Differential equations

Along this document we will often write the points in $\mathbb{R}^n$, $n \geq 2$, in bold face (as well as the vectors) in order to be consistent when handling points and vectors together.

1. | Space of continuous and bounded functions

Definition 1. Let X, Y be topological spaces. We define the following sets:

$$\mathcal{C}(X,Y) = \{f : X \longrightarrow Y : f \text{ is continuous}\}$$
$$\mathcal{C}_{\mathrm{b}}(X,\mathbb{R}^n) = \{f \in \mathcal{C}(X,\mathbb{R}^n) : f \text{ is bounded}\}$$

Theorem 2. Let X be a topological space and $f \in \mathcal{C}_{\mathrm{b}}(X,\mathbb{R}^n)$. We define the norm of f as:

$$\|f\| := \sup\{\|f(x)\| : x \in X\}$$

and a distance d in $\mathcal{C}_{\mathrm{b}}(X,\mathbb{R}^n)$ as:

$$d(f,g) := \|f-g\| \quad \forall f,g \in \mathcal{C}_{\mathrm{b}}(X,\mathbb{R}^n)$$

Then, $(\mathcal{C}_{\mathrm{b}}(X,\mathbb{R}^n), d)$ is a complete metric space.

Theorem 3. Let X be a topological space and $C \subseteq \mathbb{R}^n$ be a closed subset. Then, $(\mathcal{C}(X,C), d)$ is a complete metric space.

Corollary 4. Let $K \subset \mathbb{R}^n$ be a compact subset and $C \subseteq \mathbb{R}^n$ be a closed subset. Then, $(\mathcal{C}(K,C), d)$ is a complete metric space.

Corollary 5. Let $D \subset \mathbb{R}^n$ be a closed set and $X = \mathcal{C}([a,b], D)$. Then (X,d) is also a complete metric space.

2. | Ordinary differential equations

Definition 6. An *ordinary differential equation* (*ode*) of m unknowns and of order n in *implicit form* is an expression of the form:

$$\mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t), \ldots, \mathbf{x}^{(n)}(t)\right) = \mathbf{0}$$

where $\mathbf{x} : U \subseteq \mathbb{R} \to \mathbb{R}^m$ is a vector-valued function of one variable $t \in \mathbb{R}$ (which is called *independent variable*) and $\mathbf{f} : \Omega \subseteq \mathbb{R} \times \mathbb{R}^{m\cdot(n+1)} \to \mathbb{R}^m$, where both U and Ω are open sets. The same ordinary differential equation in *explicit form* is an expression of the form:

$$\mathbf{x}^{(n)}(t) = \mathbf{g}\left(t, \mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t), \ldots, \mathbf{x}^{(n-1)}(t)\right)$$

where $\mathbf{g} : \Omega \subseteq \mathbb{R} \times \mathbb{R}^{m\cdot n} \to \mathbb{R}^m$[1].

Definition 7. Consider the following ODE of m unknowns and of order n:

$$\mathbf{x}^{(n)}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}'(t), \ldots, \mathbf{x}^{(n-1)}(t)\right) \tag{1}$$

We say that $\boldsymbol{\varphi} : I \subseteq \mathbb{R} \to \mathbb{R}^m$ is a *solution of the ODE* if:

- $\boldsymbol{\varphi}$ is n times differentiable on I.
- $\left\{\left(t, \boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t), \ldots, \boldsymbol{\varphi}^{(n-1)}(t)\right) : t \in I\right\} \subseteq \operatorname{dom}\mathbf{f}$
- For all $t \in I$ we have:
$$\boldsymbol{\varphi}^{(n)}(t) = \mathbf{f}\left(t, \boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t), \ldots, \boldsymbol{\varphi}^{(n-1)}(t)\right)$$

The set of all solutions of an ODE is called *general solution of the ODE*.

Proposition 8. Consider an ODE of m unknowns and order n of the form of Eq. (1). Then, we can transform this ODE to an ODE of $m \cdot n$ unknowns and order 1 in the following way[2]. Define $\mathbf{y}_i = \mathbf{x}^{(i-1)}$ for $i = 1, \ldots, n$. Therefore, the functions $\mathbf{y}_i$ must satisfy:

$$\begin{cases} {\mathbf{y}_1}' = \mathbf{y}_2 \\ {\mathbf{y}_2}' = \mathbf{y}_3 \\ \quad\vdots \\ {\mathbf{y}_{n-1}}' = \mathbf{y}_{n-2} \\ {\mathbf{y}_n}' = \mathbf{f}\left(t, \mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_n(t)\right) \end{cases}$$

This is called a *system of ordinary differential equations* (of order 1) or a *differential system*.

Definition 9. We say that an ODE is *autonomous* if it doesn't depend on the independent variable, that is, if it is of the form:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

Otherwise, we say that an ODE is *non-autonomous*.

Definition 10. We say that an ODE of order n is *linear* if it is of the form:

$$a_0(t)\mathbf{x} + a_1(t)\mathbf{x}' + \cdots + a_n(t)\mathbf{x}^{(n)} = \mathbf{b}(t) \tag{2}$$

where $a_i \in \mathcal{C}(I,\mathbb{R})$ for $i = 0, \ldots, n$ and $\mathbf{b} \in \mathcal{C}(I,\mathbb{R}^m)$ are arbitrary functions which do not need to be linear. We say that the linear ODE of Eq. (2) is *homogeneous* if $\mathbf{b}(t) = \mathbf{0}$ $\forall t \in I$. We say that linear ODE of Eq. (2) is of *constant coefficients* if $a_i(t) := a_{i0} \in \mathbb{R}$ $\forall t \in I$ and $\forall i = 0, \ldots, n$.

Definition 11 (Initial value problem). Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a function. Given $(t_0, \mathbf{x}_0) \in U$, the *initial value problem* (*ivp*) (or *Cauchy problem*) consists in finding a solution of the ODE

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$

with *initial conditions* $\mathbf{x}(t_0) = \mathbf{x}_0$.

[1] Sometimes we will write $\mathbf{x}^{(n)} = \mathbf{g}\left(t, \mathbf{x}, \mathbf{x}', \ldots, \mathbf{x}^{(n-1)}\right)$ instead of $\mathbf{x}^{(n)}(t) = \mathbf{g}\left(t, \mathbf{x}(t), \mathbf{x}'(t), \ldots, \mathbf{x}^{(n-1)}(t)\right)$ in order to simplify the notation.

[2] Therefore, we will mainly study the ODEs of order 1.

Methods for solving ODEs

Proposition 12 (Separation of variables). Let $f : (a,b) \to \mathbb{R}$, $g : (c,d) \to \mathbb{R}$ be continuous functions such that $f(x) \neq 0$ $\forall x \in (a,b)$. Consider the ODE $x' = f(x)g(t)$. To find the solution of this ODE, proceed as follows:

$$x' = f(x)g(t) \iff \int \frac{\mathrm{d}x}{f(x)} = C + \int g(t)\,\mathrm{d}t$$

where the constant C is determined with the initial conditions of the ODE.

Proposition 13 (Variation of constants). Let $I \subset \mathbb{R}$ be an interval, $a, b \in \mathcal{C}(I, \mathbb{R})$. Consider the ODE $x' = a(t)x + b(t)$. To find the solution of this ODE, proceed as follows:

1. Find the solution of the associated homogeneous system with the separation of variables method. Let's say that is $\varphi(t)c$, where $c \in \mathbb{R}$.

2. Try to find a general solution of the form $\varphi(t)c(t)$:

$$\begin{aligned}(\varphi(t)c(t))' = a(t)\varphi(t)c(t) + b(t) &\iff \\ \varphi(t)'c(t) + \varphi(t)c(t)' = a(t)\varphi(t)c(t) + b(t) &\iff \\ \varphi(t)c(t)' = b(t) \iff c(t) = d + \int \varphi(t)^{-1}b(t)\,\mathrm{d}t&\end{aligned}$$

   where $d \in \mathbb{R}$. Hence, the general solution will be:

$$\varphi(t)\left(d + \int \varphi(t)^{-1}b(t)\,\mathrm{d}t\right)$$

Proposition 14 (Characteristic equation). Consider the following ODE of order n of constant coefficients:

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0x = 0 \tag{3}$$

We define the *characteristic equation* of that system as the equation:

$$p(r) := r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$$

In order to find the solution of this ODE, we need to find the solutions to $p(r) = 0$. So suppose p has s distinct real roots and $2(m-s)$ distinct complex roots.

$$\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \overline{\lambda_{s+1}}, \ldots, \lambda_m, \overline{\lambda_m}$$

Here, $\lambda_i \in \mathbb{R}$ $\forall i = 1, \ldots, s$ and $\lambda_i = \alpha_i + \mathrm{i}\beta_i \in \mathbb{C}$ $\forall i = s+1, \ldots, m$. Assume, each of these roots have multiplicity $k_i \in \mathbb{N}$. Then, the general solution to Eq. (3) is:

$$\begin{aligned}\varphi(t) = \sum_{i=1}^{s}\left(c_{i,0} + c_{i,1}t + \cdots + c_{i,k_i-1}t^{k_i-1}\right)\mathrm{e}^{\lambda_i t} + \\ + \sum_{i=s+1}^{m}\sum_{j=0}^{k_i-1} t^j \mathrm{e}^{\alpha_i t}\left(c_{i,j,1}\cos(\beta_i t) + c_{i,j,2}\sin(\beta_i t)\right)\end{aligned}$$

where $c_{i,j,k} \in \mathbb{R}$ are constants.

Proposition 15. Consider a system of the form Eq. (3) which is equivalent to:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} =: \mathbf{A}\mathbf{x}$$

Then, the characteristic equation is precisely the characteristic polynomial of $\mathbf{A}$.

Corollary 16. Consider the following ODE of order n of constant coefficients:

$$x'' + px' + q = 0 \tag{4}$$

Let λ_1, λ_2 be the roots of the polynomial $p(r) = r^2 + pr + q$. Then, the general solution to Eq. (4) is:

- If $p^2 - 4q > 0$:
$$\varphi(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t}$$
- If $p^2 - 4q = 0$, then $\lambda_1 = \lambda_2$ and the solution is:
$$\varphi(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2 t\mathrm{e}^{\lambda_1 t}$$
- If $p^2 - 4q < 0$, then $\lambda_1 = \alpha + \mathrm{i}\beta \in \mathbb{C}$ and the solution is:
$$\varphi(t) = \mathrm{e}^{\alpha t}\left[c_1\cos(\beta t) + c_2\sin(\beta t)\right]$$

Proposition 17 (Reducible linear ODE of second order). Let $I \subset \mathbb{R}$ be an interval, $a, b, c, d \in \mathcal{C}(I, \mathbb{R})$. Consider the system of ODEs:

$$\begin{cases} x' = a(t)x - b(t)y + c(t) \\ y' = b(t)x + a(t)y + d(t) \end{cases} \tag{5}$$

In order to find the solution of this ODE, consider the change of variable $z = x + \mathrm{i}y$. Then, Eq. (5) becomes:

$$z' = [a(t) + \mathrm{i}b(t)]z + c(t) + \mathrm{i}d(t)$$

which is a linear ODE of order 1 and can be easily solved.

Proposition 18 (Bernoulli differential equation). Let $p, q \in \mathcal{C}((a,b), \mathbb{R})$ and $\alpha \in \mathbb{R}$. Consider the *Bernoulli differential equation*:

$$x' + p(t)x = q(t)x^\alpha \tag{6}$$

If $\alpha = 0, 1$ the ODE is linear. So suppose $\alpha \neq 0, 1$. In order to solve it, consider the change of variable $y = x^{1-\alpha}$. Then, Eq. (6) becomes:

$$y' + (1-\alpha)p(t)y = (1-\alpha)q(t)$$

which is a linear ODE of order 1 and can be easily solved.

Proposition 19 (Riccati differential equation). Let $q_0, q_1, q_2 \in \mathcal{C}((a,b), \mathbb{R})$. Consider the *Riccati differential equation*:

$$x' = q_0(t) + q_1(t)x + q_2(t)x^2 \tag{7}$$

Suppose we have found a particular solution $x_1(t)$ of the ODE of Eq. (7). In order to find the general solution, consider the change of variable $x = x_1(t) + \frac{1}{y}$. Then, Eq. (7) becomes:

$$y' + [q_1(t) + 2q_2(t)x_1(t)]y = -q_2(t)$$

which is a linear ODE of order 1 and can be easily solved.

Proposition 20 (Integrating factor). Consider the ODE:

$$p(t,x)+q(t,x)x'=0 \iff p(t,x)\,\mathrm{d}t+q(t,x)\,\mathrm{d}x=0$$

where $p,q \in \mathcal{C}^1(U,\mathbb{R})$ and $U \subseteq \mathbb{R}^2$ is an open set. An *integrating factor* $\mu(t,x) \in \mathcal{C}^1(U)$, $\mu(t,x) \neq 0$, is a function so that

$$\mu(t,x)p(t,x)\,\mathrm{d}t+\mu(t,x)q(t,x)\,\mathrm{d}x$$

is an exact differential ($\mathrm{d}\Phi(t,x)$) of a function $\Phi(t,x)$, that is:

$$\frac{\partial \Phi}{\partial t}(t,x)=\mu(t,x)p(t,x) \tag{8}$$

$$\frac{\partial \Phi}{\partial x}(t,x)=\mu(t,x)q(t,x) \tag{9}$$

So we need that:

$$\frac{\partial}{\partial x}\left(\mu(t,x)p(t,x)\right)=\frac{\partial}{\partial t}\left(\mu(t,x)q(t,x)\right)$$

From here, in certain cases, we will be able to find $\mu(x,y)$ and, therefore, $\Phi(t,x)$ by integrating Eqs. (8) and (9).

3. | Existence and uniqueness of solutions

Proposition 21. Let $f:(a,b)\to\mathbb{R}$ be a continuous function such that $f(x)\neq 0\ \forall x\in(a,b)$. Then, the ivp

$$\begin{cases} x'=f(x)\\ x(t_0)=x_0\end{cases}$$

has a unique solution $\forall t_0\in\mathbb{R}$ and $\forall x_0\in(a,b)$.

Proposition 22. Let $f:(a,b)\to\mathbb{R}$, $g:(c,d)\to\mathbb{R}$ be continuous functions such that $f(x)\neq 0\ \forall x\in(a,b)$. Then, the ivp

$$\begin{cases} x'=f(x)g(t)\\ x(t_0)=x_0\end{cases}$$

has a unique solution $\forall t_0\in(c,d)$ and $\forall x_0\in(a,b)$.

Proposition 23. Let $I\subseteq\mathbb{R}$ be an interval and $a:I\to\mathbb{R}$ and $b:I\to\mathbb{R}$ be continuous functions. Then, the ivp

$$\begin{cases} x'=a(t)x+b(t)\\ x(t_0)=x_0\end{cases}$$

has a unique solution $\forall t_0\in I$ and $\forall x_0\in\mathbb{R}$[3].

Lipschitz continuity

Definition 24. Let $\mathbf{f}:U\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ be a function. We say that $\mathbf{f}$ is *Lipschitz continuous with respect to the second variable* if $\exists L\in\mathbb{R}_{>0}$ such that:

$$\|\mathbf{f}(t,\mathbf{x})-\mathbf{f}(t,\mathbf{y})\|\leq L\|\mathbf{x}-\mathbf{y}\| \qquad \forall (t,\mathbf{x}),(t,\mathbf{y})\in U$$

Definition 25. Let $\mathbf{f}:U\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ be a function. We say that $\mathbf{f}$ is *locally Lipschitz continuous with respect to the second variable* if $\forall(t_0,\mathbf{x}_0)\in U$ there exists a neighbourhood V of $(t_0,\mathbf{x}_0)$ such that $f|_V$ is Lipschitz continuous with respect to the second variable.

Proposition 26. Let $U\subseteq\mathbb{R}\times\mathbb{R}^n$ be an open set and $\mathbf{f}:U\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be a function. Then:

1. If $\mathbf{f}$ is locally Lipschitz continuous with respect to the second variable, then it is continuous with respect to the second variable.
2. If $\mathbf{f}$ is Lipschitz continuous with respect to the second variable, then it is uniformly continuous with respect to the second variable.
3. If $\mathbf{f}$ is continuous, U is compact and $\mathbf{f}$ is locally Lipschitz continuous with respect to the second variable, then $\mathbf{f}$ is Lipschitz continuous with respect to the second variable.

