. A *vector field* defined on U is a correspondence $\mathbf{X}$ that at each point $p \in U$ it assigns a tangent vector $\mathbf{X}(p) =: \mathbf{X}_p \in T_p\mathbb{R}^n$. We say that $\mathbf{X}$ is *differentiable* at $p \in U$ if there is a parametrization $\varphi(u^1, \dots, u^k)$ of M whose image contains p such that

$$\mathbf{X} = \sum X^i \varphi_{u^i}$$

for some functions $X^1, \dots, X^k$ differentiable at p . We say that $\mathbf{X}$ is *differentiable* if it is differentiable at each point $p \in U$. We say that $\mathbf{X}$ is *tangent* to M if $\mathbf{X}_p \in T_pM \forall p \in U$. We denote by $\mathcal{X}(U)$ the set of all differentiable vector fields on U that are tangent to M .

Definition 230. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k and $U \subseteq M$ be an open set and $(V, \varphi(u^1, \dots, u^k))$ be a local parametrization of M with $\varphi(V) = U$. A *differential ℓ -form* on U is a differentiable function $\omega : U \rightarrow \Lambda^\ell(T_pM)^* \cong \mathbb{R}^{\binom{k}{\ell}}$. We denote by $\Omega^\ell(U)$ the set of all differential ℓ -forms defined on U with the agreement that $\Omega^0(U) := \mathcal{C}^\infty(U)$ ³¹. We will denote the dual basis of $(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^k}) := (\frac{\partial \varphi}{\partial u^1}, \dots, \frac{\partial \varphi}{\partial u^k})$ by $(du^1, \dots, du^k)$. Thus, a differential ℓ -form can be written uniquely as:

$$\omega = \sum_{j_1 < \dots < j_\ell} \omega_{j_1, \dots, j_\ell} du^{j_1} \wedge \dots \wedge du^{j_\ell}$$

³¹All the definitions made in the previous section about differential forms can be applied, conveniently modified, to submanifolds.

Definition 231. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k and $U \subseteq M$ be an open set, $(V, \varphi(u^1, \dots, u^k))$ be a local parametrization of M with $\varphi(V) = U$ and $\omega \in \Omega^\ell(U)$. We define the *exterior differential* as the linear function $d : \Omega^\ell(U) \rightarrow \Omega^{\ell+1}(U)$ defined as follows: if

$$\omega = \sum_{j_1 < \dots < j_\ell} \omega_{j_1, \dots, j_\ell} dx^{j_1} \wedge \dots \wedge dx^{j_\ell}$$

then:

$$d\omega = \sum_{m, j_1 < \dots < j_\ell} \frac{\partial \omega_{j_1, \dots, j_\ell}}{\partial x^m} dx^m \wedge dx^{j_1} \wedge \dots \wedge dx^{j_\ell}$$

Proposition 232. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k and $U \subseteq \mathbb{R}^n$ be an open set and $\omega \in \Omega^\ell(U)$. Then, ω induces a differential form $\omega_M \in \Omega^\ell(V)$, where $V = U \cap M$, defined by:

$$\omega_M(\mathbf{X}_1, \dots, \mathbf{X}_\ell)(p) = \omega_p((\mathbf{X}_1)_p, \dots, (\mathbf{X}_\ell)_p)$$

for all $p \in V$ and all $\mathbf{X}_i \in \mathcal{X}(V)$. The expression in the coordinates $u^1, \dots, u^k$ from a parametrization φ is: $\omega_M = \varphi^* \omega$.

Manifolds with boundary

Definition 233. Let $k \in \mathbb{N}$. We define the set $\mathbb{H}^k$ as:

$$\mathbb{H}^k := \{(x^1, \dots, x^k) \in \mathbb{R}^k : x^k \geq 0\}$$

Note that $\partial \mathbb{H}^k = \{(x^1, \dots, x^k) \in \mathbb{R}^k : x^k = 0\}$.

