Complex analysis

1. | Complex numbers

Definition of complex numbers

Definition 1. Consider $x^2+1 \in \mathbb{R}[x]$ and the ring $R := \mathbb{R}[x]/(x^2+1)$. Then, R is a commutative field, which we will denote by $\mathbb{C}$, whose elements are of the form $a+b\overline{x} =: a+b\mathrm{i}$, $a,b \in \mathbb{R}$[1]. This field is called *field of complex numbers*.

Proposition 2. Let $a_1+b_1\mathrm{i}, a_2+b_2\mathrm{i} \in \mathbb{C}$, $a_1,a_2,b_1,b_2 \in \mathbb{R}$[2]. Then:

- $(a_1+b_1\mathrm{i})+(a_2+b_2\mathrm{i}) = (a_1+a_2)+(b_1+b_2)\mathrm{i}$
- $(a_1+b_1\mathrm{i})\cdot(a_2+b_2\mathrm{i}) = (a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)\mathrm{i}$
- $\dfrac{a_1+b_1\mathrm{i}}{a_2+b_2\mathrm{i}} = \dfrac{a_1a_2+b_1b_2}{{a_2}^2+{b_2}^2} + \dfrac{a_2b_1-a_1b_2}{{a_2}^2+{b_2}^2}\mathrm{i}$, provided that $a_2, b_2 \neq 0$.

Theorem 3. $\mathbb{C}$ is not an ordered field.

Complex conjugate, modulus and argument

Definition 4. Let $z = a+b\mathrm{i} \in \mathbb{C}$. We define the *complex conjugate* (or simply *conjugate*) of z as $\overline{z} := a - b\mathrm{i}$.

Proposition 5. Let $z,w \in \mathbb{C}$. Then:

1. $\overline{\overline{z}} = z$
2. $\overline{z+w} = \overline{z}+\overline{w}$
3. $\overline{z\cdot w} = \overline{z}\cdot\overline{w}$
4. $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$, provided that $w \neq 0$.
5. $z \in \mathbb{R} \iff \overline{z} = z$

Definition 6. Let $z = a+b\mathrm{i} \in \mathbb{C}$. We define the *real part* of z as $\operatorname{Re} z := a$. We define the *imaginary part* of z as $\operatorname{Im} z := b$.

Proposition 7. Let $z \in \mathbb{C}$. Then:

$$\operatorname{Re} z = \frac{z+\overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z-\overline{z}}{2\mathrm{i}}$$

Definition 8. Let $z = a+b\mathrm{i} \in \mathbb{C}$. We define the *modulus* of z as:

$$|z| := \sqrt{a^2+b^2}$$

Proposition 9. Let $z,w \in \mathbb{C}$. Then:

1. $|z| \geq 0$
2. $|z| = 0 \iff z = 0$
3. $z\overline{z} = |z|^2$
4. $z^{-1} = \frac{1}{|z|^2}\cdot\overline{z}$
5. $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$
6. $|z\cdot w| = |z|\cdot|w|$
7. $|z^n| = |z|^n \ \forall n \in \mathbb{Z}$
8. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$, provided that $w \neq 0$.
9. $|z \pm w|^2 = |z|^2 + |w|^2 \pm 2\operatorname{Re}(z\overline{w})$
10. $|z \pm w| \leq |z| + |w|$
11. $||z|-|w|| \leq |z \pm w|$

Corollary 10. Let $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{C}$. Then:

- $\left|\displaystyle\sum_{i=1}^n z_i\right| \leq \displaystyle\sum_{i=1}^n |z_i|$
- $|z_1\cdots z_n| = |z_1|\cdots|z_n|$
- $|\operatorname{Re}(z_1\cdots z_n)|, |\operatorname{Im}(z_1\cdots z_n)| \leq |z_1|\cdots|z_n|$

Definition 11. Let $z \in \mathbb{C}^*$. We define the *argument* of z, denoted by $\arg z$, as the real number θ satisfying:

$$z = |z|(\cos\theta + \mathrm{i}\sin\theta)$$

Note that $\arg z$ is not unique. Because of that, we say that $\arg z$ is a *multivalued function*.

Definition 12. Let $U \subseteq \mathbb{C}$ be an open set and $f: U \to \mathbb{C}^*$ be a function. A *determination of the argument of* f is a continuous function $g: U \to \mathbb{R}$ such that $f(z) = |f(z)|\mathrm{e}^{\mathrm{i}g(z)} \ \forall z \in U$.

Definition 13. Let $z \in \mathbb{C}^*$. We define the *principal argument* of z as the unique real number θ satisfying:

$$\operatorname{Arg} z := \{\theta \in (-\pi, \pi] : z = |z|(\cos\theta + \mathrm{i}\sin\theta)\}$$

Note that this determination of the argument is not continuous.

Proposition 14. Let $z = a+b\mathrm{i} \in \mathbb{C}$. Then:

$$\operatorname{Arg} z = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0, y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0, y < 0 \\ \operatorname{sgn}(y)\frac{\pi}{2} & \text{if } x = 0 \end{cases}$$

[1] Such expression of a complex number is called *Cartesian form* of a complex number.

[2] From now on, we will omit to say that these values are real numbers. If they aren't, we will explicitly remark it.

Metric topology of $\mathbb{C}$

Proposition 15. Consider the distance d defined as:

$$\begin{aligned} d: \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto |z - w| \end{aligned}$$

Then, $(\mathbb{C}, d)$ is a metric space[3].

Proposition 16. Thinking complex numbers as an ordered pair of real numbers, the topology of $\mathbb{C}$ induced by d is the same as the ordinary topology of $\mathbb{R}^2$.

Definition 17. We define the *extended complex plane* as $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. We define the *extended real numbers* as $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. The topologies added to those sets are the ones given by the one-point compactification[4].