Proposition 27. Let $U\subseteq\mathbb{R}\times\mathbb{R}^n$ be an open and convex set and $\mathbf{f}:U\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be a function of class $\mathcal{C}^1$. Then:

1. $\mathbf{f}$ is locally Lipschitz continuous with respect to the second variable.
2. $\mathbf{f}$ is Lipschitz continuous with respect to the second variable if and only if $\mathbf{Df}$ is bounded.

Picard theorem

Proposition 28. Let $U\subseteq\mathbb{R}\times\mathbb{R}^n$ be an open set and $\mathbf{f}:U\to\mathbb{R}^n$ be a continuous function. Let $I\subseteq\mathbb{R}$ be an open interval, $t_0\in I$ and $\mathbf{x}_0\in\mathbb{R}^n$ be such that $(t_0,\mathbf{x}_0)\in U$. Then, a continuous function $\boldsymbol{\varphi}:I\to\mathbb{R}^n$ is a solution of the ivp

$$\begin{cases} \mathbf{x}'=\mathbf{f}(t,\mathbf{x})\\ \mathbf{x}(t_0)=\mathbf{x}_0\end{cases} \tag{10}$$

if and only if

$$\boldsymbol{\varphi}(t)=\mathbf{x}_0+\int_{t_0}^{t} f(s,\boldsymbol{\varphi}(s))\,\mathrm{d}s \quad \forall t\in I$$

Definition 29. An *operator* is a function whose domain is a set of functions.

Definition 30. Let $U\subseteq\mathbb{R}\times\mathbb{R}^n$ be an open set, $(t_0,\mathbf{x}_0)\in U$, $\mathbf{f}:U\to\mathbb{R}^n$ be a continuous function and I be a closed interval. We define the operator

$$\begin{aligned}\mathbf{T}:\mathcal{C}(I,\mathbb{R}^n) &\longrightarrow \mathcal{C}(I,\mathbb{R}^n)\\ \boldsymbol{\varphi} &\longmapsto \mathbf{T}\boldsymbol{\varphi}(t)=\mathbf{x}_0+\int_{t_0}^{t} f(s,\boldsymbol{\varphi}(s))\,\mathrm{d}s\end{aligned}$$ [4]

Theorem 31 (Banach fixed-point theorem). Let (X,d) be a complete metric space and $f:(X,d)\to(X,d)$ be a contraction. Then, f has a unique fixed point $p\in X$[5].

[3] See Eq. (12) for the solution.

[4] Note that the fixed points of this operator are precisely the solutions of the ivp of Eq. (10).

[5] Furthermore, p can be found as follows: start with an arbitrary element $x_0\in X$ and define a sequence (x_n) by $x_n=f(x_{n-1})$ for $n\geq 1$. Then, $\lim\limits_{n\to\infty} x_n=p$.

Corollary 32. Let (X, d) be a complete metric space and $f : (X, d) \to (X, d)$ be a function. If there exists $m \in \mathbb{N}$ such that f^m is a contraction, then f has a unique fixed point $p \in X$.

Definition 33. Let $t_0 \in \mathbb{R}$, $\mathbf{x}_0 \in \mathbb{R}^n$ and $a, b \in \mathbb{R}_{>0}$. We define the following sets:

$$I_a(t_0) := [t_0 - a, t_0 + a] \subset \mathbb{R} \quad \text{and} \quad \overline{B}_b(\mathbf{x}_0) := \overline{B}(\mathbf{x}_0, b) \subset \mathbb{R}^n$$

Theorem 34 (Picard theorem). Let $t_0 \in \mathbb{R}$, $\mathbf{x}_0 \in \mathbb{R}^n$, $a, b \in \mathbb{R}_{>0}$, $\mathbf{f} : I_a(t_0) \times \overline{B}_b(\mathbf{x}_0) \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and Lipschitz continuous with respect to the second variable, and define:

$$M := \max\{\|\mathbf{f}(t, x)\| : (t, x) \in I_a(t_0) \times \overline{B}_b(\mathbf{x}_0)\}$$

Then, the ivp of Eq. (10) has a unique solution $\boldsymbol{\varphi} : I_\alpha(t_0) \to \overline{B}_b(\mathbf{x}_0)$, where $\alpha := \min\left\{a, \frac{b}{M}\right\}$.

Corollary 35. Let $I \subset \mathbb{R}$ be a closed interval, $t_0 \in I$, $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{f} : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and Lipschitz continuous with respect to the second variable. Then, the ivp of Eq. (10) has a unique solution $\boldsymbol{\varphi} : I \to \mathbb{R}^n$.

Corollary 36 (Picard iteration process). Suppose we want to solve the ivp of Eq. (10). That is, we look for a solution $\boldsymbol{\varphi}(t)$. Let $\boldsymbol{\varphi}_0$ be a fixed function (usually chosen to be $\boldsymbol{\varphi}_0 = \mathbf{x}_0$) and define

$$\boldsymbol{\varphi}_{n+1}(t) = \mathbf{T}\boldsymbol{\varphi}_n(t) = \mathbf{x}_0 + \int_{t_0}^{t} f(s, \boldsymbol{\varphi}_n(s))\, \mathrm{d}s$$

for all $n \geq 0$. Then, $\boldsymbol{\varphi}(t) = \lim_{n\to\infty} \boldsymbol{\varphi}_n(t)$.

Corollary 37. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and locally Lipschitz continuous with respect to the second variable. Then, $\forall (t_0, \mathbf{x}_0) \in U$, there exists $\alpha(t_0, \mathbf{x}_0) \in \mathbb{R}_{>0}$ and a neighbourhood $V_{t_0,\mathbf{x}_0} = I_{a(t_0,\mathbf{x}_0)}(t_0) \times \overline{B}_{b(t_0,\mathbf{x}_0)}(\mathbf{x}_0)$ of $(t_0, \mathbf{x}_0)$ in U such that the ivp of Eq. (10) has a unique solution $\boldsymbol{\varphi}_{t_0,\mathbf{x}_0}$ defined on $I_{\alpha(t_0,\mathbf{x}_0)} \subseteq I_{a(t_0,\mathbf{x}_0)}$ with $\mathrm{graph}(\boldsymbol{\varphi}_{t_0,\mathbf{x}_0}) \subset V_{t_0,\mathbf{x}_0}$.

Proposition 38. Let $I \subseteq \mathbb{R}$ be an interval and $\mathbf{f} : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and Lipschitz continuous with respect to the second variable. Then, $\forall (t_0, \mathbf{x}_0) \in I \times \mathbb{R}^n$ there is a unique solution of the ivp of Eq. (10) defined on I.

Corollary 39. Let $I \subseteq \mathbb{R}$ be an interval and $\mathbf{A} : I \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbf{b} : I \to \mathbb{R}^n$ be continuous functions. Then, for all $(t_0, \mathbf{x}_0) \in I \times \mathbb{R}^n$ the ivp

$$\begin{cases} \mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

has a unique solution defined on I.

Theorem 40. Let $f : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $x_0 \in \mathbb{R}$. Suppose that f is decreasing with respect to the second variable. Then, the ivp

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution defined on $[t_0, t_1^*]$, where $t_1^* \leq t_1$.

Peano theorem

Definition 41. Let (X, d) be a metric space and $F \subset \mathcal{C}(X, \mathbb{R}^n)$ be a subset. We say that F is *pointwise bounded* if:

$$\forall x \in X\ \exists M_x > 0 \text{ such that } \|\mathbf{f}(x)\| \leq M_x \quad \forall \mathbf{f} \in F$$

We say that F is *uniformly bounded* if:

$$\exists M > 0 \text{ such that } \|\mathbf{f}(x)\| \leq M \quad \forall \mathbf{f} \in F \text{ and } \forall x \in X$$

Definition 42. Let (X, d) be a metric space and $F \subset \mathcal{C}(X, \mathbb{R}^n)$ be a subset. We say that F is *equicontinuous at a point* $x_0 \in X$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in X$ with $d(x, x_0) < \delta$ we have:

$$\|\mathbf{f}(x) - \mathbf{f}(x_0)\| < \varepsilon \quad \forall \mathbf{f} \in F$$

We say that F is *pointwise equicontinuous* if it is equicontinuous at each point of X. Finally, we say that F is *uniformly equicontinuous* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in X$ with $d(x, y) < \delta$ we have:

$$\|\mathbf{f}(x) - \mathbf{f}(y)\| < \varepsilon \quad \forall \mathbf{f} \in F$$

Proposition 43. Let (X, d) be a metric space and $F \subset \mathcal{C}_\mathrm{b}(X, \mathbb{R}^n)$ be a subset. Suppose that $\mathbf{f}$ is Lipschitz continuous for all $\mathbf{f} \in F$. Then, F is uniformly equicontinuous.

Theorem 44 (Arzelà-Ascoli theorem). Let (X, d) be a compact metric space and $(\mathbf{f}_m)$ be a sequence of functions such that $\mathbf{f}_m \in \mathcal{C}(X, \mathbb{R}^n)$ $\forall m \geq 1$. If the sequence is pointwise equicontinuous and pointwise bounded, then there exists a subsequence $(\mathbf{f}_{m_k})$ that converges on $\mathcal{C}(X, \mathbb{R}^n)$.

Corollary 45. Let (X, d) be a compact metric space, $D \subset \mathbb{R}^n$ be a closed set and $(\mathbf{f}_m)$ be a sequence of functions such that $\mathbf{f}_m \in \mathcal{C}(X, D)$ $\forall m \geq 1$. If the sequence is pointwise equicontinuous and pointwise bounded, then there exists a subsequence $(\mathbf{f}_{m_k})$ that converges on $\mathcal{C}(X, D)$.

Theorem 46 (Peano theorem). Let $t_0 \in \mathbb{R}$, $\mathbf{x}_0 \in \mathbb{R}^n$, $a, b \in \mathbb{R}_{>0}$, $\mathbf{f} : I_a(t_0) \times \overline{B}_b(\mathbf{x}_0) \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and define:

$$M := \max\{\|\mathbf{f}(t, x)\| : (t, x) \in I_a(t_0) \times \overline{B}_b(\mathbf{x}_0)\}$$

Then, the ivp of Eq. (10) has at least one solution $\boldsymbol{\varphi} : I_\alpha(t_0) \to \mathbb{R}^n$, where $\alpha := \min\left\{a, \frac{b}{M}\right\}$.

Corollary 47. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set, $K \subset U$ be a compact set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. Then, $\exists \alpha \in \mathbb{R}_{>0}$ such that $\forall (t_0, \mathbf{x}_0) \in K$, the ivp of Eq. (10) has a solution defined in $I_\alpha(t_0)$.

Maximal solutions

Definition 48. Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set, $(t_0, \mathbf{x}_0) \in U$ and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. We define the set $\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0)$ as:

$$\begin{aligned}\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0) := \{&(I, \boldsymbol{\varphi}) : I \subseteq \mathbb{R} \text{ is an interval}, t_0 \in I \\ &\text{and } \boldsymbol{\varphi} : I \to \mathbb{R}^n \text{ is a solution of the ivp of Eq. (10)}\}\end{aligned}$$

Definition 49. We define the relation $\leq$ defined on $\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0)$ in the following way. For $(I, \boldsymbol{\varphi}), (J, \boldsymbol{\psi}) \in \mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0)$:

$$(J, \boldsymbol{\psi}) \leq (I, \boldsymbol{\varphi}) \iff J \subset I \text{ and } \boldsymbol{\varphi}|_J = \boldsymbol{\psi}[6]$$

In this case, we say that $(I, \boldsymbol{\varphi})$ is an *extension* of $(J, \boldsymbol{\psi})$.

Definition 50. Let $(A, \leq)$ be a poset. Then, $m \in A$ is a *maximal element* if and only if $\forall a \in A$ with $m \leq a$ we have $m = a$.

Definition 51. Consider the poset $(\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0), \leq)$. We say that a solution $(I, \boldsymbol{\varphi})$ is *maximal* if for all extensions $(J, \boldsymbol{\psi})$ of $(I, \boldsymbol{\varphi})$ we have $I = J$ and $\boldsymbol{\varphi} = \boldsymbol{\psi}$.

Definition 52. Let $(A, \leq)$ be a poset and $C \subseteq A$ be a subset of A. We say that C is a *chain* if it is totally ordered in the inherited order, that is, if it is partially ordered and $\forall x, y \in C$ we have either $x \leq y$ or $y \leq x$.

Definition 53. Let $(A, \leq)$ be a poset, $x \in A$ and $B \subseteq A$ be a subset. x is an *upper bound* of B if and only if $b \leq x$ $\forall b \in B$.

Definition 54. Let $(A, \leq)$ be a poset and $B \subseteq A$ be a subset. Then, $g \in A$ is a *greatest element* of B if $g \in B$ and $\forall b \in B$ we have $b \leq g$.

Lemma 55 (Zorn's lemma). Let $(A, \leq)$ be a poset. If every chain $C \subseteq A$ has an upper bound in A, then A contains at least one maximal element.

Theorem 56. Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set, $(t_0, \mathbf{x}_0) \in U$ and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. Consider the poset $(\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0), \leq)$. Then, $\mathcal{S}(U, \mathbf{f}, t_0, \mathbf{x}_0)$ has maximal elements. Furthermore, if $(I, \boldsymbol{\varphi})$ is a maximal solution, then I is open.

Proposition 57. Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be such that $\forall (t_0, \mathbf{x}_0) \in U$ the ivp of Eq. (10) has a unique solution defined in a neighbourhood of t_0. Then, $\forall (t_0, \mathbf{x}_0) \in U$ the ivp of Eq. (10) has a unique maximal solution.

Lemma 58 (Wintner lemma). Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function, $\boldsymbol{\varphi} : I \to \mathbb{R}^n$ be a solution of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ and $(b, y) \in U$ be an accumulation point of $\boldsymbol{\varphi}$. Then, $\lim\limits_{t\to b} \boldsymbol{\varphi}(t) = y$ and the solution can be extended up to b.

Corollary 59. Let $U \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and $\boldsymbol{\varphi} : (a, b) \to \mathbb{R}^n$ be a maximal solution of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$. If $b < \infty$, then for all compact set $K \subset U$, $\exists t_0 < \infty$ such that $(t, \boldsymbol{\varphi}(t)) \notin K$ $\forall t \in [t_0, b)$. In that case, we say that $\boldsymbol{\varphi}$ *tends to the boundary* of U.

[6] It can be seen that $\leq$ is a partial (but not total) order relation.

4. | Linear differential equations

Definition 60. Let $I \subseteq \mathbb{R}$ be an interval. A *system of linear differential equations* is an expression of the form:

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t) \tag{11}$$

where $\mathbf{A} : I \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbf{b} : I \to \mathbb{R}^n$ are continuous functions. We say that linear equation of Eq. (11) is *homogeneous* if $\mathbf{b}(t) = \mathbf{0}$ $\forall t \in I$. We say that linear equation of Eq. (11) is of *constant coefficients* if $\mathbf{A}(t) = \mathbf{A}$ $\forall t \in I$, where $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$.

Definition 61. Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$, $\mathbf{x}_0 \in \mathbb{R}^n$ and consider the ODE of Eq. (11). We define the *flow of the linear ODE* as the function:

$$\begin{aligned}\boldsymbol{\phi} : I \times I \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t, t_0, \mathbf{x}_0) &\longmapsto \boldsymbol{\varphi}_{(t_0,\mathbf{x}_0)}(t)\end{aligned}$$

where $\boldsymbol{\varphi}_{(t_0,\mathbf{x}_0)}$ is the solution of Eq. (11) with initial conditions $(t_0, \mathbf{x}_0)$.

Proposition 62. Let $I \subseteq \mathbb{R}$ be an interval and $a, b \in \mathcal{C}(I, \mathbb{R})$. Then, the general solution of the ivp

$$\begin{cases} x' = a(t)x + b(t) \\ x(t_0) = x_0 \end{cases}$$

is given by:

$$\varphi(t, t_0, x_0) = \mathrm{e}^{\int_{t_0}^{t} a(s)\mathrm{d}s} \left(x_0 + \int_{t_0}^{t} b(u)\mathrm{e}^{-\int_{t_0}^{u} a(s)\mathrm{d}s}\,\mathrm{d}u \right) \tag{12}$$

for all $t \in I$.