Definition 234. Let $U \subseteq \mathbb{H}^k$ be an open set and $\mathbf{f} : U \rightarrow \mathbb{R}^m$ be a function. We say that $\mathbf{f}$ is *differentiable* if $\forall p \in U$ there exists a neighbourhood $W \subseteq \mathbb{R}^k$ of p and a differentiable function $\tilde{\mathbf{f}} : W \rightarrow \mathbb{R}^m$ such that $\tilde{\mathbf{f}}|_{V \cap W} = \mathbf{f}|_{V \cap W}$. In this case, we define the *differential* of $\mathbf{f}$ at a point $p \in U$ as $d\mathbf{f}_p = d\tilde{\mathbf{f}}_p$.

Definition 235. Let $M \subseteq \mathbb{R}^n$ be a set. We say that M is a *submanifold with boundary* of dimension k if $\forall p \in M$ there is an open neighbourhood $U \subseteq \mathbb{R}^n$ of p , an open set $V \subset \mathbb{H}^k$ and a differentiable function $\varphi : V \rightarrow \mathbb{R}^n$ such that:

1. $\varphi(V) = U \cap M$ and $\varphi : V \rightarrow U \cap M$ is a homeomorphism.
2. φ is an immersion.

In these conditions, the pair (V, φ) is called *local parametrization* of M .

Proposition 236. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k , (V, φ) be a local parametrization of it, $W \subseteq \mathbb{R}^m$ be an open set and $\mathbf{f} : W \rightarrow \mathbb{R}^n$ be a differentiable function such that $\mathbf{f}(W) \subseteq M$. Then, $\varphi^{-1} \circ \mathbf{f}$ is differentiable.

Lemma 237. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k , (V, φ) be a local parametrization of it, $x \in V \cap \partial \mathbb{H}^k$ and $p := \varphi(x)$. Then the set

$$\{\alpha'(0) : \alpha : [0, \varepsilon) \rightarrow M \text{ is differentiable with } \alpha(0) = p\}$$

is equal to the image $d\varphi_x(\mathbb{H}^k) \subseteq T_p \mathbb{R}^n$.

³²From now on the submanifolds may be with boundary or not.

Definition 238. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k . We say that $p \in M$ is an *interior point* of M if there exists a local parametrization (V, φ) of M such that $p \in \varphi(V \setminus \partial \mathbb{H}^k)$. We call *boundary* of M the set:

$$\partial M = M \setminus \{p \in M : p \text{ is interior}\}$$

Proposition 239. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k . Then, ∂M is a manifold (without boundary) of dimension $k - 1$.

Proposition 240. Let $M \subseteq \mathbb{R}^n$ be a set. M is a submanifold with boundary of dimension n if and only if $\forall p \in M$ there exists an open neighbourhood $U \subseteq \mathbb{R}^n$ of p and a function $F : U \rightarrow \mathbb{R}$ such that:

1. $U \cap M = \{x \in U : F(x) \leq 0\}$
2. F is a submersion on the points of $F^{-1}(0)$.

Definition 241. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k , $p \in M$ and $(V, \varphi(u^1, \dots, u^k))$ be a local parametrization of M such that $p \in \varphi(V)$. We define the *tangent space* of M at p as the following vector subspace of $T_p \mathbb{R}^n$:

$$T_p M := \langle (\varphi_{u^1})_p, \dots, (\varphi_{u^k})_p \rangle$$

Definition 242. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k , $p \in \partial M$. We define the *set of interior vectors* $T_p^i M$ of $T_p M$ as the set:

$$\{\alpha'(0) : \alpha : [0, \varepsilon) \rightarrow M \text{ is differentiable with } \alpha(0) = p\}$$

Note that $T_p^i M$ is a closed subspace of $T_p M$ and its boundary is $T_p \partial M \cong \mathbb{R}^{k-1}$. The elements of $T_p M \setminus T_p^i M$ are called *exterior vectors*.