Definition 18 (Stereographic projection). The *stereographic projection* is the function $p: S^2 \to \mathbb{C}_\infty$ defined as:

$$p(x_1, x_2, x_3) = \begin{cases} \frac{x_1}{1-x_3} + \mathrm{i}\frac{x_2}{1-x_3} & \text{if } x_3 \neq 1 \\ \infty & \text{if } x_3 = 1 \end{cases}$$

The inverse of the stereographic projection $p^{-1}: \mathbb{C}_\infty \to S^2$ is:

$$p^{-1}(z) = \begin{cases} \left(\frac{z+\overline{z}}{1+|z|^2}, \frac{z-\overline{z}}{\mathrm{i}(1+|z|^2)}, \frac{|z|^2-1}{1+|z|^2}\right) & \text{if } z \in \mathbb{C} \\ (0,0,1) & \text{if } z = \infty \end{cases}$$

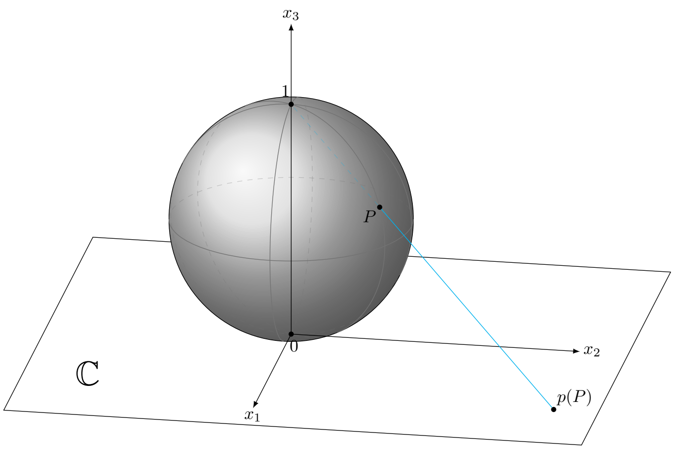

Figure 1: Stereographic projection

2. | Sequences and series

Sequences

Definition 19. A *sequence of complex numbers* is a function of the form

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{C} \\ n &\longmapsto z_n \end{aligned}$$

In general, we will denote that sequence by (z_n).

Definition 20. Let $(z_n) \in \mathbb{C}$ be a sequence. A subsequence of (z_n) is a sequence (z_{k_n}), where $(k_n) \in \mathbb{N}$ is an increasing sequence of natural numbers.

Definition 21. Let $(z_n) \in \mathbb{C}$ be a sequence. We say that (z_n) has *limit* $z \in \mathbb{C}$ (or it *converges* to z) if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \quad \forall n > n_0$$

In that case, we will write $\lim_{n\to\infty} z_n = z$ and we say that sequence is *convergent*. Otherwise, we say that the sequence is *divergent*.

Definition 22. Let $(z_n) \in \mathbb{C}$ be a sequence. We say that (z_n) is *bounded* if $\exists M \in \mathbb{R}$ such that $|z_n| \leq M \ \forall n \in \mathbb{N}$.

Definition 23. Let $(z_n) \in \mathbb{C}$ be a sequence. We say that (z_n) is *Cauchy* if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$|z_n - z_m| < \varepsilon \quad \forall n, m > n_0$$

Proposition 24. Let $(z_n) \in \mathbb{C}$ be a convergent sequence. Then, (z_n) is bounded and Cauchy.

Proposition 25. Let $(z_n) \in \mathbb{C}$ be a sequence. Then, (z_n) is convergent if and only if all its subsequences are convergent.

Proposition 26. Let $(z_n), (w_n) \in \mathbb{C}$ be two convergent sequences whose limits are $z, w \in \mathbb{C}$, respectively. Then:

1. $\lim_{n\to\infty} z_n + w_n = z + w$
2. $\lim_{n\to\infty} z_n w_n = zw$
3. $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$, provided that $w_n \neq 0 \ \forall n \in \mathbb{N}$.

Definition 27. Let $(z_n) \in \mathbb{C}$ be a sequence such that $z_n = x_n + y_n\mathrm{i} \ \forall n \in \mathbb{N}$, where $x_n, y_n \in \mathbb{R}$. Then:

1. (z_n) is convergent if and only if (x_n) and (y_n) are convergent. In that case, we have:
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \mathrm{i} \lim_{n\to\infty} y_n$$
2. (z_n) is Cauchy if and only if (x_n) and (y_n) are Cauchy.

Theorem 28. $\mathbb{C}$ is a complete metric space.

Series

Definition 29. Let $(z_n) \in \mathbb{C}$ be a sequence. A *numeric series of complex numbers* is an expression of the form

$$\sum_{n=1}^{\infty} z_n$$

We call z_n *general term of the series* and $S_N := \sum_{n=1}^{N} z_n$, for all $N \in \mathbb{N}$, *N-th partial sum of the series*[5].

[3] In order to simplify the notation we will refer to $(\mathbb{C}, d)$ simply as $\mathbb{C}$.
[4] See **??**.
[5] Sometimes we will write $\sum z_n$ to refer to $\sum_{n=1}^{\infty} z_n$.

Definition 30. We say the series of complex numbers $\sum z_n$ is *convergent* if $S = \lim_{N\to\infty} S_N$ exists and it is finite. In that case, S is called the *sum of the series*. If the previous limit doesn't exist or it is infinite, we say the series is *divergent*[6].

Definition 31. Let $\sum z_n$ be a series of complex numbers. A *reordering* of $\sum z_n$ is any series $\sum z_{\sigma(n)}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective function.

Proposition 32. Let $(z_n) \in \mathbb{C}$ be a sequence and $\sum z_n$ be a convergent series. Then, $\lim_{n\to\infty} z_n = 0$.

Proposition 33. Let $(z_n) \in \mathbb{C}$ be a sequence such that such that $z_n = x_n + y_n\mathrm{i}$ $\forall n \in \mathbb{N}$, where $x_n, y_n \in \mathbb{R}$, and $\sum z_n$ be a series. Then, $\sum z_n < \infty$ if and only if $\sum x_n < \infty$ and $\sum y_n < \infty$. In that case, we have:

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + \mathrm{i}\sum_{n=1}^{\infty} y_n$$

Proposition 34. Let $\sum z_n = z \in \mathbb{C}$ and $\sum w_n = w \in \mathbb{C}$ be two series, and $\lambda \in \mathbb{C}$. Then:

1. $\sum(z_n + w_n)$ is convergent and $\sum(z_n + w_n) = z + w$.
2. $\sum(\lambda z_n)$ is convergent and $\sum(\lambda z_n) = \lambda z$.