Homogeneous systems

Theorem 63. Let $I \subseteq \mathbb{R}$ be an interval and $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$. We define $\mathcal{A}_n$ as the set of all solutions of the linear ODE:

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \tag{13}$$

Then, $\mathcal{A}_n$ is a vector space of dimension n and for each $t_0 \in I$, the function

$$\begin{aligned}\boldsymbol{\xi}_{t_0} : \mathbb{R}^n &\longrightarrow \mathcal{A}_n \\ \mathbf{x}_0 &\longmapsto \boldsymbol{\varphi}(\cdot, t_0, \mathbf{x}_0)\end{aligned}$$

is an isomorphism.

Corollary 64. Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$, $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be a basis of $\mathbb{R}^n$ and $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n \in \mathcal{A}_n$ be such that:

$$\boldsymbol{\varphi}_i = \boldsymbol{\xi}_{t_0}(\mathbf{v}_i) \quad \text{for } i = 1, \ldots, n$$

Then, $(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n)$ is a basis of $\mathcal{A}_n$.

Corollary 65. Let $I \subseteq \mathbb{R}$ be an interval and $\boldsymbol{\psi} \in \mathcal{A}_n$. Suppose $\exists t_0 \in I$ such that $\boldsymbol{\psi}(t_0) = 0$. Then, $\boldsymbol{\psi} = \mathbf{0}$.

Corollary 66. Let $I \subseteq \mathbb{R}$ be an interval, $m, n \in \mathbb{N}$ with $m \leq n$, $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m \in \mathcal{A}_n$ and $t_0 \in I$ such that the vectors $\boldsymbol{\varphi}_1(t_0), \ldots, \boldsymbol{\varphi}_m(t_0)$ are linearly independent. Then, $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m$ are linearly independent.

Corollary 67. Let $s, t, w \in \mathbb{R}$. Consider the function

$$\begin{aligned} \boldsymbol{\phi}_s^t : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto (\boldsymbol{\xi}_s(\mathbf{x}))(t) \end{aligned}$$

Then, $\boldsymbol{\phi}_s^t$ is an isomorphism and satisfies:

1. $\boldsymbol{\phi}_s^s = \mathbf{id}$
2. $\boldsymbol{\phi}_s^t \circ \boldsymbol{\phi}_w^s = \boldsymbol{\phi}_w^t$
3. $\left[\boldsymbol{\phi}_s^t\right]^{-1} = \boldsymbol{\phi}_t^s$

Definition 68. Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$ and $\mathbf{M}(t) = (m_{ij}(t)) \in \mathcal{M}_n(\mathbb{R})$. We say that $\mathbf{M}(t)$ is a *matrix solution* of the ODE of Eq. (13) if $\boldsymbol{\varphi}_j = (m_{1j}(t), \ldots, m_{nj}(t))^{\mathrm{T}} \in \mathcal{A}_n$ for $j = 1, \ldots, n$. We say that $\mathbf{M}(t)$ is a *fundamental matrix solution* of the ODE of Eq. (13) if $\mathbf{M}(t)$ is a matrix solution and $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$ are linearly independent.

Proposition 69. Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$ and $\mathbf{M}(t) \in \mathcal{M}_n(\mathbb{R})$. Then:

1. $\mathbf{M}(t)$ is a matrix solution of the ODE of Eq. (13) $\iff \mathbf{M}'(t) = \mathbf{A}(t)\mathbf{M}(t)$[7].
2. $\mathbf{M}(t)$ is a matrix solution of the ODE of Eq. (13) $\iff \forall \mathbf{c} \in \mathbb{R}^n$, $\mathbf{M}(t)\mathbf{c} \in \mathcal{A}_n$.
3. If $\mathbf{M}(t)$ is a matrix solution of the ODE of Eq. (13), then $\forall \mathbf{C} \in \mathcal{M}_n(\mathbb{R})$, $\mathbf{M}(t)\mathbf{C}$ is a matrix solution of the ODE of Eq. (13).
4. If $\mathbf{M}(t)$ is a fundamental matrix solution of the ODE of Eq. (13), then $\det \mathbf{M}(t) \neq 0$ $\forall t \in I$.
5. $\mathbf{M}(t)$ is a fundamental matrix solution of the ODE of Eq. (13) $\iff$ $\mathbf{M}(t)$ is a matrix solution of the ODE of Eq. (13) and $\exists t_0 \in I$ such that $\det \mathbf{M}(t_0) \neq 0$.

Proposition 70. Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$ and $\boldsymbol{\Phi}(t), \boldsymbol{\psi}(t) \in \mathcal{M}_n(\mathbb{R})$ be matrix solutions of the ODE of Eq. (13) such that $\boldsymbol{\Phi}(t)$ is fundamental. Then, $\exists! \mathbf{C} \in \mathcal{M}_n(\mathbb{R})$ such that $\boldsymbol{\psi}(t) = \boldsymbol{\Phi}(t)\mathbf{C}$. Moreover, $\boldsymbol{\psi}(t)$ is fundamental if and only if $\det \mathbf{C} \neq 0$.

Non-homogeneous linear systems

Proposition 71. Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$ and $\mathbf{b} \in \mathcal{C}(I, \mathbb{R}^n)$. Suppose $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ is the flow of the ODE of Eq. (11). Then,

$$\boldsymbol{\phi}(t, t_0, \mathbf{x}_0) = \Phi(t) \left[\Phi(t_0)^{-1}\mathbf{x}_0 + \int_{t_0}^{t} \Phi(s)^{-1}\mathbf{b}(s) \, \mathrm{d}s \right]$$

where $\Phi(t)$ is a fundamental matrix of the associated homogeneous system.

[7] By definition, if $\mathbf{M}(t) = (m_{ij}(t))$, then $\mathbf{M}'(t) := ({m_{ij}}'(t))$.

Corollary 72. Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$ and $\mathbf{b} \in \mathcal{C}(I, \mathbb{R}^n)$. Then, the general solution $\boldsymbol{\varphi}(t)$ of the ODE of Eq. (13) can be written as:

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_{\mathrm{h}}(t) + \boldsymbol{\varphi}_{\mathrm{p}}(t)$$

where $\boldsymbol{\varphi}_{\mathrm{h}}(t)$ is the general solution to the associated homogeneous system and $\boldsymbol{\varphi}_{\mathrm{p}}(t)$ is a particular solution of Eq. (13).

Proposition 73 (Liouville's formula). Let $I \subseteq \mathbb{R}$ be an interval, $\mathbf{A} \in \mathcal{C}(I, \mathcal{L}(\mathbb{R}^n))$, $\boldsymbol{\Phi}(t) \in \mathcal{M}_n(\mathbb{R})$ be a matrix solution of the ODE of Eq. (13) and $t_0 \in I$. Then, for all $t \in I$ we have:

$$\det(\Phi(t)) = \det(\Phi(t_0))\mathrm{e}^{\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s)) \mathrm{d}s}$$

Constant coefficients linear systems

Lemma 74. Let $I \subseteq \mathbb{R}$ be a compact interval and $\mathbf{f} : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and Lipschitz continuous with respect to the second variable. Let $\boldsymbol{\varphi} : I \to \mathbb{R}^n$ be the solution of the ivp of Eq. (10). Then, $\forall \boldsymbol{\psi} \in \mathcal{C}(I, \mathbb{R}^n)$ the sequence $(\mathbf{T}^m\boldsymbol{\psi})$ converges uniformly to $\boldsymbol{\varphi}$ on I.

Theorem 75. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\boldsymbol{\Phi}(t) \in \mathcal{M}_n(\mathbb{R})$ be a matrix solution of the ODE

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{14}$$

such that $\Phi(0) = \mathbf{I}_n$. Then:

1. For all $t, s \in \mathbb{R}$, then $\Phi(t + s) = \Phi(t)\Phi(s)$.
2. $\Phi(t)^{-1} = \Phi(-t)$.
3. The series $\displaystyle\sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$ converges uniformly on compact sets.

Definition 76. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $t \in \mathbb{R}$. We define the *matrix exponential* $\mathrm{e}^{\mathbf{A}t}$ as:

$$\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \tag{15}$$

Proposition 77. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $t, s \in \mathbb{R}$. Then, the matrix exponential $\mathrm{e}^{\mathbf{A}t}$ is a fundamental matrix of the ODE of Eq. (14) and has the following properties:

1. $\mathrm{e}^{\mathbf{A} \cdot 0} = \mathbf{I}_n$
2. $\mathrm{e}^{\mathbf{A}(t+s)} = \mathrm{e}^{\mathbf{A}t}\mathrm{e}^{\mathbf{A}s}$
3. $\left(\mathrm{e}^{\mathbf{A}t}\right)^{-1} = \mathrm{e}^{-\mathbf{A}t}$
4. $\left(\mathrm{e}^{\mathbf{A}t}\right)' = \mathbf{A}\mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{\mathbf{A}t}\mathbf{A}$
5. If $\boldsymbol{\Phi}(t)$ is an arbitrary fundamental matrix of the ODE of Eq. (14), then:
$$\mathrm{e}^{\mathbf{A}t} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}$$

Lemma 78. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{M}_n(\mathbb{R})$. Then:

1. If $\mathbf{BC} = \mathbf{CA}$, then:

$$\mathrm{e}^{\mathbf{B}t}\mathbf{C} = \mathbf{C}\mathrm{e}^{\mathbf{A}t}$$

2. If $\mathbf{AB} = \mathbf{BA}$, then:

$$\mathrm{e}^{\mathbf{A}t}\mathbf{B} = \mathbf{B}\mathrm{e}^{\mathbf{A}t} \quad \text{and} \quad \mathrm{e}^{(\mathbf{A}+\mathbf{B})t} = \mathrm{e}^{\mathbf{A}t}\mathrm{e}^{\mathbf{B}t}$$

Corollary 79. Let $t \in \mathbb{R}$, $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\mathbf{J} \in \mathcal{M}_n(\mathbb{R})$ be the Jordan form of $\mathbf{A}$ such that $\mathbf{A} = \mathbf{CJC}^{-1}$ for some matrix $\mathbf{C} \in \mathrm{GL}_n(\mathbb{R})$. Then:

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{C}\mathrm{e}^{\mathbf{J}t}\mathbf{C}^{-1}$$

Proposition 80. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $t \in \mathbb{R}$. If λ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{v}$, then $\mathrm{e}^{\lambda t}$ is an eigenvalue of $\mathrm{e}^{\mathbf{A}t}$ with associated eigenvector $\mathbf{v}$. That is, $\mathrm{e}^{\mathbf{A}t}\mathbf{v} = \mathrm{e}^{\lambda t}\mathbf{v}$. Hence, $\boldsymbol{\varphi}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$ is a solution of the ivp:

$$\begin{cases} \mathbf{x}' = \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) = \mathbf{v} \end{cases}$$

Corollary 81. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $t \in \mathbb{R}$ and consider the linear ODE of Eq. (14). If $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of eigenvectors with associated eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, then $(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n)$, where $\boldsymbol{\varphi}_i = \mathrm{e}^{\lambda_i t}\mathbf{v}_i$ for $i = 1, \ldots, n$, is a basis of $\mathcal{A}_n$.

Lemma 82. Let $\mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{M}_n(\mathbb{R})$ and $t \in \mathbb{R}$. Then:

$$\mathrm{e}^{\mathbf{A}t} = \mathrm{diag}(\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_n t})$$

Proposition 83. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C} \setminus \mathbb{R}$ be an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{v} = \mathbf{u} + \mathrm{i}\mathbf{w} \in \mathbb{C}^n$. Then:

$$\mathrm{e}^{\mathbf{A}t}\mathbf{v} = \mathrm{e}^{\mathbf{A}t}\mathbf{u} + \mathrm{i}\mathrm{e}^{\mathbf{A}t}\mathbf{w} = \mathrm{e}^{\alpha t}\left[\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}\right] + \\ + \mathrm{i}\mathrm{e}^{\alpha t}\left[\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}\right]$$

and $\mathrm{e}^{\mathbf{A}t}\mathbf{u}$, $\mathrm{e}^{\mathbf{A}t}\mathbf{w}$ are linearly independent solutions of the ODE of Eq. (14) with initial conditions $\mathbf{x}(0) = \mathbf{u}$ and $\mathbf{x}(0) = \mathbf{w}$, respectively.

Definition 84. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$. A vector $\mathbf{w} \in \mathbb{R}^n$ is a *generalized eigenvector* of rank m of $\mathbf{A}$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$ if:

$$(\mathbf{A} - \lambda\mathbf{I}_n)^m\mathbf{w} = 0 \quad \text{but} \quad (\mathbf{A} - \lambda\mathbf{I}_n)^{m-1}\mathbf{w} \neq 0$$

The set spanned by all generalized eigenvectors of λ is called *generalized eigenspace* of λ.

Proposition 85. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\lambda \in \sigma(A)$. Then, the dimension of the generalized eigenspace is the algebraic multiplicity of λ.

Lemma 86. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\mathbf{v}_1 \in \mathbb{R}^n$ be an eigenvector of $\mathbf{A}$ with associated eigenvalue λ. We define $\mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ in the following way:

$$(\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{v}_k = \mathbf{v}_{k-1} \qquad k = 2, \ldots, m$$

That is, $\mathbf{v}_k$ is a generalized eigenvector of rank k of $\mathbf{A}$ with associated eigenvalue λ. Then,

$$\begin{cases} \boldsymbol{\varphi}_1 = \mathrm{e}^{\lambda t}\mathbf{v}_1 \\ \boldsymbol{\varphi}_2 = \mathrm{e}^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ \boldsymbol{\varphi}_3 = \mathrm{e}^{\lambda t}\left(\mathbf{v}_3 + t\mathbf{v}_2 + \dfrac{t^2}{2}\mathbf{v}_1\right) \\ \quad \vdots \\ \boldsymbol{\varphi}_m = \mathrm{e}^{\lambda t}\left(\mathbf{v}_m + t\mathbf{v}_{m-1} + \cdots + \dfrac{t^{m-1}}{(m-1)!}\mathbf{v}_1\right) \end{cases}$$

are solutions of the ODE of Eq. (14). Furthermore, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, then so are $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_k$.

Corollary 87. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ and $\sigma(\mathbf{A}) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of $\mathbf{A}$ such that:

- $\lambda_1, \ldots, \lambda_{2k} \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_{k+i} = \overline{\lambda_i}$ and $\lambda_i = \alpha_i + \mathrm{i}\beta_i$, $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \ldots, k$.
- $\lambda_{2k+1}, \ldots, \lambda_n \in \mathbb{R}$

Then, the general solution of the ODE of Eq. (14) is of the form:

$$\boldsymbol{\varphi}(t) = \sum_{i=1}^{k} \mathrm{e}^{\alpha_i t}\left(\mathbf{P}_i(t)\cos(\beta_i t) + \mathbf{Q}_i(t)\sin(\beta_i t)\right) + \\ + \sum_{i=2k+1}^{n} \mathrm{e}^{\lambda_i t}\mathbf{R}_i(t)$$

where $\mathbf{P}_i, \mathbf{Q}_i, \mathbf{R}_i \in \mathbb{R}^n[t]$ and $\deg\mathbf{P}_i, \deg\mathbf{Q}_i, \deg\mathbf{R}_i < n$ $\forall i$.

5. | Dependence on initial conditions and parameters

Definition 88. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. Suppose that the ivp:

$$\begin{cases} \mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\lambda}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{16}$$

has a unique maximal solution $\boldsymbol{\varphi}_{(t_0,\mathbf{x}_0,\boldsymbol{\lambda})}(t)$ defined on an interval $I_{(t_0,\mathbf{x}_0,\boldsymbol{\lambda})}$. We define the *flow* of the ODE $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\lambda})$ as:

$$\begin{aligned} \boldsymbol{\phi} : I_{(t_0,\mathbf{x}_0,\boldsymbol{\lambda})} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p &\longrightarrow \mathbb{R}^n \\ (t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) &\longmapsto \boldsymbol{\varphi}_{(t_0,\mathbf{x}_0,\boldsymbol{\lambda})}(t) \end{aligned}$$

Continuous and Lipschitz continuous dependence

Lemma 89. Let X be a compact metric space and $(\boldsymbol{\varphi}_m)$ be a sequence of functions $\boldsymbol{\varphi}_m : X \to \mathbb{R}^n$ such that they are pointwise equicontinuous and pointwise bounded. Suppose that all convergent partial subsequences of $(\boldsymbol{\varphi}_m)$ have the same limit $\boldsymbol{\varphi}$. Then, $(\boldsymbol{\varphi}_m)$ converges uniformly to $\boldsymbol{\varphi}$.