Lemma 243. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k , $p \in M$ and $(V, \varphi(u^1, \dots, u^k))$ be a local parametrization of M such that $p \in \varphi(V)$. A vector $\mathbf{w} \in T_p M$ can be written as:

$$\mathbf{w} = a^1 \left(\frac{\partial \varphi}{\partial u^1} \right)_p + \dots + a^k \left(\frac{\partial \varphi}{\partial u^k} \right)_p$$

Then:

$$\begin{cases} \mathbf{w} \text{ is interior} & \iff a^k \geq 0 \\ \mathbf{w} \text{ is exterior} & \iff a^k < 0 \end{cases}$$

Orientability

Definition 244. Let $M \subseteq \mathbb{R}^n$ be a submanifold³² of dimension k . We say that M is *orientable* if we can assign an orientation to each tangent space $T_p M$ of M in a way that $\forall p \in M$, there is a local parametrization (V, φ) of M such that $p \in \varphi(V)$ and $d\varphi_x : T_x \mathbb{R}^k \rightarrow T_{\varphi(x)} M$ is positively-oriented (or negatively-oriented). In these conditions, we say that (V, φ) is *compatible* with the orientation of M .

Proposition 245. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k . M is orientable if and only if there exists an atlas $\{(V_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that $\forall \alpha, \beta \in A$ we have:

$$J(\varphi_\beta^{-1} \circ \varphi_\alpha) > 0$$

Proposition 246. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k . Then, there exists a unique differential k -form $\eta_M \in \Omega^k(M)$ such that if $(\mathbf{e}_1, \dots, \mathbf{e}_k)$ is an orthonormal positive basis of $T_p M$, then $\eta_M(\mathbf{e}_1, \dots, \mathbf{e}_k) = 1$. This form η_M is called *volume element* of M .

Proposition 247. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k . M is orientable if and only if there exists a differential k -form $\eta \in \Omega^k(M)$ that isn't zero at each point of M .

Lemma 248. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k with $\partial M \neq \emptyset$. Then, there is a global exterior vector field ν defined on ∂M , that is, a vector field such that $\forall p \in \partial M$, $\nu_p \notin T_p M$ (in particular $\nu_p \neq 0$).

Definition 249. Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k with $\partial M \neq \emptyset$. We call *unit normal exterior vector field* the field $\nu_{\partial M}$ univocally determined by being unit, exterior and perpendicular to $T_p \partial M$ at each point $p \in M$.

Proposition 250. Let $M \subseteq \mathbb{R}^n$ be a submanifold with boundary of dimension k . If M is orientable, so it is ∂M .

Definition 251. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k with $\partial M \neq \emptyset$. We say that a basis $(\mathbf{e}_1, \dots, \mathbf{e}_{k-1})$ of $T_p \partial M$ is *positive* if $(\nu_{\partial M}, \mathbf{e}_1, \dots, \mathbf{e}_{k-1})$ is a positive basis of $T_p M$. This choice determines an orientation on ∂M , which is called *orientation induced by M* .

Proposition 252. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k with $\partial M \neq \emptyset$, η_M be the volume element of M and $\nu_{\partial M}$ be the unit normal exterior vector field. Then, the volume element of ∂M associated with the orientation of ∂M induced by the one of M is:

$$\eta_{\partial M} = \iota_{\nu_{\partial M}} \eta_M$$

Proposition 253. Let $S \subseteq \mathbb{R}^3$ be a regular surface oriented with a vector field ν_S . Then, the area element of S is given by $\eta_S = \iota_{\nu_S} \eta$, where $\eta = dx \wedge dy \wedge dz$. Moreover, if $\varphi(u, v)$ is a local parametrization of S compatible with the orientation, then:

$$\eta_S = \varphi^* \eta_S = \sqrt{E_\varphi G_\varphi - F_\varphi^2} du \wedge dv$$