Lemma 35 (Abel's summation formula). Let $(z_n), (w_n) \in \mathbb{C}$ be two sequence. Let $S_N := \sum_{n=1}^{N} z_n$. Then:

$$\sum_{n=1}^{N} z_n w_n = S_N w_N + \sum_{n=1}^{N-1} S_n(w_n - w_{n+1})$$

Definition 36. Let $\sum z_n$ be a series of complex numbers. We say that $\sum z_n$ is *absolutely convergent* if $\sum |z_n| < \infty$[7].

Proposition 37. Let $\sum z_n$ be a series of complex numbers.

1. $\sum |z_n| < \infty \implies \sum z_n < \infty$
2. $\sum |z_n| = z < \infty \implies \forall \sigma \in S(\mathbb{N})$, $\sum z_{\sigma(n)} = w < \infty$ for some $w \in \mathbb{C}$.

Definition 38 (Cauchy product). Let $\sum z_n$, $\sum w_n$ be absolutely convergent series of complex numbers. We define the *product* of $\sum z_n$ and $\sum w_n$ as the series $\sum p_n$, where $p_n = \sum_{k=0}^{n} z_k w_{n-k}$.

Proposition 39. Let $\sum z_n$, $\sum w_n$ be absolutely convergent series of complex numbers. Then, the product of these series is absolutely convergent and satisfy:

$$\sum_{n=1}^{\infty} p_n = \left(\sum_{n=1}^{\infty} z_n\right)\left(\sum_{n=1}^{\infty} w_n\right)$$

3. | Complex functions

Continuity

Definition 40. Let $D \subseteq \mathbb{C}$ be a set. We define a *complex function*[8] as a function of the form:

$$\begin{aligned} f : D &\longrightarrow \mathbb{C} \\ z &\longmapsto f(z) \end{aligned}$$

Definition 41. Let $D \subseteq \mathbb{C}$ be a set and $f : D \to \mathbb{C}$ be a function. We say that f is *continuous* at $z_0 \in D$ if and only if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. We say that f is continuous on D if it is continuous at z $\forall z \in D$.

Definition 42. Let $D \subseteq \mathbb{C}$ be a set and $f : D \to \mathbb{C}$ be a function. We define the function $\operatorname{Re} f$ as the function:

$$\begin{aligned} \operatorname{Re} f : D &\longrightarrow \mathbb{C} \\ z &\longmapsto \operatorname{Re}(f(z)) \end{aligned}$$

Analogously, we define the function $\operatorname{Im} f$ as the function:

$$\begin{aligned} \operatorname{Im} f : D &\longrightarrow \mathbb{C} \\ z &\longmapsto \operatorname{Im}(f(z)) \end{aligned}$$

Proposition 43. Let $D \subseteq \mathbb{C}$ be a set, $f : D \to \mathbb{C}$ be a function and $z_0 \in D$. Then, $f = \operatorname{Re} f + \mathrm{i}\operatorname{Im} f$ is continuous at z_0 if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0.

Proposition 44. Let $D \subseteq \mathbb{C}$ be a set, $f : D \to \mathbb{C}$ be a function and $z_0 \in D$. Then, f is continuous at z_0 if and only if for all sequence $(w_n) \in D$ convergent to z_0, the sequence $(f(w_n))$ converges to $f(z_0)$.

Proposition 45. Let $D \subseteq \mathbb{C}$ be a set, $f, g : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous function at a point $z_0 \in D$ and $\lambda \in \mathbb{C}$. Then, λf, $f + g$, and fg are continuous at z_0. Moreover, if $g(z_0) \neq 0$, then f/g is continuous at z_0.

Sequences of functions

Definition 46. Let $D \subseteq \mathbb{C}$. A set

$$(f_n(z)) = \{f_1(z), f_2(z), \ldots, f_n(z), \ldots\}$$

is a *sequence of complex functions* if $f_i : D \to \mathbb{C}$ is a complex function $\forall i \in \mathbb{N}$. In this case we say the sequence $(f_n(z))$, or simply (f_n), is well-defined on D[9].

Definition 47. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions and $f : D \to \mathbb{C}$. We say (f_n) *converges pointwise* to f on D if $\forall z \in D$, $\lim_{n\to\infty} f_n(z) = f(z)$.

Definition 48. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions and $f : D \to \mathbb{C}$. We say (f_n) *converges uniformly* to f on D if $\forall \varepsilon > 0$, $\exists n_0$ such that $|f_n(z) - f(z)| < \varepsilon$ $\forall n \geq n_0$ and $\forall z \in D$.

[6] We will use the notation $\sum z_n < \infty$ or $\sum z_n = +\infty$ to express that the series converges or diverges, respectively.

[7] Note that since $\sum |z_n|$ is a sequence of real numbers, all the criteria for convergence of numeric series of real numbers are, thus, applicable.

[8] Due to the close similarity between these kind of functions and the multivariate functions, we will only expose the most remarkable results about continuity of complex functions.

[9] The majority of definitions and results of sequences of real-valued functions can be extended conveniently to sequences of complex functions. So in this document, we will only expose the most important ones.

Lemma 49. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions. (f_n) converges uniformly to $f : D \to \mathbb{C}$ on D if and only if $\lim_{n\to\infty} \sup \{|f_n(z) - f(z)| : z \in D\} = 0$.

Theorem 50 (Cauchy's test). A sequence of functions $(f_n) \in D \subseteq \mathbb{C}$ converges uniformly to $f : D \to \mathbb{C}$ on $D \subseteq \mathbb{C}$ if and only if $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0$, we have:

$$\sup \{|f_n(z) - f_m(z)| : z \in D\} < \varepsilon$$

Theorem 51. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of continuous functions. If (f_n) converges uniformly to $f : D \to \mathbb{C}$ on D, then f is continuous on D.

Series of functions

Definition 52. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions. The expression

$$\sum_{n=1}^{\infty} f_n(z)$$

is the *series of functions* associated with (f_n)[10].

Definition 53. A series of functions $\sum f_n(z)$ defined on $D \subseteq \mathbb{C}$ *converges pointwise* on D if the sequence of partials sums

$$F_N(z) = \sum_{n=1}^{N} f_n(z)$$

converges pointwise on D. If the pointwise limit of (F_N) is $F(z)$, we say F is the *sum of the series in a pointwise sense*.