Proposition 90. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $\mathbf{f}_m : U \to \mathbb{R}^n$ be continuous function for $m \in \mathbb{N}$ and such that for all compact $K \subset U$, the sequence $(\mathbf{f}_m|_K)$ converge uniformly to a function $\mathbf{f}_0|_K$. Let $((t_m, \mathbf{x}_m)) \subset U$ be a sequence such that $\lim_{m\to\infty}(t_m, \mathbf{x}_m) = (t_0, \mathbf{x}_0)$. Suppose that for all $m \geq 0$ the ivp

$$\begin{cases}\mathbf{x}' = \mathbf{f}_m(t, \mathbf{x})\\ \mathbf{x}(t_m) = \mathbf{x}_m\end{cases}$$

has a unique maximal solution $\boldsymbol{\varphi}_m$ defined on I_m. Then, for all $[a, b] \subset I_0$ with $t_0 \in (a, b)$, $\exists m_0 \in \mathbb{N}$ such that $[a, b] \subset I_m$ $\forall m > m_0$. Furthermore, the sequence $\left(\boldsymbol{\varphi}_m|_{[a,b]}\right)_{m>m_0}$ converges uniformly to $\boldsymbol{\varphi}_0|_{[a,b]}$.

Theorem 91. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. Suppose that each ivp of the form of Eq. (10) has a unique maximal solution. Then, the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ is a continuous function defined in an open set.

Theorem 92. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function. Suppose that each ivp of the form of Eq. (16) has a unique maximal solution. Then, the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ is a continuous function defined in an open set $V \subseteq I_{(t_0,\mathbf{x}_0,\boldsymbol{\lambda})} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$.

Lemma 93 (Grönwall's lemma). Let $u, v, w : [a, b) \to \mathbb{R}$ be continuous functions such that $v(t) \geq 0$ $\forall t \in [a, b)$ and satisfying:

$$u(t) \leq w(t) + \int_a^t v(s)u(s)\,\mathrm{d}s \quad \forall t \in [a, b)$$

Then:

$$u(t) \leq w(t) + \int_a^t w(s)v(s)\mathrm{e}^{\int_s^t v(r)\mathrm{d}r}\,\mathrm{d}s \quad \forall t \in [a, b)$$

If, moreover, $w \in \mathcal{C}^1((a, b))$, then:

$$u(t) \leq w(a)\mathrm{e}^{\int_a^t v(r)\mathrm{d}r} + \int_a^t w'(s)\mathrm{e}^{\int_s^t v(r)\mathrm{d}r}\,\mathrm{d}s \quad \forall t \in [a, b)$$

Proposition 94. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and Lipschitz continuous with respect to the second variable with Lipschitz constant L. Let $\boldsymbol{\phi}$ be the flow of the ODE $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$. Then, $\forall (t_0, \mathbf{x}_1), (t_0, \mathbf{x}_2) \in U$ and $\forall t \in I_{(t_0,\mathbf{x}_1)} \cap I_{(t_0,\mathbf{x}_2)}$, we have:

$$\|\boldsymbol{\phi}(t, t_0, \mathbf{x}_2) - \boldsymbol{\phi}(t, t_0, \mathbf{x}_1)\| \leq \mathrm{e}^{L|t-t_0|}\|\mathbf{x}_2 - \mathbf{x}_1\|$$

Thus, $\boldsymbol{\phi}$ is locally Lipschitz continuous with respect to the third variable.

Differentiable dependence

Theorem 95 (Dependence on $\mathbf{x}_0$). Let $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and of class $\mathcal{C}^1$ with respect to the second variable. Suppose that the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\lambda})$ is defined on an open set $V \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Then, $\forall (t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) \in V$, $\boldsymbol{\phi}$ is differentiable with respect to $\mathbf{x}_0$ and $\mathbf{D}_3\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$[8] is continuous on V. Furthermore, $\mathbf{D}_3\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ satisfies the following ivp:

$$\begin{cases}\mathbf{M}' = \mathbf{D}_2\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})\mathbf{M}\\ \mathbf{M}(t_0) = \mathbf{I}_n\end{cases}$$

Or, equivalently, $\frac{\partial \boldsymbol{\phi}}{\partial {\mathbf{x}_0}_i}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) = \mathbf{D}_3\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})\mathbf{e}_i$ satisfies the following ivp:

$$\begin{cases}\mathbf{y}' = \mathbf{D}_2\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})\mathbf{y}\\ \mathbf{y}(t_0) = \mathbf{e}_i\end{cases} \quad \text{for } i = 1, \ldots, n$$

These kinds of equations are called *variational equations*.

Theorem 96 (Dependence on t_0). Let $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and of class $\mathcal{C}^1$. Suppose that the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\lambda})$ is defined on an open set $V \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Then, $\forall (t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) \in V$, $\boldsymbol{\phi}$ is differentiable with respect to t_0 and $\mathbf{D}_2\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ is continuous on V. Furthermore, $\mathbf{D}_2\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ satisfies the following ivp:

$$\begin{cases}\mathbf{y}' = \mathbf{D}_2\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})\mathbf{y}\\ \mathbf{y}(t_0) = -\mathbf{f}(t_0, \mathbf{x}_0, \boldsymbol{\lambda})\end{cases}$$

Theorem 97 (Dependence on $\boldsymbol{\lambda}$). Let $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and of class $\mathcal{C}^1$ with respect to the second and third variable. Suppose that the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\lambda})$ is defined on an open set $V \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Then, $\forall (t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) \in V$, $\boldsymbol{\phi}$ is differentiable with respect to $\boldsymbol{\lambda}$ and $\mathbf{D}_4\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ is continuous on V. Furthermore, $\mathbf{D}_4\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ satisfies the following ivp:

$$\begin{cases}\mathbf{M}' = \mathbf{D}_2\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})\mathbf{M} + \mathbf{B}\\ \mathbf{M}(t_0) = \mathbf{0}\end{cases}$$

where $\mathbf{B} = \mathbf{D}_3\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})$. Equivalently, $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\lambda}_i}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ satisfies the ivp:

$$\begin{cases}\mathbf{y}' = \mathbf{D}_2\mathbf{f}(t, \boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}), \boldsymbol{\lambda})\mathbf{y} + \mathbf{B}\mathbf{e}_i\\ \mathbf{y}(t_0) = \mathbf{0}\end{cases} \quad \text{for } i = 1, \ldots, n$$

Higher order dependence

Theorem 98. Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a continuous function and of class $\mathcal{C}^k$, $k \in \mathbb{N}$. Suppose that the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ is defined on an open set $V \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. Then, $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ is of class $\mathcal{C}^k$ on V.

[8] Here, $\mathbf{D}_3\boldsymbol{\phi}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda}) = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}_0}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})$ denotes the matrix $\left(\frac{\partial \phi_i}{\partial {\mathbf{x}_0}_j}(t, t_0, \mathbf{x}_0, \boldsymbol{\lambda})\right) \in \mathcal{M}_n(\mathbb{R})$, where ${\mathbf{x}_0}_j$ denotes the j-th component of $\mathbf{x}_0$ and ϕ_i denotes the i-th component of $\boldsymbol{\phi}$.

6. | Qualitative theory of autonomous systems

Introduction to dynamical systems

Definition 99. A *dynamical system* is a triplet (X, G, Π), where G is a topological abelian group[9], X is a topological space and $\Pi : G \times X \to X$ is a function such that:

- $\Pi(t, \cdot)$ is continuous $\forall t \in G$.
- $\Pi(0, x) = x \ \forall x \in X$.
- $\Pi(s, \Pi(t, x)) = \Pi(t + s, x) \ \forall s, t \in G$ and $\forall x \in X$.

We say that a dynamical system (X, G, Π) is *discrete* if $G = \mathbb{Z}$ and we say that it is *continuous* if $G = \mathbb{R}$. If we have defined our system for $G_{\geq 0}$, we will say that we have a *semidynamical system*.

Definition 100. Let (G, X, Ψ) be a dynamical system and $x \in X$. The *orbit* through x is defined as:

$$\gamma(x) = \gamma_\Psi(x) := \{\Psi(t, x) : t \in G\}^{10}$$

Moreover if $G = \mathbb{Z}$ or $G = \mathbb{R}$ we define the *positive semi-orbit* through x and the *negative semi-orbit* through x as the following respective sets:

$$\gamma^+(x) = {\gamma_\Psi}^+(x) := \{\Psi(t, x) : t \in G_{\geq 0}\}$$
$$\gamma^-(x) = {\gamma_\Psi}^-(x) := \{\Psi(t, x) : t \in G_{\leq 0}\}$$

Definition 101. Let (G, X, Ψ) be a dynamical system. Then, we have an equivalence relation $\sim$ on X given by

$$x \sim y \iff \gamma(x) = \gamma(y) \quad \forall x, y \in X$$

which creates a partition of X, called *phase portrait*.

Definition 102. The *phase space* of an ODE or system of ODEs is the space in which all possible states of a system are represented with each possible state corresponding to one unique point in the phase space.

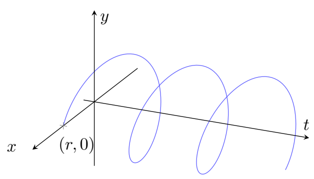


Figure 1: Phase space of the system $\{x' = -y, y' = x : (x(0), y(0)) = (r, 0)\}$.

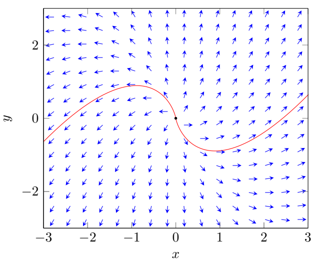


Figure 2: Vector field of the system $\{x' = x, y' = x + y\}$ together with two orbits.

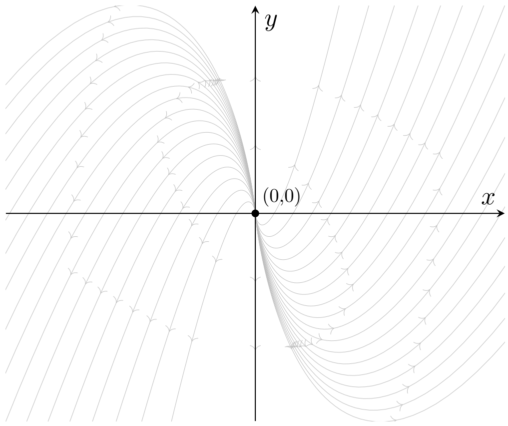


Figure 3: Phase portrait of the system $\{x' = x/2, y' = x + y/2\}$.

Definition 103. Let (G, X, Ψ) be a dynamical system and $x \in X$. We define the following function:

$$\begin{aligned} \Psi_x : G &\longrightarrow \gamma(x) \\ t &\longmapsto \Psi(t, x) \end{aligned}$$

Lemma 104. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that the flow $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ of the ODE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is defined for all $t \in \mathbb{R}$. Then, $(\mathbb{R}, \mathbb{R}^n, \boldsymbol{\Psi})$ is a dynamical system, where $\boldsymbol{\Psi}(t, \mathbf{x}) = \boldsymbol{\phi}(t, 0, \mathbf{x})$. Furthermore, note that $\boldsymbol{\gamma}(\mathbf{x}) = \operatorname{im}(\boldsymbol{\phi}(\cdot, 0, \mathbf{x}))$.

Lemma 105. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $\exists M, N \in \mathbb{R}_{\geq 0}$ with $\|\mathbf{f}(\mathbf{x})\| \leq M\|\mathbf{x}\| + N$. Then, the solutions of the ODE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ are defined for all $t \in \mathbb{R}$.

Definition 106. Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ be continuous functions and $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x}' = \mathbf{g}(\mathbf{x})$ be two ODEs for which we have existence and uniqueness of solutions. We say that these two ODEs are *equivalent* if there exists $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{h}(\mathbf{x}) \geq 0$ and $\mathbf{f}(\mathbf{x}) = \mathbf{h}(\mathbf{x})\mathbf{g}(\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{R}^n$. Therefore, $\mathbf{f}$ and $\mathbf{g}$ have the same orbits oriented in the same way.

Corollary 107. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that the ODE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has existence and uniqueness of solutions for all initial conditions. Then, there exists a continuous function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that the autonomous ODEs induced by $\mathbf{f}$ and $\mathbf{g}$ are equivalent and the flow of the ODE $\mathbf{x}' = \mathbf{g}(\mathbf{x})$ is defined $\forall t \in \mathbb{R}$.

Lemma 108. Let H be a proper subgroup of $\mathbb{R}$ which is closed. Then, $\exists T \in \mathbb{R}_{\geq 0}$ such that $H = T\mathbb{Z}$.

Proposition 109. Let $(\mathbb{R}, \mathbb{R}^n, \boldsymbol{\Psi})$ be a dynamical system and $\boldsymbol{\gamma}(\mathbf{x})$ be an orbit. Then, there are 3 possible cases for $\boldsymbol{\gamma}(\mathbf{x})$:

1. $\boldsymbol{\gamma}(\mathbf{x}) = \{\mathbf{x}\}$.
2. $\boldsymbol{\gamma}(\mathbf{x}) \cong S^1$.
3. $\boldsymbol{\gamma}(\mathbf{x})$ is homeomorphic to an injective and continuous image of $\mathbb{R}$.

[9] That is, G is an abelian group with an inherited topological structure.
[10] In general, if the context is clear we will use the notation $\gamma(x)$ instead of $\gamma_\Psi(x)$.

Definition 110. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{p} \in \mathbb{R}^n$. We say that $\mathbf{p} \in \mathbb{R}^n$ is a *critical point* or *singular point* if $\boldsymbol{\gamma}(\mathbf{p}) = \{\mathbf{p}\}$. Otherwise, we say that $\mathbf{p}$ is *non-singular* or *regular*.

Definition 111. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\boldsymbol{\gamma}(\mathbf{x})$ be an orbit of $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$. We say that $\boldsymbol{\gamma}(\mathbf{x})$ is *periodic* of period $T > 0$ if $\boldsymbol{\gamma}(\mathbf{x}) \cong S^1$ and $\ker \mathbf{\Psi}_{\mathbf{x}} = T\mathbb{Z}$.

Proposition 112. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system such that $\mathbf{\Psi}(t, \mathbf{x}) = \boldsymbol{\phi}(t, 0, \mathbf{x})$, where $\boldsymbol{\phi}(t, t_0, \mathbf{x}_0)$ is the flow of the ODE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. Let $\mathbf{p} \in \mathbb{R}^n$. Then, the following statements are equivalent:

1. $\{\mathbf{p}\}$ is a critical point.
2. $\boldsymbol{\phi}(t, 0, \mathbf{p}) = \mathbf{p}$.
3. $\mathbf{f}(\mathbf{p}) = 0$.

Definition 113. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{y} \in \mathbb{R}^n$ is an *α-limit point* of $\mathbf{x}$ if there exists a sequence $(t_n) \subset \mathbb{R}$ such that $\lim_{n\to\infty} t_n = -\infty$ and $\lim_{n\to\infty} \mathbf{\Psi}(t_n, \mathbf{x}) = \mathbf{y}$. The set of all α-limit points of $\mathbf{x}$ is called *α-limit set*, and it is denoted by $\alpha(\mathbf{x})$. For an orbit $\boldsymbol{\gamma}$ of $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$, we say that $\mathbf{y}$ is an *ω-limit point* of $\boldsymbol{\gamma}$, it is a ω-limit point of some point on the orbit $\boldsymbol{\gamma}$. The set of such α-limit points will be denoted as $\alpha(\boldsymbol{\gamma})$.

Definition 114. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{y} \in \mathbb{R}^n$ is an *ω-limit point* of $\mathbf{x}$ if there exists a sequence $(t_n) \subset \mathbb{R}$ such that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} \mathbf{\Psi}(t_n, \mathbf{x}) = \mathbf{y}$. The set of all ω-limit points of $\mathbf{x}$ is called *ω-limit set*, and it is denoted by $\omega(\mathbf{x})$. For an orbit $\boldsymbol{\gamma}$ of $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$, we say that $\mathbf{y}$ is an *ω-limit point* of $\boldsymbol{\gamma}$, it is a ω-limit point of some point on the orbit $\boldsymbol{\gamma}$. The set of such ω-limit points will be denoted as $\omega(\boldsymbol{\gamma})$.