Integration of differential forms

Definition 254. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k and $\omega \in \Omega^k(M)$. We define the support of ω as:

$$\text{supp}(\omega) := \overline{\{p \in M : \omega_p \neq 0\}}$$

Definition 255. Let $U \subseteq \mathbb{R}^k$ be an open set and $\omega = h du^1 \wedge \dots \wedge du^k \in \Omega^k(\mathbb{R}^k)$ where $h = h(u^1, \dots, u^k)$ and such that $\text{supp}(\omega) \subset U$ is compact. We define the *integral* of ω on U as:

$$\int_U \omega = \int_U h du^1 \wedge \dots \wedge du^k := \int_U h du^1 \dots du^k$$

Definition 256. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k , $(U, \varphi(u^1, \dots, u^k))$ be a local parametrization of M compatible with the orientation and $\omega \in \Omega^k(M)$ be such that $\text{supp}(\omega) \subset \varphi(U)$ is compact. We define the *integral* of ω on M as:

$$\int_M \omega := \int_U \varphi^* \omega = \int_U \omega \left(\frac{\partial \varphi}{\partial u^1}, \dots, \frac{\partial \varphi}{\partial u^k} \right) du^1 \dots du^k \quad 33$$

If $h : U \rightarrow M$ is a differentiable function and $\text{supp}(h) \subset \varphi(U)$, then we define the *integral* of h on M as:

$$\int_M h := \int_M h \eta_M = \int_U (h \circ \varphi) \varphi^* \eta_M$$

where η_M is the volume element of M .

Definition 257. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k , $(U, \varphi(u^1, \dots, u^k))$ be a local parametrization of M compatible with the orientation and $R \subset M$ be a compact region contained in $\varphi(U)$. Let $Q := \varphi^{-1}(R) \subset U$. We define the volume of R as:

$$\text{vol}(R) = \int_R \eta_M := \int_Q \varphi^* \eta_M$$

Proposition 258. Let $K \subseteq \mathbb{R}^n$ be a compact set, $\{V_\alpha : \alpha \in A\}$ be an open cover of K . Then, there exist differentiable and non-negative functions $\rho_1, \dots, \rho_m \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that:

1. $\sum_{i=1}^m \rho_i(x) = 1 \quad \forall x \in K$
2. For all $i \in \{1, \dots, m\}$, $\exists \alpha \in A$ such that $\text{supp}(\rho_i) \subset V_\alpha$

In these conditions the set $\{\rho_i : i = 1, \dots, m\}$ is called a *partition of unity* of K subordinated to $\{V_\alpha : \alpha \in A\}$.

Definition 259. Let $k \geq 1$ and $M \subseteq \mathbb{R}^n$ be an orientated submanifold. We define the set $\Omega_c^k(M)$ as the vector space of all differential k -forms of M with compact support.

Definition 260. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k , $\{(V_\alpha, \varphi_\alpha) : \alpha \in A\}$ and an atlas of M compatible with the orientation. Given $\omega \in \Omega_c^k(M)$, let $\{\rho_1, \dots, \rho_m\}$ be a partition of unity of K subordinated to $\{\varphi(V_\alpha)\}$. We define the *integral* of ω on M as:

$$\int_M \omega = \sum_{i=1}^m \int_M \rho_i \omega = \sum_{i=1}^m \int_{U_\alpha} \varphi_\alpha^* (\rho_i \omega)$$

where $\alpha \in A$ is such that $\text{supp}(\rho_i) \subset \varphi(V_\alpha)$ ³⁴.

³³It can be seen that this definition does not depend on the parametrization $(U, \varphi(u^1, \dots, u^k))$.

³⁴It can be seen that this definition doesn't depend on either the atlas or the partition of unity chosen.

Proposition 261. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k , $\{(V_i, \varphi_i) : i = 1, \dots, m\}$ be a finite set of parametrizations of M compatibles with the orientation and such that:

1. $M \setminus \bigcup_{i=1}^m \varphi_i(U_i)$ is union of submanifolds of dimension less than k .
2. $\varphi_i(U_i) \cap \varphi_j(U_j) = \emptyset \ \forall i \neq j$.