Definition 54. A series of functions $\sum f_n(z)$ defined on $D \subseteq \mathbb{C}$ *converges uniformly* on D if the sequence of partials sums

$$F_N(z) = \sum_{n=1}^{N} f_n(z)$$

converges uniformly on D. If the uniform limit of (F_N) is $F(z)$, we say F is the *sum of the series in an uniform sense*.

Theorem 55 (Cauchy's test). A series of functions $\sum f_n(z)$ defined on $D \subseteq \mathbb{C}$ converges uniformly on D if and only if $\forall \varepsilon > 0\ \exists n_0$ such that $\forall M, N \geq n_0$ (with $N \leq M$), we have:

$$\sup \left\{ \left| \sum_{n=N}^{M} f_n(z) \right| : z \in D \right\} < \varepsilon$$

Corollary 56. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions. If $\sum f_n(z)$ is uniformly convergent on $D \subseteq \mathbb{C}$, then (f_n) converges uniformly to zero on D.

Theorem 57. Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of continuous functions. If $\sum f_n(z)$ is uniformly convergent on $D \subseteq \mathbb{C}$, then its sum function is also continuous on D.

Theorem 58 (Weierstraß M-test). Let $(f_n) \in D \subseteq \mathbb{C}$ be a sequence of functions such that $|f_n(z)| \leq M_n\ \forall z \in D$ and suppose that $\sum M_n < \infty$. Then, $\sum f_n(z)$ converges uniformly on D.

Theorem 59 (Dirichlet's test). Let $(f_n) \in X \subseteq \mathbb{C}$ and $(g_n) \in Y \subseteq \mathbb{C}$ be two sequences of functions and $F_N := \sum_{n=1}^{N} f_n(z)$. Suppose:

1. (F_N) is uniformly bounded on X.
2. $(g_n(z))$ is a monotone sequence of real numbers and converges uniformly to 0 on Y.

Then, $\sum f_n(z)g_n(z)$ converges uniformly on $X \times Y$.

Theorem 60 (Abel's test). Let $(f_n) \in X \subseteq \mathbb{C}$ and $(g_n) \in Y \subseteq \mathbb{C}$ be two sequences of functions. Suppose:

1. $\sum f_n(z)$ is uniformly convergent on X.
2. (g_n) is a monotone and bounded sequence of real numbers.

Then, $\sum f_n(z)g_n(z)$ converges uniformly on $X \times Y$.

Theorem 61 (Dedekind's test). Let $(f_n) \in X \subseteq \mathbb{C}$ and $(g_n) \in Y \subseteq \mathbb{C}$ be two sequences of functions and $F_N := \sum_{n=1}^{N} f_n(z)$. Suppose:

1. (F_N) is uniformly bounded on X.
2. (g_n) converges uniformly to 0 on Y.
3. $\sum |g_n(z) - g_{n+1}(z)|$ converges uniformly on Y.

Then, $\sum f_n(z)g_n(z)$ converges uniformly on $X \times Y$.

Theorem 62 (Du Bois-Reymond's test). Let $(f_n) \in X \subseteq \mathbb{C}$ and $(g_n) \in Y \subseteq \mathbb{C}$ be two sequences of functions. Suppose:

1. $\sum f_n(z)$ is uniformly convergent on X.
2. $\sum |g_n(z) - g_{n+1}(z)| < \infty\ \forall z \in Y$.

Then, $\sum f_n(z)g_n(z)$ converges uniformly on $X \times Y$.

Power series

Definition 63. Let $(a_n) \in \mathbb{C}$ be a sequence and $z_0 \in \mathbb{C}$. A *complex power series* centered at z_0 is a series of functions of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem 64 (Cauchy-Hadamard theorem). Let $\sum a_n (z - z_0)^n$ be a complex power series. Then, $\exists! R \in [0, \infty]$ defined as

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1} \in [0, \infty]$$

that satisfies the following properties:

1. If $|z - z_0| < R \implies \sum a_n (z - z_0)^n$ converges absolutely.
2. If $0 \leq r < R \implies \sum a_n (z - z_0)^n$ converges absolutely and uniformly on $\{z \in \mathbb{C} : |z - z_0| \leq r\}$.

[10] The majority of definitions and results of series of real-valued functions can be extended conveniently to series of complex functions. So in this document, we will only expose the most important ones.

3. If $|z-z_0|>R \implies \sum |a_n||z-z_0|^n$ diverges.

The number R is called *radius of convergence* of series.

Theorem 65 (Abel's theorem). Let $\sum a_n z^n$ be a complex power series that converges uniformly on $D\subseteq\mathbb{C}$. Then, the series $\sum a_n z^n$ converges uniformly on $\{r\zeta\in\mathbb{C}: r\in[0,1]\wedge\zeta\in D\}$, which is a *cone* with base D.

Corollary 66 (Abel's theorem). Let $\sum a_n z^n$ be a complex power series with radius of convergence $R\in(0,\infty)$ and $I\subseteq[0,2\pi)$ be a non-empty connected set. Suppose that the series $\sum a_n z^n$ is uniformly convergent on $D:=\{R\mathrm{e}^{\mathrm{i}\theta}\in\mathbb{C}:\theta\in I\}$. Then, $f(z):=\sum a_n z^n$ converges uniformly on the cone C with base D. In particular, we have:

$$\lim_{\substack{z\to R\mathrm{e}^{\mathrm{i}\theta}\\ z\in C}} f(z)=\sum_{n=0}^{\infty} a_n R^n \mathrm{e}^{\mathrm{i}n\theta}\qquad \forall\theta\in I$$

Proposition 67. Let $f:\mathbb{C}\to\mathbb{C}$ be the sum function of a complex power series. Then f is continuous on the domain of convergence of the series.

Exponential and logarithmic functions

Definition 68. For all $z\in\mathbb{C}$, we define the *complex exponential function* as:

$$\mathrm{e}^z:=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

Proposition 69. The radius of convergence of e^z is infinite, and its image is $\mathbb{C}^*$.