Proposition 115. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{x} \in \mathbb{R}^n$. Then:

$$\mathrm{Cl}(\boldsymbol{\gamma}(\mathbf{x})) = \alpha(\mathbf{x}) \cup \boldsymbol{\gamma}(\mathbf{x}) \cup \omega(\mathbf{x})$$

Definition 116. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $A \subseteq \mathbb{R}^n$ be a subset. We say that A is *invariant* if $\boldsymbol{\gamma}(\mathbf{x}) \subseteq A \ \forall \mathbf{x} \in A$. We say that A is *positively invariant* if $\boldsymbol{\gamma}^+(\mathbf{x}) \subseteq A \ \forall \mathbf{x} \in A$. Analogously, we say that A is *negatively invariant* if $\boldsymbol{\gamma}^-(\mathbf{x}) \subseteq A \ \forall \mathbf{x} \in A$.

Proposition 117. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system, $\mathbf{p} \in \mathbb{R}^n$ and $\boldsymbol{\gamma}$ be an orbit of the system such that $\boldsymbol{\gamma}^+$ is contained in a compact set. Then:

- $\omega(\mathbf{p}) \neq \varnothing$.
- $\omega(\mathbf{p})$ is compact.
- $\omega(\mathbf{p})$ is invariant.
- $\omega(\mathbf{p})$ is connected.
- If $\omega(\boldsymbol{\gamma}) \subseteq \boldsymbol{\gamma} \implies \omega(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is either a critical point or a period orbit.

Proposition 118. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system, $\mathbf{p} \in \mathbb{R}^n$ and $\boldsymbol{\gamma}$ be an orbit of the system such that $\boldsymbol{\gamma}^-$ is contained in a compact set. Then:

- $\alpha(\mathbf{p}) \neq \varnothing$.
- $\alpha(\mathbf{p})$ is compact.
- $\alpha(\mathbf{p})$ is invariant.
- $\alpha(\mathbf{p})$ is connected.
- If $\alpha(\boldsymbol{\gamma}) \subseteq \boldsymbol{\gamma} \implies \alpha(\boldsymbol{\gamma}) = \boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is either a critical point or a period orbit.

Definition 119. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $K \subset \mathbb{R}^n$ be a compact set. We say that K is *positively stable* if for all neighbourhood U of K, there exists a neighbourhood V of K with $V \subseteq U$ and such that $\forall \mathbf{x} \in V$, $\boldsymbol{\gamma}^+(\mathbf{x}) \subset U$. Analogously, we say that K is *negatively stable* if for all neighbourhood U of K, there exists a neighbourhood V of K with $V \subseteq U$ and such that $\forall \mathbf{x} \in V$, $\boldsymbol{\gamma}^-(\mathbf{x}) \subset U$.

Definition 120. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $K \subset \mathbb{R}^n$ be a compact set. We say that K is *attracting* if there exists a neighbourhood U of K such that $\forall \mathbf{x} \in U$, $\omega(\mathbf{x}) \subset K$. We say that K is *repelling* if there exists a neighbourhood U of K such that $\forall \mathbf{x} \in U$, $\alpha(\mathbf{x}) \subset K$. We say that K is *asymptotically stable* if it is both attracting and positively stable.

Proposition 121. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $K \subset \mathbb{R}^n$ be a compact set. Suppose that K is positively stable. Then, K is positively invariant.

Definition 122. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{p} \in \mathbb{R}^n$. We say that $\mathbf{p}$ is a *center* if there exists a neighbourhood U of $\mathbf{p}$ such that if $\boldsymbol{\gamma}(\mathbf{x}) \subset U$, then $\boldsymbol{\gamma}(\mathbf{x})$ is periodic. The largest neighbourhood with this property is called *basin* of the center.

Proposition 123. Let $(\mathbb{R}, \mathbb{R}^n, \mathbf{\Psi})$ be a dynamical system and $\mathbf{p} \in \mathbb{R}^n$ be a center. Then:

1. $\mathbf{p}$ is positively and negatively stable.
2. $\mathbf{p}$ is not attracting.

Equivalence and conjugacy of dynamical systems

Definition 124. Let (G, X, Ψ_1) and (G, X, Ψ_2) be dynamical systems and $r \in \mathbb{N} \cup \{0, \infty\}$. We say that (G, X, Ψ_1) and (G, X, Ψ_2) are *equivalent dynamical systems* of class $\mathcal{C}^r$ if there exists a diffeomorphism $h : X \to Y$ of class $\mathcal{C}^r$ such that $\forall x \in X$, $h(\gamma_{\Psi_1}(x)) = \gamma_{\Psi_2}(h(x))$ and preserving the orientation of the orbits. In particular, if $r = 0$ we say that (G, X, Ψ_1) and (G, X, Ψ_2) are *topologically equivalent*. That diffeomorphism h is called an *equivalence* (of class $\mathcal{C}^r$) between (G, X, Ψ_1) and (G, X, Ψ_2).

Definition 125. Let (G, X, Ψ_1) and (G, X, Ψ_2) be dynamical systems and $r \in \mathbb{N} \cup \{0, \infty\}$. We say that (G, X, Ψ_1) and (G, X, Ψ_2) are *conjugate dynamical systems* of class $\mathcal{C}^r$ if there exists a diffeomorphism $h : X \to Y$ of class $\mathcal{C}^r$ such that $\forall (t, x) \in G \times X$, $h(\Psi_1(t, x)) = \Psi_2(t, h(x))$. In particular, if $r = 0$ we say that (G, X, Ψ_1) and (G, X, Ψ_2) are *topologically conjugate*. That diffeomorphism h is called a *conjugacy* (of class $\mathcal{C}^r$) between (G, X, Ψ_1) and (G, X, Ψ_2).

Proposition 126. Let (G, X, Ψ_1) and (G, X, Ψ_2) be dynamical systems and h be a conjugacy of class $\mathcal{C}^r$ between them. Then, h is an equivalence of class $\mathcal{C}^r$ between (G, X, Ψ_1) and (G, X, Ψ_2).

Proposition 127. Two dynamical systems induced by two equivalent ODEs are equivalent (as a dynamical systems).

Proposition 128. Let (G, X, Ψ_1) and (G, X, Ψ_2) be dynamical systems and $h : X \to Y$ be an equivalence of class $\mathcal{C}^r$ between them. Then:

1. h preserves the type of orbit. More precisely if $p \in X$, we have:
   i) If p is a critical point, then so it is $h(p)$.
   ii) If $\gamma(p)$ is a periodic orbit, then so it is $h(\gamma(p))$[11].
   iii) If $\gamma(p)$ is the injective and continuous image of $\mathbb{R}$, then so it is $h(\gamma(p))$.
2. If $p \in X$ is a critical point of Ψ_1, we have:
   i) If p is attracting for (G, X, Ψ_1), then so it is $h(p)$ for (G, X, Ψ_2).
   ii) If p is repelling for (G, X, Ψ_1), then so it is $h(p)$ for (G, X, Ψ_2).
   iii) If p is positively stable for (G, X, Ψ_1), then so it is $h(p)$ for (G, X, Ψ_2).
   iv) If p is asymptotically stable for (G, X, Ψ_1), then so it is $h(p)$ for (G, X, Ψ_2).

Proposition 129. A conjugacy between two dynamical systems preserves the period of periodic orbits.

Proposition 130. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha\beta > 0$. Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by:

$$h(x) = \begin{cases} x^{\beta/\alpha} & \text{if } x \geq 0 \\ -|x|^{\beta/\alpha} & \text{if } x < 0 \end{cases}$$

Then, h is a topological conjugation between the systems induced by the ODEs $x' = \alpha x$ and $y' = \beta y$.

Proposition 131. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n(\mathbb{R})$ be similar matrices, that is, $\exists \mathbf{P} \in \mathcal{M}_n(\mathbb{R})$ such that $\mathbf{B} = \mathbf{PAP}^{-1}$. Then, the function

$$\begin{aligned} \mathbf{h} : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{Px} \end{aligned}$$

is a conjugation between the systems induced by the ODEs $\mathbf{x}' = \mathbf{Ax}$ and $\mathbf{y}' = \mathbf{By}$.

Local equivalence and conjugacy of dynamical systems

Definition 132. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$. For all $\mathbf{x}_0 \in U$, let $\boldsymbol{\varphi}_{\mathbf{x}_0}(t)$ be the maximal solution to the ivp

$$\begin{cases} \mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

We define the *flow* of the vector field $\mathbf{f}$ as $\boldsymbol{\phi}(t, \mathbf{x}) := \boldsymbol{\varphi}_{\mathbf{x}}(t)$.

Proposition 133. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$. Then, the flow $\boldsymbol{\phi}(t, \mathbf{x})$ of $\mathbf{f}$ defines locally a dynamical system.

Definition 134. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields of class $\mathcal{C}^1$, and $r \in \mathbb{N} \cup \{0, \infty\}$. We say that $\mathbf{f}$ and $\mathbf{g}$ are *equivalent* of class $\mathcal{C}^r$ if there exists a diffeomorphism $\mathbf{h} : U \to V$ of class $\mathcal{C}^r$ such that $\forall \mathbf{x} \in U$, $\mathbf{h}(\boldsymbol{\gamma}_{\mathbf{f}}(\mathbf{x})) = \boldsymbol{\gamma}_{\mathbf{g}}(\mathbf{h}(\mathbf{x}))$ and preserving the orientation of the orbits. In particular, if $r = 0$ we say that $\mathbf{f}$ and $\mathbf{g}$ are *locally topologically equivalent*. That diffeomorphism $\mathbf{h}$ is called a *local equivalence* (of class $\mathcal{C}^r$) between $\mathbf{f}$ and $\mathbf{g}$.

Definition 135. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields of class $\mathcal{C}^1$ with flows $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, respectively, and $r \in \mathbb{N} \cup \{0, \infty\}$. We say that $\mathbf{f}$ and $\mathbf{g}$ are *conjugate* of class $\mathcal{C}^r$ if there exists a diffeomorphism $\mathbf{h} : U \to V$ of class $\mathcal{C}^r$ such that $\mathbf{h}(\boldsymbol{\phi}(t, \mathbf{x})) = \boldsymbol{\psi}(t, \mathbf{h}(\mathbf{x}))$ $\forall (t, \mathbf{x}) \in \operatorname{dom}(\boldsymbol{\phi})$ when the equation is well-defined[12]. In particular, if $r = 0$ we say that $\mathbf{f}$ and $\mathbf{g}$ are *locally topologically conjugate*. That diffeomorphism $\mathbf{h}$ is called a *local conjugacy* (of class $\mathcal{C}^r$) between $\mathbf{f}$ and $\mathbf{g}$.

Lemma 136. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields of class $\mathcal{C}^1$ and $\mathbf{h} : U \to V$ be a diffeomorphism of class $\mathcal{C}^1$. Then, $\mathbf{h}$ is a conjugacy of class $\mathcal{C}^1$ between them if and only if $\mathbf{Dh}(\mathbf{x})(\mathbf{f}(\mathbf{x})) = \mathbf{g}(\mathbf{h}(\mathbf{x}))$ $\forall \mathbf{x} \in U$.

Proposition 137. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields. Suppose that $\mathbf{h}$ is a conjugacy between $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and $\mathbf{y}' = \mathbf{g}(\mathbf{y})$. Then, $\mathbf{x}' = -\mathbf{f}(\mathbf{x})$ and $\mathbf{y}' = -\mathbf{g}(\mathbf{y})$ are conjugate by $\mathbf{h}$.

Definition 138. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^r$, $r \geq 1$, and $A \subseteq \mathbb{R}^{n-1}$ be an open set. We say that function $\mathbf{s} : A \to U$ of class $\mathcal{C}^r$ is a *local transversal section* of $\mathbf{f}$ of class $\mathcal{C}^r$ if $\forall a \in A$, $\langle \operatorname{im} \mathbf{Dh}(a), \mathbf{f}(\mathbf{s}(\mathbf{a})) \rangle = \mathbb{R}^n$[13]. Take $\Sigma := \mathbf{s}(A)$. If $\mathbf{s} : A \to \Sigma$ is a homeomorphism, we say that Σ is a *transversal section* of $\mathbf{f}$ of class $\mathcal{C}^r$.

Theorem 139 (Flow box theorem). Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^r$, $r \geq 1$, $\mathbf{p} \in U$ be a non-singular point of $\mathbf{f}$ and $\mathbf{s} : A \to \Sigma$ be a transversal section of $\mathbf{f}$ of class $\mathcal{C}^r$ with $\mathbf{s}(\mathbf{0}) = \mathbf{p}$. Then, there exists a neighbourhood $V \subseteq U$ of $\mathbf{p}$ and a diffeomorphism $\mathbf{h} : V \to (-\varepsilon, \varepsilon) \times B$ of class $\mathcal{C}^r$, where $\varepsilon > 0$ and $B \subseteq \mathbb{R}^{n-1}$ is an open ball centered at $\mathbf{0} = \mathbf{s}^{-1}(\mathbf{p})$, such that:

- $\mathbf{h}(\Sigma \cap V) = \{0\} \times B$.
- $\mathbf{h}$ is a conjugacy of class $\mathcal{C}^r$ between $\mathbf{f}|_V$ and $\mathbf{g} : (-\varepsilon, \varepsilon) \times B \to \mathbb{R}^n$ defined as $\mathbf{g}(\mathbf{x}) = (1, 0 \ldots, 0)$ $\forall \mathbf{x} \in \operatorname{dom} \mathbf{g}$.

[11] Note that the period of $\gamma(p)$ and $h(\gamma(p))$ may be different.
[12] That is, $\forall (t, \mathbf{x}) \in \operatorname{dom}(\boldsymbol{\phi})$ such that $(t, \mathbf{h}(\mathbf{x})) \in \operatorname{dom}(\boldsymbol{\psi})$.
[13] A plane version would be that $\mathbf{s} : A \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ is a *local transversal section* of $\mathbf{f}$ if $\forall a \in A$, $\mathbf{s}'(a)$ and $\mathbf{f}(\mathbf{s}(\mathbf{a}))$ are linearly independent.

Lemma 140. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. Suppose that $\mathbf{h}$ is a conjugacy of class $\mathcal{C}^1$ between $\mathbf{f}$ and $\mathbf{g}$. Then, the vector fields $\mathbf{Df}(\mathbf{p})$ and $\mathbf{Dg}(\mathbf{h}(\mathbf{p}))$ are conjugate by $\mathbf{Dh}(\mathbf{p})$. In particular, $\sigma(\mathbf{Df}(\mathbf{p})) = \sigma(\mathbf{Dg}(\mathbf{h}(\mathbf{p})))$.

Definition 141. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. Suppose $\sigma(\mathbf{Df}(\mathbf{p})) = \{\lambda_1, \lambda_2\}$. We say that $\mathbf{p}$ is a

- *stable node* if $\lambda_1, \lambda_2 \in \mathbb{R}_{<0}$ and $\lambda_1 \neq \lambda_2$ (see Fig. 10a).
- *stable degenerated node* if $\lambda_1, \lambda_2 \in \mathbb{R}_{<0}$, $\lambda_1 = \lambda_2$ and $\mathbf{Df}(\mathbf{p}) \sim \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$ (see Fig. 10b).
- *stable star* if $\lambda_1, \lambda_2 \in \mathbb{R}_{<0}$, $\lambda_1 = \lambda_2$ and $\mathbf{Df}(\mathbf{p}) \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ (see Fig. 10c).
- *unstable node* if $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ and $\lambda_1 \neq \lambda_2$ (see Fig. 10d).
- *unstable degenerated node* if $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$, $\lambda_1 = \lambda_2$ and $\mathbf{Df}(\mathbf{p}) \sim \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$ (see Fig. 10e).
- *unstable star* if $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$, $\lambda_1 = \lambda_2$ and $\mathbf{Df}(\mathbf{p}) \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ (see Fig. 10f).
- *saddle point* if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 < 0$ (see Fig. 10j).
- *stable focus* (or *sink*) if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\mathrm{Re}(\lambda_1) < 0$ (see Fig. 10h).
- *unstable focus* (or *source*) if $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\mathrm{Re}(\lambda_1) > 0$ (see Fig. 10i).
- *center* if $\lambda_1, \lambda_2 \in \mathbb{C}$, $\mathrm{Re}(\lambda_1) = 0$ and $\mathbf{p}$ is surrounded by periodic orbits (see Fig. 10g).

Definition 142. Let $\mathbf{A} \in \mathcal{M}_2(\mathbb{R})$ and consider the linear system induced by $\mathbf{A}$ such that the origin is a saddle point, and E_1, E_2 be the eigenspaces of $\mathbf{A}$. We say that the four orbits contained in $E_1 \cup E_2$ (without taking into account the singular point $\mathbf{0}$) are the *saddle separatrices* of the linear system.