Then, given $\omega \in \Omega_c^k(M)$, we have:

$$\int_M \omega = \sum_{i=1}^m \int_{U_i} \varphi_i^* \omega$$

Theorem 262 (Change of variables). Let $M \subseteq \mathbb{R}^n$, $M' \subseteq \mathbb{R}^m$ be two orientated submanifolds of dimension k , $\mathbf{F} : M \rightarrow M'$ be an orientation-preserving diffeomorphism and $\omega \in \Omega_c^k(M')$. Then:

$$\int_{M'} \omega = \int_M \mathbf{F}^* \omega$$

Stokes theorem

Proposition 263. Let $\omega \in \Omega_c^{k-1}(\mathbb{H}^k)$ and consider $\partial \mathbb{H}^k$ with the orientation induced by the one of $\mathbb{H}^k$. Then:

1. If $\text{supp } \omega \cap \partial \mathbb{H}^k = \emptyset$, then $\int_{\mathbb{H}^k} d\omega = 0$.
2. If $\text{supp } \omega \cap \partial \mathbb{H}^k \neq \emptyset$, then $\int_{\mathbb{H}^k} d\omega = \int_{\partial \mathbb{H}^k} \omega$.

Theorem 264 (Stokes theorem). Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k and $\omega \in \Omega_c^{k-1}(M)$. Then:

$$\int_M d\omega = \int_{\partial M} \omega$$

Corollary 265. Let $M \subseteq \mathbb{R}^n$ be an orientated submanifold of dimension k with $\partial M = \emptyset$ and $\omega \in \Omega_c^{k-1}(M)$. Then:

$$\int_M d\omega = 0$$

Corollary 266 (Green's formula). Let $D \subseteq \mathbb{R}^2$ be a regular domain (manifold of dimension 2 with boundary) and $P, Q : D \rightarrow \mathbb{R}$ be differentiable functions. Then:

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} P dx + Q dy$$

Vector calculus

Definition 267. Let $U \subseteq \mathbb{R}^3$, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. We define the following differential forms on U :

$$\begin{aligned} \omega_{\mathbf{X}}^1 &= X^1 dx + X^2 dy + X^3 dz \\ \omega_{\mathbf{X}}^2 &= X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy \\ \omega_f^3 &= f dx \wedge dy \wedge dz \end{aligned}$$

Lemma 268. Let $U \subseteq \mathbb{R}^3$, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{X}(U)$. Then:

1. $\omega_{\mathbf{X}}^1(\mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle$
2. $\omega_{\mathbf{X}}^2(\mathbf{Y}, \mathbf{Z}) = \langle \mathbf{X}, \mathbf{Y} \times \mathbf{Z} \rangle = \det(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$
3. $\omega_f^3(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = f \det(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$

Proposition 269. Let $U \subseteq \mathbb{R}^3$, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$df = \omega_{\nabla f}^1 \quad d\omega_{\mathbf{X}}^1 = \omega_{\text{rot } \mathbf{X}}^2 \quad d\omega_{\mathbf{X}}^2 = \omega_{\text{div } \mathbf{X}}^3$$

Corollary 270. Let $U \subseteq \mathbb{R}^3$, $f \in \mathcal{C}^\infty(U)$ and $\mathbf{X} \in \mathcal{X}(U)$. Then:

1. $\text{rot}(\nabla f) = 0$
2. $\text{div}(\text{rot } \mathbf{X}) = 0$

Definition 271. Let $C \subset \mathbb{R}^3$ be a compact regular curve, U be an open neighbourhood of C and $\mathbf{X} \in \mathcal{X}(U)$. We define the *line integral* (or *circulation*) of $\mathbf{X}$ along C as the integral:

$$\int_C \mathbf{X} := \int_C \mathbf{X} \cdot d\ell := \int_C \omega_{\mathbf{X}}^1$$

where $d\ell = \eta_C$ is the length element of C .