Proposition 70. Let $z,w\in\mathbb{C}$. Then:

1. $\mathrm{e}^{z+w}=\mathrm{e}^z\mathrm{e}^w$
2. $\overline{\mathrm{e}^z}=\mathrm{e}^{\overline{z}}$
3. $|\mathrm{e}^z|=\mathrm{e}^{\operatorname{Re} z}$

Corollary 71 (Euler's formula). Let $x\in\mathbb{R}$. Then:

$$\mathrm{e}^{\mathrm{i}x}=\cos x+\mathrm{i}\sin x$$

Proposition 72. Let $z,w\in\mathbb{C}$ and $n\in\mathbb{Z}$. Then:

1. $\arg(zw)=\arg z+\arg w$
2. $\arg(z^n)=n\arg z$

Definition 73 (Polar form). Let $z\in\mathbb{C}$, $r=|z|$ and $\theta=\arg z$. We define the *polar form* of z as:

$$z=r\mathrm{e}^{\mathrm{i}\theta}=r(\cos\theta+\mathrm{i}\sin\theta)$$

Corollary 74. Let $z,w\in\mathbb{C}$. Then:

1. $\mathrm{e}^z=1\iff z=2\pi k\mathrm{i}$, $k\in\mathbb{Z}$.
2. e^z is periodic of period $2\pi i$.
3. $\mathrm{e}^z=\mathrm{e}^w\iff z=w+2\pi\mathrm{i}k$, $k\in\mathbb{Z}$.

Corollary 75. Let $x\in\mathbb{R}$. Then:

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\qquad \sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Proposition 76 (De Moivre's formula). Let $\theta\in\mathbb{R}$ and $n\in\mathbb{Z}$. Then:

$$(\cos\theta+\mathrm{i}\sin\theta)^n=\cos(n\theta)+\mathrm{i}\sin(n\theta)$$

Theorem 77. Let $n\in\mathbb{N}$. Then, there are n n-th roots of any complex number $z\in\mathbb{C}^*$. Assuming $z=r(\cos\theta+\mathrm{i}\sin\theta)$, these roots are:

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)+\mathrm{i}\sin\left(\frac{\theta}{n}+\frac{2\pi}{n}k\right)\right]$$

for $k=0,\ldots,n-1$.

Proposition 78. Let $z\in\mathbb{C}^*$. Then, the equation $\mathrm{e}^w=z$ has infinitely many solutions.

Definition 79. Let $z\in\mathbb{C}$. We define a *complex natural logarithm* of z as a solution to the equation $\mathrm{e}^w=z$. That is:

$$\ln z:=\ln|z|+\mathrm{i}\arg z$$

Note that $\ln z$ is a multivalued function. We define the *principal value* of $\ln z$ as:

$$\operatorname{Ln} z:=\ln|z|+\mathrm{i}\operatorname{Arg} z$$

Trigonometric functions

Definition 80. Let $z\in\mathbb{C}$. We define the *complex sine* and *complex cosine* respectively as:

$$\cos z=\frac{\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z}}{2}=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$
$$\sin z=\frac{\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Proposition 81. Let $z,w\in\mathbb{C}$. Then:

1. $(\cos z)^2+(\sin z)^2=1$
2. $\cos(-z)=\cos z$, $\sin(-z)=-\sin z$
3. $\cos(z\pm w)=\cos z\cos w\mp\sin z\sin w$
4. $\sin(z\pm w)=\sin z\cos w\pm\cos z\sin w$

Proposition 82. The functions $\cos z$, $\sin z$ are unbounded and periodic of period 2π.

Proposition 83. Let $z\in\mathbb{C}$. Then:

- $\cos z=0\implies z=\frac{\pi}{2}+\pi k\in\mathbb{R}$, for some $k\in\mathbb{Z}$.
- $\sin z=0\implies z=\pi k\in\mathbb{R}$, for some $k\in\mathbb{Z}$.

Definition 84. We define the *complex tangent*, *complex secant*, *complex cosecant* and *complex cotangent* respectively as:

$$\tan z=\frac{\sin z}{\cos z}=-\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}}{\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z}}\quad\forall z\in\mathbb{C}\setminus\bigcup_{k\in\mathbb{Z}}\left\{\frac{\pi}{2}+\pi k\right\}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \left\{\frac{\pi}{2} + \pi k\right\}$$
$$\csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \{\pi k\}$$
$$\cot z = \frac{1}{\tan z} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \{\pi k\}$$

Definition 85. Let $z \in \mathbb{C}$. We define the *complex hyperbolic sine* and *complex hyperbolic cosine* respectively as:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$
$$\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Proposition 86. Let $z, w \in \mathbb{C}$. Then:

1. $(\cosh z)^2 - (\sinh z)^2 = 1$
2. $\cosh(-z) = \cosh z$, $\sinh(-z) = -\sinh z$
3. $\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sin w$
4. $\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sin w$

Proposition 87. Let $z = x + iy \in \mathbb{C}$. Then:

1. $\cos z = \cos x \cosh y - i \sin x \sinh y$
2. $\sin z = \sin x \cosh y + i \cos x \sinh y$

Definition 88. We define the *complex hyperbolic tangent*, *complex hyperbolic secant*, *complex hyperbolic cosecant* and *complex hyperbolic cotangent* respectively as:

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \left\{i\left(\frac{\pi}{2} + \pi k\right)\right\}$$
$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \left\{i\left(\frac{\pi}{2} + \pi k\right)\right\}$$
$$\operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \{\pi k i\}$$
$$\coth z = \frac{1}{\tanh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}} \quad \forall z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \{\pi k i\}$$

4. | Complex differentiation

Holomorphic functions

Definition 89. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U \to \mathbb{C}$ be a function. We say that f is $\mathbb{C}$-*differentiable* at z_0 if the following limit exists:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

In that case, the limit is called *derivative of f at z_0* and it is denoted by $f'(z_0)$.

Proposition 90. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U \to \mathbb{C}$ be a function. Then, f is $\mathbb{C}$-differentiable at z_0 if and only if $\exists a + bi \in \mathbb{C}$ such that $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$|f(z_0 + h) - f(z_0) - (a + bi)h| \leq \varepsilon|h|$$

whenever $|h| < \delta$.