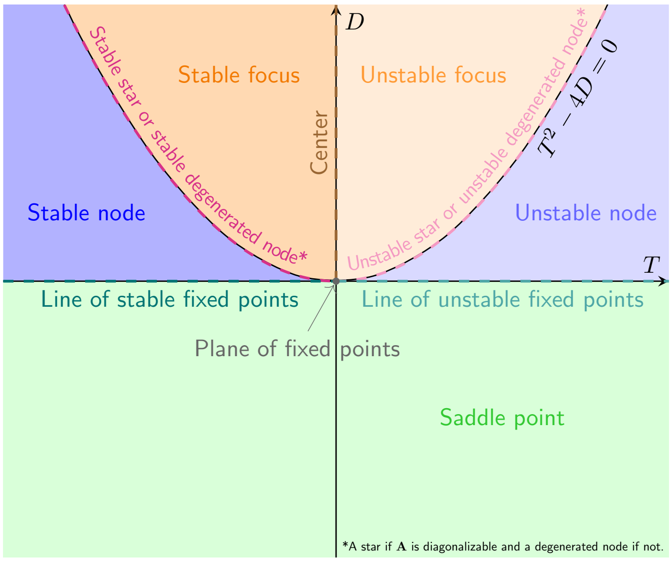


Figure 4: Classification of singular points of a linear dynamical system of dimension 2, induced by the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{A} \in \mathcal{M}_2(\mathbb{R})$ in terms of $D = \det \mathbf{A}$ and $T = \operatorname{tr} \mathbf{A}$.

Definition 143. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field and consider the differential system induced by $\mathbf{f}$. Let $\boldsymbol{\gamma}$ be an orbit of that system. We say that $\boldsymbol{\gamma}$ is a *homoclinic orbit* if it joins a saddle point to itself. We say that $\boldsymbol{\gamma}$ is a *heteroclinic orbit* if it joins joins two different singular points.

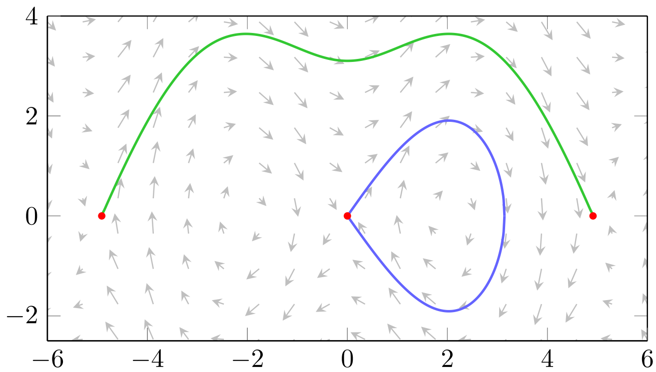


Figure 5: A homoclinic orbit (blue) and a heteroclinic orbit (green) of the differential system $x'' = \sin(x) + x\cos(x)$.

Definition 144. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. We say that $\mathbf{p}$ has

- an *elliptic sector* if a side of $\mathbf{p}$ is locally as Fig. 6.
- a *hyperbolic sector* if a side of $\mathbf{p}$ is locally as Fig. 7.
- an *attracting parabolic sector* if a side of $\mathbf{p}$ is locally as Fig. 8.
- a *repelling parabolic sector* if a side of $\mathbf{p}$ is locally as Fig. 9.

The union of all sectors that form a neighbourhood of $\mathbf{p}$ is called *sectorial decomposition*.

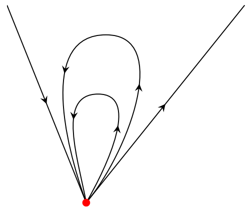


Figure 6: Elliptic sector

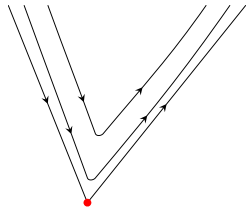


Figure 7: Hyperbolic sector

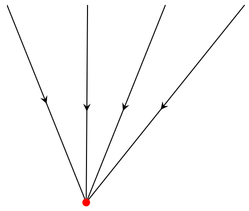


Figure 8: Attracting parabolic sector

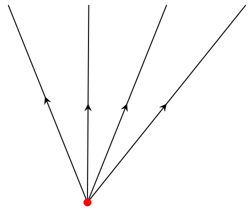


Figure 9: Repelling parabolic sector

Proposition 145. Any critical point of an analytic differential system in the plane can either be:

- A focus.
- A center.
- A finite collection of elliptic sectors, hyperbolic sectors and/or parabolic sectors.

Hamiltonian systems

Definition 146. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field and $H : U \to \mathbb{R}$ be a non-constant function. We say that H is a *first integral* for the differential system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if for each solution $\boldsymbol{\varphi}(t)$ of that system, we have $H(\boldsymbol{\varphi}(\mathbf{t})) = \text{const}$. Thus, the phase trajectory of a solution $\boldsymbol{\varphi}(t)$ to $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ lies on a level surface of H. In particular, if $n = 2$, $\boldsymbol{\varphi}(t)$ will be a level curve of H.

Proposition 147. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field such that $\mathbf{f} = (f_1, \ldots, f_n)$ and $H : U \to \mathbb{R}$ be a non-constant function. Then, H is a first integral for the differential system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if and only if:

$$\frac{\partial H}{\partial x_1}(\mathbf{x}) f_1(\mathbf{x}) + \cdots + \frac{\partial H}{\partial x_n}(\mathbf{x}) f_n(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in U$$

Definition 148. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field and $H_1, \ldots, H_k : U \to \mathbb{R}$ be $k \leq n-1$ first integrals for the differential system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. We say that $H_1, \ldots, H_k$ are *functionally independent* (or simply *independent*) if for all $\mathbf{x} \in U$ (except for maybe a set of zero area), we have:

$$\operatorname{rank} \begin{pmatrix} \frac{\partial U_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial U_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial U_k}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial U_k}{\partial x_n}(\mathbf{x}) \end{pmatrix} = k$$

Proposition 149. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field and $H_1, \ldots, H_k$ be $k \leq n-1$ functionally independent first integrals for the differential system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. Then, the number of unknowns of the system can be reduced to $n - k$.

Definition 150. Let $U \subseteq \mathbb{R}^{2n}$ be an open set, $H : U \to \mathbb{R}$ be a function and $(\mathbf{x}, \mathbf{y}) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$. The differential system

$$\begin{cases} {x_1}' = -\dfrac{\partial H}{\partial y_1}(\mathbf{x}) \\ \quad\vdots \\ {x_n}' = -\dfrac{\partial H}{\partial y_n}(\mathbf{x}) \\ {y_1}' = \dfrac{\partial H}{\partial x_1}(\mathbf{x}) \\ \quad\vdots \\ {y_n}' = \dfrac{\partial H}{\partial x_1}(\mathbf{x}) \end{cases}$$

is called *Hamiltonian system* in $2n$ unknowns and H is called *Hamiltonian* of the system.

Proposition 151. Let $U \subseteq \mathbb{R}^{2n}$ be an open set and $H : U \to \mathbb{R}$ be the Hamiltonian of its Hamiltonian associated system. Then, H is a first integral of that differential system.

Theorem 152. Let $U \subseteq \mathbb{R}^2$ be an open set, $H : U \to \mathbb{R}$ be the Hamiltonian of its Hamiltonian associated system and $\mathbf{p} \in U$ be a singular point. Then, $\mathbf{p}$ is either a saddle point or a center.

Local structure of hyperbolic critical points

Definition 153. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^r$ with $r \geq 1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. We say that $\mathbf{p}$ is *hyperbolic critical point* if $\operatorname{Re}(\lambda) \neq 0$ $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$.

Theorem 154 (Hartman-Grobman theorem). Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a hyperbolic critical point of $\mathbf{f}$. Let $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ be the vector field defined as $\mathbf{g}(\mathbf{x}) = \mathbf{Df}(\mathbf{p})(\mathbf{x})$. Then, there exist neighbourhoods $V \subseteq U$ of $\mathbf{p}$ and $W \subseteq \mathbb{R}^n$ of $\mathbf{f}(\mathbf{p}) = \mathbf{0}$ such that $\mathbf{f}|_V$ and $\mathbf{g}|_W$ topologically conjugate.

Corollary 155. Let $U, V \subseteq \mathbb{R}^n$ be open sets and $\mathbf{f} : U \to \mathbb{R}^n$, $\mathbf{g} : V \to \mathbb{R}^n$ be vector fields of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a hyperbolic critical point of $\mathbf{f}$. Suppose that $\mathbf{h}$ is a conjugacy of class $\mathcal{C}^1$ between $\mathbf{f}$ and $\mathbf{g}$. Then, $\mathbf{h}(\mathbf{p})$ is a hyperbolic critical point of $\mathbf{g}$.

Definition 156. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$. We say that $\mathbf{A}$ is *hyperbolic matrix* if $\operatorname{Re}(\lambda) \neq 0$ $\forall \lambda \in \sigma(\mathbf{A})$.

Proposition 157. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ be a hyperbolic matrix. Then, $\mathbf{0} \in \mathbb{R}^n$ is the unique critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and it is hyperbolic.

Definition 158. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$. We define the *stability number* of $\mathbf{A}$ as:

$$\iota(\mathbf{A}) := |\{\lambda \in \sigma(\mathbf{A}) : \operatorname{Re}(\lambda) < 0\}|$$

Theorem 159. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ be a hyperbolic matrix such that $\iota(\mathbf{A}) = n$. Then, $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{y}' = -\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^n$, are topologically conjugate. In particular, the origin is attracting.

Corollary 160. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ be a hyperbolic matrix such that $\iota(\mathbf{A}) = 0$. Then, $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{y}' = \mathbf{y}$, $\mathbf{y} \in \mathbb{R}^n$, are topologically conjugate. In particular, the origin is repelling.

Corollary 161. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ be a hyperbolic matrix such that $\iota(\mathbf{A}) = k$. Then, $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and

$$\{\mathbf{y}' = -\mathbf{y}, \mathbf{z}' = \mathbf{z} : \mathbf{y} \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}\}$$

are topologically conjugate. In particular, the origin is neither attracting nor repelling.

Theorem 162. Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n(\mathbb{R})$ be hyperbolic matrices. Then, $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{y}' = \mathbf{B}\mathbf{y}$ are topologically conjugate if and only if $\iota(\mathbf{A}) = \iota(\mathbf{B})$.

Corollary 163. Let $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ be a hyperbolic matrix. Then:

- $\mathbf{0}$ is attracting for $\mathbf{x}' = \mathbf{A}\mathbf{x} \iff \iota(\mathbf{A}) = n$.
- $\mathbf{0}$ is repelling for $\mathbf{x}' = \mathbf{A}\mathbf{x} \iff \iota(\mathbf{A}) = 0$.

Theorem 164. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. Then:

1. If $\iota(\mathbf{Df}(\mathbf{p})) = n$, then $\mathbf{p}$ is asymptotically stable for the dynamical system induced by $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
2. If $\iota(\mathbf{Df}(\mathbf{p})) = 0$, then $\mathbf{p}$ is repelling and negatively stable for the dynamical system induced by $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
3. If $\mathbf{p}$ is positively stable, then $\mathrm{Re}(\lambda) \leq 0$ $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$.
4. If $\mathbf{p}$ is negatively stable, then $\mathrm{Re}(\lambda) \geq 0$ $\forall \lambda \in \sigma(\mathbf{Df}(\mathbf{p}))$.

[14]Note that $\deg k \leq n - 1$.

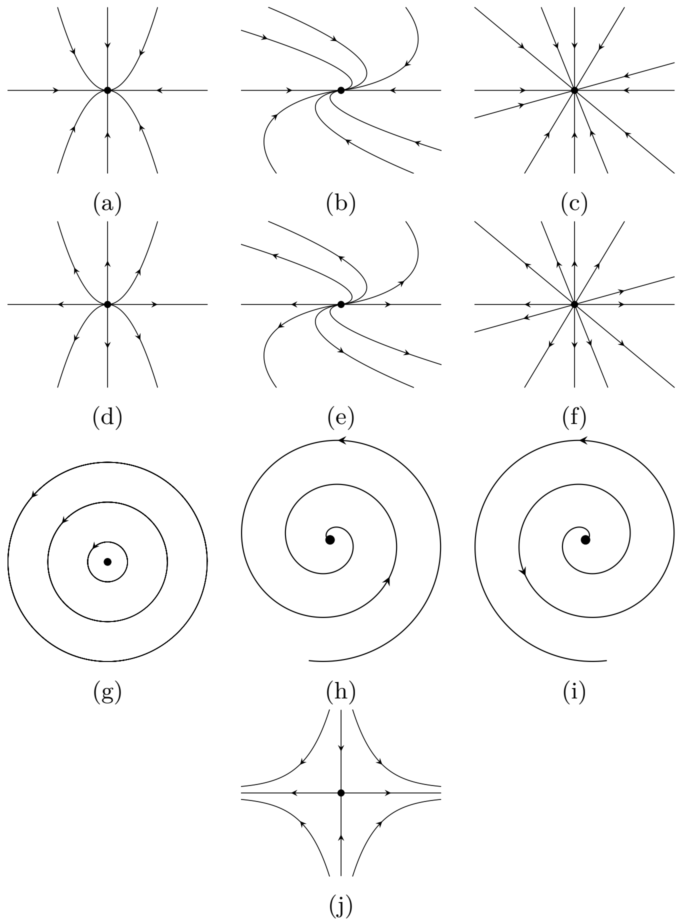


Figure 10: Phase portraits of hyperbolic singular points

7. | Qualitative theory of planar differential systems

Polynomial vectors fields

Definition 165. Let $p, q \in \mathbb{R}[x, y]$. The system of ODEs

$$\begin{cases} x' = p(x, y) \\ y' = q(x, y) \end{cases} \tag{17}$$

is called a *polynomial system*. The field $\mathbf{f} = (p, q)$ is called *polynomial vector field*. We define the *degree* of that system as $n := \max\{\deg p, \deg q\}$. Another commonly used notation for expressing the vector field is through the operator

$$\mathbf{X} := p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y} \tag{18}$$

Definition 166. Let $f \in \mathbb{R}[x, y]$ be a polynomial. An *algebraic curve* is the set of points satisfying the equation $f(x, y) = 0$.

Definition 167. Let $f(x, y) = 0$ be an algebraic curve, $p, q \in \mathbb{R}[x, y]$ and consider the polynomial system of degree n of Eq. (17). We say that $f(x, y) = 0$ is an *invariant algebraic curve* under the system of Eq. (17) if

$$\frac{\partial f}{\partial x}(x, y)p(x, y) + \frac{\partial f}{\partial y}q(x, y) = k(x, y)f(x, y) \tag{19}$$

where $k \in \mathbb{R}[x, y]$ is called *cofactor* of the invariant curve $f(x, y) = 0$[14]. The Eq. (19) can be written as:

$$\mathbf{X}f = kf$$

where $\mathbf{X}$ is the operator defined in Eq. (18).

Proposition 168. Let $f(x, y) = 0$ be an algebraic curve, $p, q \in \mathbb{R}[x, y]$ and consider the polynomial system of degree n of Eq. (17). Then, the invariant curve $f(x, y) = 0$ is a set of orbits of the differential system of Eq. (17).

Local structure of periodic orbits

Definition 169 (Poincaré map). Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ with flow $\boldsymbol{\phi}(t, \mathbf{x})$, $\mathbf{p} \in U$ and $\boldsymbol{\gamma}(\mathbf{p})$ be a periodic orbit of period T that passes through $\mathbf{p}$. Let Σ be a transversal section at $\mathbf{p}$. For each $\mathbf{q} \in \Sigma$ (close enough to $\mathbf{p}$) such that the trajectory $\boldsymbol{\phi}(t, \mathbf{q})$ intersects Σ in a distinct point from $\mathbf{q}$, we define the *Poincaré map* as the function $\pi : \Sigma_0 \subset \Sigma \to \Sigma$ sending $\mathbf{q}$ to the first point where $\boldsymbol{\phi}(t, \mathbf{q})$ intersects Σ (different from $\mathbf{q}$).

Definition 170. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\boldsymbol{\gamma}$ be a periodic orbit. We say that $\boldsymbol{\gamma}$ is a *limit cycle* if there exists a neighbourhood V of $\boldsymbol{\gamma}$ such that $\boldsymbol{\gamma}$ is the only periodic orbit in V.