Definition 272. Let $S \subset \mathbb{R}^3$ be a compact oriented surface, U be an open neighbourhood of S and $\mathbf{X} \in \mathcal{X}(U)$. We define the *surface integral* (or *flux*) of $\mathbf{X}$ through S as the integral:

$$\int_S \mathbf{X} := \int_S \mathbf{X} \cdot dS := \int_S \omega_{\mathbf{X}}^2$$

where $dS = \eta_S$ is the area element of S .

Theorem 273 (Curl theorem). Let $S \subset \mathbb{R}^3$ be a compact oriented surface with boundary, U be an open neighbourhood of S and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$\int_S \text{rot } \mathbf{X} \cdot dS = \int_{\partial S} \mathbf{X} \cdot d\ell$$

Theorem 274 (Divergence theorem on $\mathbb{R}^3$). Let $D \subset \mathbb{R}^3$ be a compact submanifold with boundary of dimension 3, U be an open neighbourhood of D and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$\int_D \text{div } \mathbf{X} \cdot dV = \int_{\partial D} \mathbf{X} \cdot dS$$

where $dV = \eta$ is the volume element of D .

Theorem 275 (Divergence theorem). Let $M \subseteq \mathbb{R}^n$ be a submanifold of dimension k , ν be a unit normal exterior vector field of M , U be an open neighbourhood of M and $\mathbf{X} \in \mathcal{X}(U)$. Then:

$$\int_M \text{div } \mathbf{X} \cdot \eta = \int_{\partial M} \langle \mathbf{X}, \nu \rangle \eta_{\partial M}$$

8. | Gauß-Bonnet theorem

Local Gauß-Bonnet theorem

Definition 276. Let $S \subset \mathbb{R}^3$ be an oriented surface and $R \subseteq S$ be a subset. We say that R is a *simple region* if it is homeomorphic to a closed disk and ∂R is a piecewise curve of class $\mathcal{C}^1$ which is closed and simple.

Definition 277. Let $S \subset \mathbb{R}^3$ be an oriented surface, $(V, \varphi(u, v))$ be a parametrization of S compatible with the orientation and $R \subseteq S$ be a simple region such that $R \subseteq \varphi(V)$ and that ∂R is positively-oriented. Let $\alpha : [0, \ell] \rightarrow \partial R$ be a parametrization of ∂R which is differentiable on the intervals $[t_i, t_{i+1}]$, $i = 0, \dots, n$. For $i = 0, \dots, n$, let $\alpha'(t_i^-)$ and $\alpha'(t_i^+)$ denote the respective lateral derivatives of α at the point t_i . The arches $\alpha([t_i, t_{i+1}])$ are called *edges* of R ; the points $\alpha(t_i)$, *vertices* of R , and the angle $\theta_i \in [-\pi, \pi]$ between $\alpha'(t_i^-)$ and $\alpha'(t_i^+)$, *exterior angle*.

Theorem 278. Let $S \subset \mathbb{R}^3$ be an oriented surface, $(V, \varphi(u, v))$ be a parametrization of S compatible with the orientation and $R \subseteq S$ be a simple region such that $R \subseteq \varphi(V)$ and that ∂R is positively-oriented. Let $\alpha : [0, \ell] \rightarrow \partial R$ be a parametrization of ∂R which is differentiable on the intervals $[t_i, t_{i+1}]$, $i = 0, \dots, n$. Suppose $\tau_i[t_i, t_{i+1}]$ is a differentiable determination of the angle between φ_u and $\alpha(t)$, $t \in [t_i, t_{i+1}]$. Then:

$$\sum_{i=0}^n [\tau_i(t_{i+1}) - \tau_i(t_i)] + \sum_{i=0}^n \theta_i = \pm 2\pi$$

And the sign in the right-hand side of the equation is positive if and only if the parametrization α of ∂R is positive.