Proposition 91. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function at z_0. Then, f is continuous at z_0.

Proposition 92. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$, $f, g : U \to \mathbb{C}$ be two $\mathbb{C}$-differentiable functions at z_0 and $\alpha, \beta \in \mathbb{C}$. Then:

1. $\alpha f + \beta g$ is $\mathbb{C}$-differentiable at z_0 and:
$$(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$$
2. fg is $\mathbb{C}$-differentiable at z_0 and:
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$
3. If $g(z_0) \neq 0$, then f/g is $\mathbb{C}$-differentiable at z_0 and:
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

Theorem 93 (Chain rule). Let $U, V \subseteq \mathbb{C}$ be open sets, $z_0 \in U$, $f : U \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function at z_0 such that $f(U) \subseteq V$, and $g : V \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function at $f(z_0)$. Then, $g \circ f$ is $\mathbb{C}$-differentiable at z_0 and:

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

Proposition 94. Let $z \in \mathbb{C}$. Then:

- $(e^z)' = e^z$
- $(\cos z)' = -\sin z$
- $(\sin z)' = \cos z$
- $(\tan z)' = 1 + (\tan z)^2$
- $(\cosh z)' = \sinh z$
- $(\sinh z)' = \cosh z$
- $(\tanh z)' = 1 - (\tanh z)^2$

Definition 95. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. We say that f is *holomorphic* on U if f is $\mathbb{C}$-differentiable at each $z \in U$. We denote the set of all holomorphic functions on U by $\mathcal{H}(U)$.

Definition 96. We say that $f : \mathbb{C} \to \mathbb{C}$ is an *entire function* if f is holomorphic on the whole complex plane.

Proposition 97. Let $n \in \mathbb{N} \cup \{0\}$ and $f : \mathbb{C} \to \mathbb{C}$ be a function defined as $f(z) = z^n$. Then, f is holomorphic and $f'(z) = nz^{n-1}\ \forall z \in \mathbb{C}$.

Corollary 98. Let $p(z) = \frac{f(z)}{g(z)} \in \mathbb{C}(z)$ be a rational function. Then, $p(z)$ is a holomorphic on the open set $\mathbb{C} \setminus Z(g)$, where $Z(g) = \{z \in \mathbb{C} : g(z) = 0\}$. In particular, if $p(z) \in \mathbb{C}[z]$ is a polynomial, $p(z)$ is holomorphic on $\mathbb{C}$.

Theorem 99. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence $R \in [0,\infty]$. Then, the series $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ has the same radius of convergence R and if $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0|<R$, then f is holomorphic and:

$$f'(z)=\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} \quad \text{for } |z-z_0|<R$$

Corollary 100. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence $R \in [0,\infty]$. Then, the series $\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}$ has the same radius of convergence R for all $k \in \mathbb{N}\cup\{0\}$ and if $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0|<R$, then $f^{(k)}$ is holomorphic and:

$$f^{(k)}(z)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}$$

for $|z-z_0|<R$ and $\forall k \in \mathbb{N}\cup\{0\}$. In particular $f^{(k)}(z_0)=k!a_k$ $\forall k \in \mathbb{N}\cup\{0\}$.

Proposition 101. Let $U \subseteq \mathbb{C}$ be a connected open set and $f \in \mathcal{H}(U)$ such that $f'(z)=0$ $\forall z \in U$. Then, f is constant.

Determination of the logarithm and n-th roots

Definition 102. Let $U \subseteq \mathbb{C}$ be an open set and $f: U \longrightarrow \mathbb{C}^*$ and $g: U \to \mathbb{C}$ be functions. We say that g is a *determination* of $\ln f(z)$ if g is continuous on U and $e^{g(z)}=f(z)$ $\forall z \in U$. In particular, we say that g is a *determination of the logarithm* if g is continuous on U and $e^{g(z)}=z$ $\forall z \in U$.

Proposition 103. Let $\theta \in [0,2\pi)$, $L_\theta := \{re^{i\theta} : r \in \mathbb{R}_{\geq 0}\}$ and $U=\mathbb{C}\setminus L_\theta$. Then, there exists a determination of the logarithm $g: U \to \mathbb{C}$ defined as:

$$g(z)=\ln|z|+i\arg(z) \quad \arg(z)\in(\theta,\theta+2\pi)$$

Theorem 104. Let $U, V \subseteq \mathbb{C}$ be open sets, $f: U \longrightarrow \mathbb{C}^*$ be a function and $g \in \mathcal{H}(V)$. Suppose $f(U) \subseteq V$, $g(f(z))=z$ and $g'(f(z)) \neq 0$ $\forall z \in U$. Then, f is holomorphic on U and:

$$f'(z)=\frac{1}{g'(f(z))} \quad \forall z \in U$$

Proposition 105. Let $U, V_1, V_2 \subseteq \mathbb{C}$ be open sets, $f: U \to \mathbb{C}^*$ be a continuous function and $g_1: V_1 \longrightarrow \mathbb{C}^*$, $g_2: V_2 \longrightarrow \mathbb{C}^*$ be two determinations of the logarithm. If $W \subseteq V_1 \cap V_2 \neq \varnothing$ is connected, then $\exists k \in \mathbb{Z}$ such that:

$$g_2(z)=g_1(z)+2\pi ik \qquad \forall z \in W$$

Corollary 106. Let $U \subseteq \mathbb{C}$ be an open set, $f: U \longrightarrow \mathbb{C}^*$ be a holomorphic function and $g: U \to \mathbb{C}$ be a determination of $\ln f$. Then, g is holomorphic and:

$$g'(z)=\frac{f'(z)}{f(z)} \quad \forall z \in U$$

In particular, if g is a determination of the logarithm, then $g'(z)=\frac{1}{z}$.

Definition 107. Let $U \subseteq \mathbb{C}$ be an open set and $f: U \longrightarrow \mathbb{C}^*$ be a continuous function. A *determination* of $\sqrt[n]{f}$ is a continuous function $g: U \to \mathbb{C}$ such that $g^n=f$.