Definition 171. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\boldsymbol{\gamma}$ be a periodic orbit. We denote by $\mathrm{Ext}(\boldsymbol{\gamma})$ the set of points which belong to the unbounded component of $\mathbb{R}^2 \setminus \boldsymbol{\gamma}$, and by $\mathrm{Int}(\boldsymbol{\gamma})$ the set of points which belong to the bounded component of $\mathbb{R}^2 \setminus \boldsymbol{\gamma}$.

Proposition 172. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$, $\boldsymbol{\gamma}$ be a limit cycle and V be a neighbourhood of $\boldsymbol{\gamma}$. Then, $\boldsymbol{\gamma}$ is exactly one of the following three types of limit cycles:

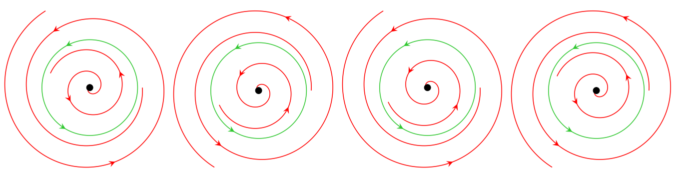
- $\boldsymbol{\gamma}$ is *stable* if $\omega(\mathbf{q}) = \boldsymbol{\gamma}$ $\forall \mathbf{q} \in V$ (Fig. 11a).
- $\boldsymbol{\gamma}$ is *unstable* if $\alpha(\mathbf{q}) = \boldsymbol{\gamma}$ $\forall \mathbf{q} \in V$ (Fig. 11b).
- $\boldsymbol{\gamma}$ is *semi-stable* if either

  $$\{\omega(\mathbf{q}) = \boldsymbol{\gamma} \quad \forall \mathbf{q} \in V \cap \mathrm{Ext}(\boldsymbol{\gamma})\} \wedge \\ \wedge \{\alpha(\mathbf{q}) = \boldsymbol{\gamma} \quad \forall \mathbf{q} \in V \cap \mathrm{Int}(\boldsymbol{\gamma})\}$$

  or

  $$\{\omega(\mathbf{q}) = \boldsymbol{\gamma} \quad \forall \mathbf{q} \in V \cap \mathrm{Int}(\boldsymbol{\gamma})\} \wedge \\ \wedge \{\alpha(\mathbf{q}) = \boldsymbol{\gamma} \quad \forall \mathbf{q} \in V \cap \mathrm{Ext}(\boldsymbol{\gamma})\}$$

  (Figs. 11c and 11d)



(a) (b) (c) (d)

Figure 11: Stability of limit cycles

Definition 173. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\boldsymbol{\gamma}$ be a periodic orbit of period T. We say that $\boldsymbol{\gamma}$ is a *hyperbolic periodic orbit* if

$$I(\boldsymbol{\gamma}) := \int_0^T \mathbf{div}\,\mathbf{f}(\boldsymbol{\gamma}(t))\,\mathrm{d}t \neq 0$$

Theorem 174. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\boldsymbol{\gamma}$ be a periodic orbit of period T. Then:

- $I(\boldsymbol{\gamma}) > 0 \implies \boldsymbol{\gamma}(t)$ is an unstable limit cycle.
- $I(\boldsymbol{\gamma}) < 0 \implies \boldsymbol{\gamma}(t)$ is a stable limit cycle.

Poincaré-Bendixson theorem

Lemma 175. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$, Σ be a transversal section of $\mathbf{f}$, $\boldsymbol{\gamma}$ be an orbit of $\mathbf{f}$ and $\mathbf{p} \in \Sigma \cap \omega(\boldsymbol{\gamma})$. Suppose that $\boldsymbol{\varphi}(t)$ is the flux of the system. Then, $\exists (t_n) \in \mathbb{R}$ such that $\boldsymbol{\varphi}(t_n) \in \Sigma$ and $\lim_{n\to\infty} \boldsymbol{\varphi}(t_n) = \mathbf{p}$.

Lemma 176. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$, Σ be a transversal section of $\mathbf{f}$, $\boldsymbol{\gamma}$ be an orbit of $\mathbf{f}$ and $\mathbf{p} \in \Sigma \cap \omega(\boldsymbol{\gamma})$. Then, $\boldsymbol{\gamma}^+(\mathbf{p})$ intersect Σ in a (finite or infinite) monotone sequence of points.

Lemma 177. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$, Σ be a transversal section of $\mathbf{f}$ and $\mathbf{p} \in U$. Then, $|\Sigma \cap \omega(\mathbf{p})|$ is either 0 or 1.

Lemma 178. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$, $\mathbf{p} \in U$ be such that $\boldsymbol{\gamma}^+(\mathbf{p})$ is contained in a compact set, and $\boldsymbol{\gamma}$ be an orbit such that $\boldsymbol{\gamma} \subseteq \omega(\mathbf{p})$. If $\omega(\mathbf{p})$ contains only non-singular points, then $\omega(\mathbf{p})$ is a periodic orbit and $\boldsymbol{\gamma} = \omega(\mathbf{p})$.

Theorem 179 (Poincaré-Bendixson theorem). Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be such that $\boldsymbol{\gamma}^+(\mathbf{p})$ is contained in a compact set. Suppose that $\mathbf{f}$ has a finite number of singular points. Then:

1. If $\omega(\mathbf{p})$ contains only non-singular points, then $\omega(\mathbf{p})$ is a periodic orbit.
2. If $\omega(\mathbf{p})$ contains only singular points, then $\omega(\mathbf{p})$ is a singular point.
3. If $\omega(\mathbf{p})$ contains both singular and non-singular points, then $\omega(\mathbf{p})$ is a collection of singular points together with homoclinic and heteroclinic orbits connecting those points.

Corollary 180 (Poincaré-Bendixson theorem). Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f} : U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be such that $\boldsymbol{\gamma}^-(\mathbf{p})$ is contained in a compact set. Suppose that $\mathbf{f}$ has a finite number of singular points. Then:

1. If $\alpha(\mathbf{p})$ contains only non-singular points, then $\alpha(\mathbf{p})$ is a periodic orbit.

2. If $\alpha(\mathbf{p})$ contains only singular points, then $\alpha(\mathbf{p})$ is a singular point.
3. If $\alpha(\mathbf{p})$ contains both singular and non-singular points, then $\alpha(\mathbf{p})$ is a collection of singular points together with homoclinic and heteroclinic orbits connecting those points.

Corollary 181. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{f}: U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ and $\boldsymbol{\gamma}$ be a periodic orbit of $\mathbf{f}$. Then, there is at least one singular point in $\mathrm{Int}(\boldsymbol{\gamma})$.

Lyapunov stability

Definition 182. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f}: U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. We say that $\mathbf{p}$ is *Lyapunov stable* if the set $\{\mathbf{p}\}$ is positively stable.

Definition 183. Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f}: U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$. We say that a function $V: U \to \mathbb{R}$ of class $\mathcal{C}^1$ is a *Lyapunov function* for $\mathbf{p}$ if there exists a neighbourhood $\tilde{U} \subseteq U$ of $\mathbf{p}$ such that:

- $V(\mathbf{p}) = 0$ and $V(\mathbf{x}) > 0 \ \forall \mathbf{x} \in \tilde{U} \setminus \{\mathbf{p}\}$
- $\boldsymbol{\nabla} V(\mathbf{q}) \cdot \mathbf{f}(\mathbf{q}) \leq 0 \ \forall \mathbf{q} \in \tilde{U}$

If instead of the second condition we have

- $\boldsymbol{\nabla} V(\mathbf{q}) \cdot \mathbf{f}(\mathbf{q}) < 0 \ \forall \mathbf{q} \in \tilde{U} \setminus \{\mathbf{p}\}$

we say that V is a *strict Lyapunov function* for $\mathbf{p}$.

Theorem 184 (Lyapunov's theorem). Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f}: U \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ and $\mathbf{p} \in U$ be a critical point of $\mathbf{f}$.

- If there exists a Lyapunov function for $\mathbf{p}$ in a neighbourhood of $\mathbf{p}$, then $\mathbf{p}$ is Lyapunov stable.
- If there exists a strict Lyapunov function for $\mathbf{p}$ in a neighbourhood of $\mathbf{p}$, then $\mathbf{p}$ is asymptotically stable.

Theorem 185 (Bendixson's theorem). Let $U \subseteq \mathbb{R}^2$ be an open set and $\mathbf{f}: U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$ such that $\mathbf{div}\,\mathbf{f}$ has constant sign in a simply connected region R and is not identically zero on any subregion of R with positive area. Then, the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ does not have periodic orbits that lie entirely on R.

Theorem 186 (Bendixson-Dulac theorem). Let $U \subseteq \mathbb{R}^2$ be an open set and $\mathbf{f}: U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$. Suppose that there exists a simply connected region R and a function $h: R \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\mathbf{div}(h\mathbf{f})$ has constant sign on R and is not identically zero on any subregion of R with positive area. Then, the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ doesn't have periodic orbits that lie entirely on R.

Theorem 187 (Generalized Bendixson-Dulac theorem). Let $U \subseteq \mathbb{R}^2$ be an open set, $n \in \mathbb{N} \cup \{0\}$ and $\mathbf{f}: U \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$. Suppose that there exists a subset $R \subseteq U$ homeomorphic to a disk with n holes and a function $h: R \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\mathbf{div}(h\mathbf{f})$ has constant sign on R and is not identically zero on any subregion of R with positive area. Then, the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has at most n periodic orbits that lie entirely on R.

Poincaré compactification

Definition 188. Let $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class $\mathcal{C}^1$. Consider the sphere S^2 and the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 1\} \cong \mathbb{R}^2$. Let

$$\begin{aligned} H_+ &:= S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \\ H_- &:= S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\} \end{aligned}$$

For each point $p \in \Pi$, the line joining p and $(0,0,0)$ intersects S^2 in two points. We define the following functions

$$\begin{aligned} \mathbf{g}_+ : \quad & \Pi \longrightarrow H_+ \\ & (x_1, x_2, 1) \longmapsto \left(\frac{x_1}{\sqrt{1+x_1^2+x_2^2}}, \frac{x_2}{\sqrt{1+x_1^2+x_2^2}}, \frac{1}{\sqrt{1+x_1^2+x_2^2}} \right) \\ \mathbf{g}_- : \quad & \Pi \longrightarrow H_- \\ & (x_1, x_2, 1) \longmapsto \left(\frac{-x_1}{\sqrt{1+x_1^2+x_2^2}}, \frac{-x_2}{\sqrt{1+x_1^2+x_2^2}}, \frac{-1}{\sqrt{1+x_1^2+x_2^2}} \right) \end{aligned}$$

which are diffeomorphisms. The induced vector field $\tilde{\mathbf{f}}$ defined in $S^2 \setminus S^1 := H_+ \cup H_-$ is[15]:

$$\tilde{\mathbf{f}}(\mathbf{y}) = \begin{cases} \mathbf{Dg}_+(\mathbf{x})\mathbf{f}(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{g}_+(\mathbf{x}) \in H_+ \\ \mathbf{Dg}_-(\mathbf{x})\mathbf{f}(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{g}_-(\mathbf{x}) \in H_- \end{cases}$$

Proposition 189. Let $\mathbf{f} = (p, q): \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial vector field of degree d, $\tilde{\mathbf{f}}$ be the induced vector field on $S^2 \setminus S^1$ and $\rho: S^2 \to \mathbb{R}$ be the function defined as $\rho(y_1, y_2, y_3) = {y_3}^{d-1}$. Then, the field $\rho\tilde{\mathbf{f}}$ can be extended analytically to S^2 with the equator of S^2 remaining invariant.

Corollary 190. Let $\mathbf{f} = (p, q): \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial vector field of degree d and consider the local charts $(U_i, \boldsymbol{\phi}_i)$, $(V_i, \boldsymbol{\psi}_i)$ for $i = 1, 2, 3$ defined as:

$$\begin{aligned} U_i &= \{(x_1, x_2, x_3) \in S^2 : x_i > 0\} \\ V_i &= \{(x_1, x_2, x_3) \in S^2 : x_i < 0\} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\phi}_i : \quad & U_i \longrightarrow \mathbb{R}^2 \\ & (y_1, y_2, y_3) \longmapsto \left(\tfrac{y_j}{y_i}, \tfrac{y_k}{y_i} \right) \\ \boldsymbol{\psi}_i : \quad & V_i \longrightarrow \mathbb{R}^2 \\ & (y_1, y_2, y_3) \longmapsto \left(\tfrac{y_j}{y_i}, \tfrac{y_k}{y_i} \right) \end{aligned}$$

with $j, k \neq i$, $j < k$ and $i = 1, 2, 3$. Then, the extended vector field defined on each $(U_i, \boldsymbol{\phi}_i)$ is:

- On $(U_1, \boldsymbol{\phi}_1)$, if $(u, v) = \left(\frac{x_2}{x_1}, \frac{1}{x_1} \right)$, then:

$$\begin{cases} u' = v^d \left[-up\left(\dfrac{1}{v}, \dfrac{u}{v} \right) + q\left(\dfrac{1}{v}, \dfrac{u}{v} \right) \right] \\ v' = -v^{d+1} p\left(\dfrac{1}{v}, \dfrac{u}{v} \right) \end{cases}$$

[15] The idea behind this concept is to study the asymptotic behaviour of the orbits of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. In order to do so, we would like to extend the field $\tilde{\mathbf{f}}$ to the equator of S^2 ($\{(x_1, x_2, x_3) \in S^2 : x_3 = 0\}$). And that set would correspond to the infinity in $\mathbb{R}^2$.

- On $(U_2, \boldsymbol{\phi}_2)$, if $(u,v) = \left(\frac{x_3}{x_2}, \frac{1}{x_2}\right)$, then:

$$\begin{cases} u' = v^d \left[p\left(\frac{u}{v}, \frac{1}{v}\right) - uq\left(\frac{u}{v}, \frac{1}{v}\right) \right] \\ v' = -v^{d+1} q\left(\frac{u}{v}, \frac{1}{v}\right) \end{cases}$$

- On $(U_3, \boldsymbol{\phi}_3)$, if $(u,v) = \left(\frac{x_1}{x_3}, \frac{1}{x_3}\right)$, then:

$$\begin{cases} u' = p(u,v) \\ v' = q(u,v) \end{cases}$$

The extended vector field defined on each $(V_i, \boldsymbol{\psi}_i)$ is the one defined on each $(U_i, \boldsymbol{\phi}_i)$ multiplied by $(-1)^{d-1}$. This extension is called *Poincaré compactification* of $\mathbf{f}$.

Definition 191. We define the *Poincaré disk* as the orthogonal projection $\boldsymbol{\pi} : \overline{H_+} \to \overline{D(0,1)}$.

Integrability theory of polynomial systems

Definition 192. Let $U \subseteq \mathbb{R}^2$ be an open set, $p, q \in \mathbb{R}[x,y]$ and consider the polynomial system of Eq. (17). We say that a function $R : U \to \mathbb{R}$ is an *integrating factor* if

$$\mathbf{div}(Rp, Rq) = \frac{\partial(Rp)}{\partial x} + \frac{\partial(Rq)}{\partial y} = 0$$

Lemma 193. Let $U \subseteq \mathbb{R}^2$ be an open set, $R : U \to \mathbb{R}$ be a differentiable function, $p, q \in \mathbb{R}[x,y]$ and consider the polynomial system of Eq. (17). Then, R is an integrating factor if and only if:

$$\mathbf{X}R = p\frac{\partial R}{\partial x} + q\frac{\partial R}{\partial y} = -R\,\mathbf{div}(p,q)$$

where $\mathbf{X}$ is the operator defined in Eq. (18).