Theorem 279 (Local Gauß-Bonnet theorem). Let $S \subset \mathbb{R}^3$ be an oriented surface, $(V, \varphi(u, v))$ be an orthogonal parametrization of S compatible with the orientation and $R \subseteq S$ be a simple region such that $R \subseteq \varphi(V)$ and that ∂R is positively-oriented. Let $\alpha : [0, \ell] \rightarrow \partial R$ be an arclength parametrization of ∂R which is differentiable on the intervals $[t_i, t_{i+1}]$, $i = 0, \dots, n$. Then:

$$\sum_{i=0}^n \int_{t_i}^{t_{i+1}} k_g(s) ds + \int_R K dS + \sum_{i=0}^n \theta_i = 2\pi\chi(R) = 2\pi$$

Definition 280. Let $S \subset \mathbb{R}^3$ be an oriented surface and $T \subseteq S$ be simple region. We say that T is a *triangle* if it has 3 edges.

Corollary 281. Let $S \subset \mathbb{R}^3$ be an oriented surface, $T \subseteq S$ be a triangle such that their edges are geodesics. Then, in the notation of Theorem 279 we have:

$$\int_T K dS = \sum_{i=0}^n \phi_i - \pi$$

where $\phi_i = \pi - \theta_i$ are the interior angles of the triangle.

³⁵Recall ??

Global Gauß-Bonnet theorem

Definition 282. Let $S \subset \mathbb{R}^3$ be an oriented surface and $R \subseteq S$ be a compact subset. We say that R is a *regular region* if ∂R is a finite union of regular curves which are closed and simple and such that they don't intersect themselves.

Theorem 283 (Global Gauß-Bonnet theorem). Let $S \subset \mathbb{R}^3$ be an oriented surface and $R \subseteq S$ be a regular region. Suppose that $C_1, \dots, C_n$ are the connected components of ∂R and let $\theta_1, \dots, \theta_m$ be the exterior angles of the curves C_i . Then:

$$\sum_{i=0}^n \int_{C_i} k_g(s) ds + \int_R K dS + \sum_{i=0}^m \theta_i = 2\pi\chi(R)$$

Corollary 284. Let $S \subset \mathbb{R}^3$ be a compact oriented surface without boundary. Then:

$$\int_R K dS = 2\pi\chi(S)$$

Poincaré-Hopf theorem

Definition 285. Let $S \subset \mathbb{R}^3$ be an oriented surface and $\mathbf{X}$ be a vector field tangent to S . We say that $p \in S$ is a *singular point* of $\mathbf{X}$ if $\mathbf{X}_p = 0$. We say that p is an *isolated singular point* (or and *isolated singularity*) if there is a neighbourhood V of p such that $\mathbf{X}_q \neq 0 \forall q \in V \setminus \{p\}$.

Proposition 286. Let $S \subset \mathbb{R}^3$ be an oriented surface, $(V, \varphi(u, v))$ be a parametrization of S compatible with the orientation, $\mathbf{X}$ be a vector field tangent to S , p be an isolated singular point and $R \subseteq \varphi(V)$ be a simple region that contains p in its interior (and no other singular point). Let $\alpha : [0, \ell] \rightarrow \partial R$ be a positive parametrization of ∂R and $\tau : [0, \ell]$ be a differentiable determination of the angle between φ_u and $\mathbf{X}_{\alpha(t)}$. Then, $\exists n \in \mathbb{Z}$ such that:

$$\int_0^\ell \frac{d\tau}{dt} dt = \tau(\ell) - \tau(0) = 2\pi n$$

This value of n is called *index* of p .

Theorem 287 (Poincaré-Hopf theorem). Let $S \subset \mathbb{R}^3$ be a compact oriented surface without boundary and $\mathbf{X}$ be a vector field tangent to S with isolated singularities whose indexes are n_i , $i \in I$. Then:

$$\sum_{i \in I} n_i = \frac{1}{2\pi} \int_S K dS = \chi(S)$$

Corollary 288. Let $S \subset \mathbb{R}^3$ be a compact oriented surface without boundary and $\mathbf{X}$ be a vector field tangent to S . Then:

- If $S \cong S^2$ or $S \cong S_g$ with $g \geq 2$, then $\mathbf{X}$ has singularities³⁵.
- If $\mathbf{X}$ doesn't have singularities, then $S \cong T^2$.