Proposition 108. Let $U \subseteq \mathbb{C}$ be an open set and $f: U \longrightarrow \mathbb{C}^*$ be a continuous function. Suppose that there exists a determination of $\log f$. Then, there exists a determination of the $\sqrt[n]{f}$ given by:

$$\sqrt[n]{f}=e^{\frac{1}{n}\log f}$$

Definition 109. Let $z \in \mathbb{C}\setminus(-\infty,0]$ and $\alpha \in \mathbb{C}$. We define z^α as $z^\alpha := e^{\alpha \log z}$.

Cauchy-Riemann equations

Definition 110. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f: U \to \mathbb{C}$ be a function. We define the partial derivatives of f at z_0 as

$$\frac{\partial f}{\partial x}(z_0)=\lim_{t\to 0}\frac{f(z_0+t)-f(z_0)}{t}$$
$$\frac{\partial f}{\partial y}(z_0)=\lim_{t\to 0}\frac{f(z_0+it)-f(z_0)}{t}$$

whenever the limits exist.

Proposition 111. Let $U \subseteq \mathbb{C}$ be an open set and $f: U \to \mathbb{C}$ be a function such that $f(z)=u(z)+iv(z)$ $\forall z \in U$ with $u, v: U \to \mathbb{R}$. Then, if $z=x+iy$, we have:

$$\frac{\partial f}{\partial x}(z)=\frac{\partial u}{\partial x}(z)+i\frac{\partial v}{\partial x}(z)$$
$$\frac{\partial f}{\partial y}(z)=\frac{\partial u}{\partial y}(z)+i\frac{\partial v}{\partial y}(z)$$

Definition 112. Let $U \subseteq \mathbb{C}$ be set and $f: U \to \mathbb{C}$ be a function such that $f(z)=u(z)+iv(z)$ $\forall z \in U$ with $u, v: U \to \mathbb{R}$. We define the *associated multivalued function* of f as the function $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x,y)=(u(x,y),v(x,y)) := (u(x+iy),v(x+iy))$.

Definition 113. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$, $f: U \to \mathbb{C}$ be a function. We say that f is $\mathbb{R}$-*differentiable* at $z_0 \in U$ if and only if there exists a $\mathbb{R}$-linear function $\mathbf{L}: \mathbb{C} \to \mathbb{R}^2$ such that:

$$\lim_{h+ki\to 0}\frac{|f(z_0+h+ki)-f(z_0)-\mathbf{L}(h+ki)|}{|h+ki|}=0$$ [11]

In that case, the function $\mathbf{L}$ is called *differential* of f at z_0 and it is denoted by $\mathbf{D}f(z_0)$[12].

Proposition 114. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f: U \to \mathbb{C}$ be a $\mathbb{R}$-differentiable function at z_0 such that $f(z)=u(z)+iv(z)$ $\forall z \in U$ with $u, v: U \to \mathbb{R}$. Then:

$$\mathbf{D}f(z_0)=\begin{pmatrix}\frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0)\\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0)\end{pmatrix}$$ [13]

[11] Here we shall think the outcomes of $\mathbf{L}$ inside $\mathbb{C}$ instead of $\mathbb{R}^2$.

[12] That is, the $\mathbb{R}$-differentiability is the usual one if we think f inside $\mathbb{R}^2$ instead of inside $\mathbb{C}$.

[13] From now on, if we use the matrix notation in the complex plane, that should be interpreted as the first row being the real part and the second row being the imaginary part.

Theorem 115 (Cauchy-Riemann theorem). Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a function and $z_0 \in U$. Then, f is $\mathbb{C}$-differentiable and $f'(z_0) = a + b\mathrm{i}$ if and only if f is $\mathbb{R}$-differentiable and:

$$\mathbf{D}f(z_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Which is equivalent to:

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0) \tag{1}$$

These equations are called *Cauchy-Riemann equations*.

Corollary 116. Let $U \subseteq \mathbb{C}$ be an open set, $u, v : U \to \mathbb{C}$ be functions such that their partial derivatives exist, they are continuous and they satisfy Eq. (1). Then, $f = u + \mathrm{i}v$ is holomorphic on U.

Definition 117. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a function. We define the *Wirtinger operators* as:

$$\partial f := \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right)$$
$$\overline{\partial} f := \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)$$

Proposition 118. Let $U \subseteq \mathbb{C}$ be an open set, $f, g : U \to \mathbb{C}$ be $\mathbb{R}$-differentiable functions, $z \in U$ and $w = g(z)$. Then:

1. $\partial(f \circ g)(z) = \partial f(w)\partial g(z) + \overline{\partial} f(w)\partial\overline{g}(z)$
2. $\overline{\partial}(f \circ g)(z) = \partial f(w)\overline{\partial} g(z) + \overline{\partial} f(w)\overline{\partial}\overline{g}(z)$

Proposition 119. Let $U \subseteq \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U \to \mathbb{C}$ be a $\mathbb{C}$-differentiable function at z_0. Then, the Cauchy-Riemann equations of Eq. (1) can also be written as:

$$\overline{\partial} f(z_0) = 0$$

Proposition 120. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a function and $z_0 \in U$. We say that f is $\mathbb{R}$-differentiable at $z_0 \in U$ if and only if there exist $x, y \in \mathbb{C}$ such that:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - xh - y\overline{h}}{h} = 0$$

In that case, we have $x = \partial f(z_0)$ and $y = \overline{\partial} f(z_0)$.

Proposition 121. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$ such that $f = u + \mathrm{i}v$ with $u, v : U \to \mathbb{R}$. Then:

$$|f'(z)|^2 = \det \mathbf{D}(u, v)(z) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Proposition 122. Let $U \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{H}(U)$ such that it satisfies one of the following properties:

- $\operatorname{Re} f = \text{const.}$
- $\operatorname{Im} f = \text{const.}$
- $|f| = \text{const.}$
- $f(U)$ is contained in a line.
- $f(U)$ is contained in a circle.

Then, f is constant.

[14] Recall **??**.