Proposition 194. Let $U \subseteq \mathbb{R}^2$ be an open set, $p, q \in \mathbb{R}[x,y]$ and consider the polynomial system of Eq. (17). Suppose that system admits an integrating factor $R : U \to \mathbb{R}$. Then, the system admits a first integral $H : U \to \mathbb{R}$ given by:

$$H(x,y) = -\int R(x,y)p(x,y)\,\mathrm{d}y + h(x)$$

where $h(x)$ satisfy:

$$h'(x) = R(x,y)q(x,y) + \frac{\partial}{\partial x}\left(\int R(x,y)p(x,y)\,\mathrm{d}y\right)$$

Definition 195. Let $p, q, g, h \in \mathbb{R}[x,y]$ and consider the polynomial system of Eq. (17) of degree d and let $\mathbf{X}$ be the vector field operator of that system (defined by Eq. (18)). We say that $\mathrm{e}^{\frac{g(x,y)}{h(x,y)}}$ is an *exponential factor* with cofactor $k(x,y) \in \mathbb{R}[x,y]$ if $\deg k \leq d-1$ and:

$$\mathbf{X}\mathrm{e}^{\frac{g(x,y)}{h(x,y)}} = k(x,y)\mathrm{e}^{\frac{g(x,y)}{h(x,y)}}$$

Theorem 196 (Darboux theorem). Let $p, q \in \mathbb{R}[x,y]$ and consider the polynomial system of Eq. (17) of degree d, $f_i(x,y) = 0$ be invariant algebraic curves with cofactors $k_i(x,y)$ for $i = 1, \ldots, r$ and $\mathrm{e}^{\frac{g_j(x,y)}{h_j(x,y)}}$ be exponential factors with cofactors $\ell_j(x,y)$ for $j = 1, \ldots, s$. Then:

1. If $\exists \lambda_i, \mu_j \in \mathbb{R}$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$, not all zero such that $\sum_{i=1}^r \lambda_i k_i + \sum_{j=1}^s \mu_j \ell_j = 0$, then
$$H = {f_1}^{\lambda_1} \cdots {f_r}^{\lambda_r} \mathrm{e}^{\mu_1 \frac{g_1(x,y)}{h_1(x,y)}} \cdots \mathrm{e}^{\mu_s \frac{g_s(x,y)}{h_s(x,y)}} \tag{20}$$
is a first integral for the system.
2. If $r + s \geq \frac{d(d+1)}{2} + 1$, then $\exists \lambda_i, \mu_j \in \mathbb{R}$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$, not all zero such that $\sum_{i=1}^p \lambda_i k_i + \sum_{j=1}^q \mu_j \ell_j = 0$. And so, the system has the first integral defined in Eq. (20).
3. If $r + s \geq \frac{d(d+1)}{2} + 2$, then the system has a rational first integral. Consequently all trajectories of the system are contained in invariant algebraic curves.
4. If $\exists \lambda_i, \mu_j \in \mathbb{R}$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$, not all zero such that $\sum_{i=1}^p \lambda_i k_i + \sum_{j=1}^q \mu_j \ell_j = -\mathbf{div}(p,q)$, then
$$R = {f_1}^{\lambda_1} \cdots {f_r}^{\lambda_r} \mathrm{e}^{\mu_1 \frac{g_1(x,y)}{h_1(x,y)}} \cdots \mathrm{e}^{\mu_s \frac{g_s(x,y)}{h_s(x,y)}}$$
is an integrating factor for the system. And so the system also admits a first integral by Theorem 194.

Index of paths and homotopy

Definition 197. Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^2$ be a closed path, $q \in \mathbb{R} \setminus \boldsymbol{\gamma}^*$ and L be a ray with vertex at q. Consider a continuous determination $\varphi : [a,b] \to \mathbb{R}$ of the angle (measured counterclockwisely) between $\boldsymbol{\gamma}(t)$ and L. Then, we define the *index* of q with respect to $\boldsymbol{\gamma}$ as:

$$\mathrm{Ind}(\boldsymbol{\gamma}, q) := \frac{\varphi(b) - \varphi(a)}{2\pi}$$

Proposition 198. Let $q_1, q_2 \in \mathbb{R}^2$ and $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed path such that the segment $\overline{q_1 q_2}$ does not intersect $\boldsymbol{\gamma}^*$. Then, $\mathrm{Ind}(\boldsymbol{\gamma}, q_1) = \mathrm{Ind}(\boldsymbol{\gamma}, q_2)$.

Corollary 199. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed path. Then, all points in the same conencted component of $\mathbb{R}^2 \setminus \boldsymbol{\gamma}^*$ have the same index.

Proposition 200. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : I \to \mathbb{R}^2$ be two closed paths and $q \in \mathbb{R}^2$ be such that $q \notin \overline{\boldsymbol{\gamma}_1(t)\boldsymbol{\gamma}_2(t)}$ $\forall t \in I$. Then, $\mathrm{Ind}(\boldsymbol{\gamma}_1, q) = \mathrm{Ind}(\boldsymbol{\gamma}_2, q)$.

Proposition 201. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : I \to \mathbb{R}^2$ be two closed paths and $q \in \mathbb{R}^2$ be such that $q \notin {\boldsymbol{\gamma}_1}^* \cup {\boldsymbol{\gamma}_2}^*$ and $\|\boldsymbol{\gamma}_1(t) - \boldsymbol{\gamma}_2(t)\| < \|q - \boldsymbol{\gamma}_2(t)\|$ $\forall t \in I$. Then, $\mathrm{Ind}(\boldsymbol{\gamma}_1, q) = \mathrm{Ind}(\boldsymbol{\gamma}_2, q)$.

Definition 202. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : I \to \mathbb{R}^2$ be two closed paths. We say that the are *homotopic*, and we denote it by $\boldsymbol{\gamma}_1 \sim \boldsymbol{\gamma}_2$, if there exists a continuous function $\mathbf{h} : I \times [0,1] \to \mathbb{R}^2$ such that:

1. $\boldsymbol{\gamma}_1(t) = \mathbf{h}(t, 0)$ $\forall t \in I$
2. $\boldsymbol{\gamma}_2(t) = \mathbf{h}(t, 1)$ $\forall t \in I$
3. $\mathbf{h}(0, s) = \mathbf{h}(1, s)$ $\forall s \in [0,1]$

Such function $\mathbf{h}$ is called the *homotopy* between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$.

Lemma 203. Being homotopic is an equivalence relation.

Proposition 204. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : I \to \mathbb{R}^2$ be two closed homotopic paths, $\mathbf{h} : I \times [0,1] \to \mathbb{R}^2$ be the respective homotopy and $q \in \mathbb{R}^2$ be such that $q \notin \operatorname{im}(\mathbf{h})$[16]. Then, $\operatorname{Ind}(\boldsymbol{\gamma}_1, q) = \operatorname{Ind}(\boldsymbol{\gamma}_2, q)$.

Definition 205. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed path. We say that $\boldsymbol{\gamma}$ is *contractible* if it is homotopic to the constant path $\boldsymbol{\alpha}(t) = a \in \mathbb{R}^2$.

Proposition 206. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed contractible path and $q \in \mathbb{R}^2 \setminus \overline{D(\boldsymbol{\gamma})}$, where $D(\boldsymbol{\gamma})$ is the domain enclosed by $\boldsymbol{\gamma}$. Then, $\operatorname{Ind}(\boldsymbol{\gamma}, q) = 0$.

Proposition 207. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed path which is homotopic to the path $\boldsymbol{\alpha}_n := q + e^{2\pi i n t}$ (thought in $\mathbb{R}^2$), $n \in \mathbb{Z}$ and $q \in \mathbb{R}^2 \setminus \overline{D(\boldsymbol{\gamma} - \boldsymbol{\alpha}_n)}$. Then, $\operatorname{Ind}(\boldsymbol{\gamma}, q) = n$.

Proposition 208. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ be a closed path and $q \in \mathbb{R}^2 \setminus \boldsymbol{\gamma}^*$ be such that $\operatorname{Ind}(\boldsymbol{\gamma}, q) = n$. Then, $\boldsymbol{\gamma} \sim \boldsymbol{\alpha}_n$.

Theorem 209. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : I \to \mathbb{R}^2$ be two closed paths and $q \in \mathbb{R}^2 \setminus \overline{D(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)}$. Then, $\boldsymbol{\gamma}_1 \sim \boldsymbol{\gamma}_2 \iff \operatorname{Ind}(\boldsymbol{\gamma}_1, q) = \operatorname{Ind}(\boldsymbol{\gamma}_2, q)$

Theorem 210. Let $\mathbf{f} : \overline{D(0,1)} = [0,1] \times [0, 2\pi] \to \mathbb{R}^2$ be a continuous function, $\boldsymbol{\gamma}(t) = \mathbf{f}(1, 2\pi t)$ with $t \in [0,1]$ and $q \in \mathbb{R}^2 \setminus \mathbf{f}(S^1)$ be such that $\operatorname{Ind}(\boldsymbol{\gamma}, q) \neq 0$. Then, $q \in \operatorname{im} \mathbf{f}$.

Poincaré-Hopf theorem

Definition 211. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{X} : U \to \mathbb{R}^2$ be a differentible vector field and $\boldsymbol{\alpha}$ be the path defined on the boundary of a closed disk $D \subsetneq U$. Let $\boldsymbol{\gamma}(t) := (\mathbf{X} \circ \boldsymbol{\alpha})(t)$ and $q \in \operatorname{Int}(\boldsymbol{\gamma})$. We define the *index* of $\mathbf{X}$ on ∂D as:

$$\operatorname{Ind}_{\partial D}(\mathbf{X}) := \operatorname{Ind}(\boldsymbol{\gamma}, q)\text{[17]}$$

Definition 212. Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{X} : U \to \mathbb{R}^2$ be a differentible vector field and $p \in U$ be an isolated singular point (on the set of all singular points). Let D be a disk that surrounds only that singular point p. We define the *index* of p as:

$$\operatorname{Ind}_p(\mathbf{X}) := \operatorname{Ind}_{\partial D}(\mathbf{X})\text{[18]}$$

Proposition 213. Let $\overline{D} \subseteq \mathbb{R}^2$ be a closed disk and $\mathbf{X} : \overline{D} \to \mathbb{R}^2$ be a continuous vector field such that $\mathbf{X}(q) \neq 0\ \forall q \in \partial \overline{D}$. Suppose that $\mathbf{X}$ has a finite number of singular points $p_1, \ldots, p_n$. Then:

$$\sum_{i=1}^n \operatorname{Ind}_{p_i}(\mathbf{X}) = \operatorname{Ind}_{\partial D}(\mathbf{X})$$

Definition 214. Let $U \subseteq S^2$ be an open set. A *tangent vector field* defined on S^2 is a vector field $\mathbf{X}$ such that $\mathbf{X}(q) \in T_p S^2\ \forall q \in U$[19].

Definition 215. Let $U \subseteq S^2$ be an open set and $\mathbf{X} : U \to \mathbb{R}^3$ be a tangent vector field and p be a singular point of $\mathbf{X}$. Suppose (rotating the sphere if necessary) that p is in one of its poles. Let $\tilde{\mathbf{X}}$ be the field created from the stereographic projection from $-p$ to the equator plane. We define the index of p with respect to the field $\mathbf{X}$ as:

$$\operatorname{Ind}_p(X) = \operatorname{Ind}_0(\tilde{\mathbf{X}})$$

Theorem 216 (Poincaré-Hopf theorem). Consider a continuous vector field $\mathbf{X}$ on a compact manifold M with a finite number of singular points. Then, the sum of their indices is $\chi(M)$.

Corollary 217 (Poincaré-Hopf theorem on S^2). Consider a continuous vector field $\mathbf{X}$ on S^2 with a finite number of singular points. Then, the sum of their indices is 2.

Proposition 218 (Poincaré index formula). Let $U \subseteq \mathbb{R}^2$ be an open set, $\mathbf{X} : U \to \mathbb{R}^2$ be a differentible vector field and p be a singular point with a finite finite sectorial decomposition. Denote by e the number of elliptic sectors; by h, the number of hyperbolic sectors, and by p, the number of parabolic sectors. Then:

$$\operatorname{Ind}_p(\mathbf{X}) = \frac{e-h}{2} + 1$$

Corollary 219. Every tangent vector field $\mathbf{X}$ defined on S^2 has singular points.

8. | Introduction to partial differential equations

Definition 220. Let $U \subseteq \mathbb{R}^n$ be an open set. A *partial differential equation* (*PDE*) of order k is an expression of the form

$$F\left(\mathbf{x}, u(\mathbf{x}), \frac{\partial u}{\partial \mathbf{x}}, \ldots, \frac{\partial^k u}{\partial \mathbf{x}^k}\right) = 0$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $F : U \times \mathbb{R} \times \mathbb{R}^{n^1} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function and $u : U \to \mathbb{R}$ is an unknown function. The function u is called *solution* of the PDE defined by F.

Quasilinear partial differential equations

Definition 221. Let $U \subseteq \mathbb{R}^n$ be an open set and $u : U \to \mathbb{R}$ be a function. A *quasilinear PDE* is an expression of the form:

$$p_1(\mathbf{x}, u)\frac{\partial u}{\partial x_1} + \cdots + p_n(\mathbf{x}, u)\frac{\partial u}{\partial x_n} = q(\mathbf{x}, u) \qquad (21)$$

Theorem 222. Let $U \subseteq \mathbb{R}^n$ be an open set and $u : U \to \mathbb{R}$ be a function and consider the PDE of Eq. (21). Let $H_1, \ldots, H_n$ be the n independent first integrals of the system:

$$\begin{cases} {x_1}' = p_1(x_1, \ldots, x_n, u) \\ \quad\vdots \\ {x_n}' = p_n(x_1, \ldots, x_n, u) \\ u' = q(x_1, \ldots, x_n, u) \end{cases}$$

[16] From now on we will denote $q \notin \operatorname{im}(\mathbf{h})$ as $q \in \mathbb{R}^2 \setminus \overline{D(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)}$, where $D(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)$ is the domain enclosed between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$.
[17] It can be seen that this definition doesn't depend on the point q inside $\boldsymbol{\gamma}$ chosen.
[18] It can be seen that this definition doesn't depend on the disk D chosen.
[19] Recall **??**.

Then, for any function $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^1$, the implicit equation

$$F(H_1(\mathbf{x}, u), \ldots, H_n(\mathbf{x}, u)) = 0$$

is a solution to Eq. (21).

Heat, wave and Laplace equations

Definition 223 (Heat equation). Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an unknown function. The *heat equation* is the PDE defined by

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

where $k \in \mathbb{R}$.

Proposition 224. Consider a bar of line $L \in \mathbb{R}_{>0}$ whose temperature can be modeled by a function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $f : [0, L] \to \mathbb{R}$ be a function. Then, the solution $u(x, t)$ to the heat equation with boundary conditions $u(x, 0) = f(x)$ and $u(0, t) = u(L, t) = 0$ is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \mathrm{e}^{-\frac{n^2 \pi^2 k}{L^2} t}$$

where $b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin\left(\frac{\pi n x}{L}\right) \mathrm{d}x$.

Definition 225 (Wave equation). Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an unknown function. The *wave equation* is the PDE defined by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c \in \mathbb{R}$.

Proposition 226. Consider a string of line $L \in \mathbb{R}_{>0}$ whose position can be modeled by a function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $f, g : [0, L] \to \mathbb{R}$ be functions. Then, the solution $u(x, t)$ to the wave equation with boundary conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ and $u(0, t) = u(L, t) = 0$ is:

$$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{\pi n x}{L}\right) \left[a_n \cos\left(\frac{\pi n c}{L} t\right) + b_n \sin\left(\frac{\pi n c}{L} t\right)\right]$$

where:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{\pi n x}{L}\right) \mathrm{d}x$$

$$b_n = \frac{1}{\pi n c} \int_{-L}^{L} g(x) \sin\left(\frac{\pi n x}{L}\right) \mathrm{d}x$$

Proposition 227. Let $u(x, t)$ be a solution to the wave equation. Then, $\exists F, G : \mathbb{R} \to \mathbb{R}$ such that:

$$u(x, t) = F(x + ct) + G(x - ct)$$

Proposition 228 (D'Alembert formula). Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. The solution $u(x, t)$ to the wave equation with boundary conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ is:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, \mathrm{d}s$$

Definition 229 (Laplace equation). Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an unknown function. The *Laplace equation* is the PDE defined by:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$$

Proposition 230. The Laplacian of a function $u : (0, \infty) \times [0, 2\pi] \to \mathbb{R}$ in polar coordinates (r, θ) is:

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

Proposition 231 (Dirichlet problem). Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function such that $f(0) = f(2\pi)$. Then, there exists a continuous function $v : \overline{D(0, \rho)} \to \mathbb{R}$ such that:

1. $v(r, 0) = v(r, 2\pi)$ $\forall r \in [0, \rho]$
2. $v \in \mathcal{C}^2(D(0, \rho) \setminus \{0\})$ and $\Delta v = 0$.
3. $v(\rho, \theta) = f(\theta)$ $\forall \theta \in [0, 2\pi]$

An example of such function is:

$$v(r, \theta) = \sum_{n=0}^{\infty} \frac{r^n}{\rho^n} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

where:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \, \mathrm{d}\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, \mathrm{d}\theta$$