5. | Complex integration

Definition 123. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$ be a function such that $f = u + \mathrm{i}v$ with $u, v : [a, b] \to \mathbb{R}$. We define the *integral* of f on the interval $[a, b]$ as:

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + \mathrm{i}\int_a^b v(t)\,\mathrm{d}t$$

Proposition 124. Let $f, g : [a, b] \subset \mathbb{R} \to \mathbb{C}$ be functions and $\alpha \in \mathbb{C}$. Then:

1. $$\int_a^b (f + g)(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t$$
2. $$\int_a^b (\alpha f)(t)\,\mathrm{d}t = \alpha\int_a^b f(t)\,\mathrm{d}t$$
3. $$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t$$
4. $$\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$$

Proposition 125. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$ be a function and $\varphi : [a, b] \to \mathbb{R}$ be a change of variable of class $\mathcal{C}^1$ such that $\varphi([a, b]) \subseteq [a, b]$. Then:

$$\int_a^b f(\varphi(t))\varphi'(t)\,\mathrm{d}t = \int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t$$

Curves

Definition 126. Let $\gamma_1 : [a, b] \to \mathbb{C}$, $\gamma_2 : [c, d] \to \mathbb{C}$ be two curves[14] of class $\mathcal{C}^1$ such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0 \in \mathbb{C}$ for some $t_1 \in [a, b]$ and $t_2 \in [c, d]$. Suppose that $\gamma_1{}'(t_1), \gamma_2{}'(t_2) \neq 0$. We define the angle between γ_1 and γ_2 at z_0 as

$$\arg \gamma_1{}'(t_1) - \arg \gamma_2{}'(t_2)$$

which does not depend on the determination of the argument.

Definition 127. Let $\gamma : [a, b] \to U$ be a piecewise path of class $\mathcal{C}^1$. We define the inverse path $\gamma^- : [a, b] \to \mathbb{C}$ as $\gamma^-(t) = \gamma(a + b - t)$.

Definition 128. Let $\gamma : [a, b] \to U$ be an injective curve. We say that γ is a triangular path if $\gamma^* = \partial T$, where $T \subset \mathbb{C}$ is a triangle. The domain delimited by T is denoted by $\overline{D(\gamma)} := \overline{T}$.

Line integration

Definition 129. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function, $\gamma : [a,b] \to U$ be a rectifiable curve and $\{t_0, \ldots, t_n\}$ be a partition of $[a,b]$. We define the *line integral* of f along γ as:

$$\int_\gamma f(z)\,\mathrm{d}z := \lim_{n\to\infty} \sum_{j=0}^{n-1} f(\gamma(\eta_j))(\gamma(t_{j+1}) - \gamma(t_j))$$

where $\eta_j \in [t_j, t_{j+1}]$ $\forall j \in \{0, 1, \ldots, n-1\}$.

Definition 130. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to U$ be a piecewise curve of class $\mathcal{C}^1$. Suppose $\{t_0, \ldots, t_n\}$ is partition of $[a,b]$ with the property that $\gamma \in \mathcal{C}^1([t_j, t_{j+1}])$ $\forall j \in \{0, 1, \ldots, n-1\}$. Then:

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

In particular, if $\gamma \in \mathcal{C}^1([a,b])$, then:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

Theorem 131. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to U$ be a piecewise curve of class $\mathcal{C}^1$. Suppose there exists a function $F \in \mathcal{H}(U)$ such that $F'(z) = f(z)$ $\forall z \in U$. Then:

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

In particular if γ is a closed curve, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Corollary 132. There is no determination of the logarithm in any set of the form $\{z \in \mathbb{C} : 0 < r \leq |z| \leq R\}$ for $r, R \in \mathbb{R}_{>0}$ with $r < R$.

Proposition 133. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to U$ be a piecewise curve of class $\mathcal{C}^1$. Let $L(\gamma)$ be the length of γ. Then:

$$L(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Proposition 134. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function, $\gamma_1 : [a,b] \to U$ be a piecewise curve of class $\mathcal{C}^1$ and γ_2 be a reparametrization of γ_1.

- If the parametrization is positive[15], then:

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

- If the parametrization is negative, then:

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = -\int_{\gamma_2} f(z)\,\mathrm{d}z$$

  In particular, for any piecewise curve of class $\mathcal{C}^1$ $\gamma : [a,b] \to U$ we have:

$$\int_\gamma f(z)\,\mathrm{d}z = -\int_{\gamma^-} f(z)\,\mathrm{d}z$$

Definition 135. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise curve of class $\mathcal{C}^1$ with respect to the partition $\{t_0, \ldots, t_n\}$ of $[a,b]$. We define the line integral with respect to the length as:

$$\int_\gamma f(z)|\mathrm{d}z| := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t$$

Definition 136. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise curve of class $\mathcal{C}^1$. We define the following line integral with respect to $\overline{z}$:

$$\int_\gamma f(z)\,\mathrm{d}\overline{z} := \overline{\int_\gamma \overline{f(z)}\,\mathrm{d}z}$$

Proposition 137. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \subset \mathbb{R} \to \mathbb{C}$ be a continuous function and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise curve of class $\mathcal{C}^1$. Then:

1. $\displaystyle\int_\gamma |\mathrm{d}z| = L(\gamma)$
2. $\displaystyle\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f(z)||\mathrm{d}z|$

6. | Local Cauchy theory

Theorem 138 (Goursat's theorem). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$ and γ be a triangular path such that $\overline{D(\gamma)} \subseteq U$. Then:

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Theorem 139 (Local Cauchy's integral theorem). Let $U \subseteq \mathbb{C}$ be an open convex set, $f \in \mathcal{H}(U)$ and $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise curve of class $\mathcal{C}^1$ with $\gamma^* \subset U$. Then:

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

[15] Recall ??.

Lemma 140. Let $U \subseteq \mathbb{C}$ be an open convex set, $z_0 \in U$, $f \in \mathcal{H}(U \setminus \{z_0\})$ and $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise curve of class $\mathcal{C}^1$ such that $\gamma^* \subseteq U \setminus \{z_0\}$. Suppose $\lim_{z\to z_0} f(z)(z-z_0) = 0$. Then:

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Definition 141. We define the *Fresnel integrals* $S(z)$ and $C(z)$ as:

$$S(z) = \int_0^z \sin(\zeta^2)\,\mathrm{d}\zeta \qquad C(z) = \int_0^z \cos(\zeta^2)\,\mathrm{d}\zeta$$

Index of a curve

Theorem 142. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve and $z_0 \notin \gamma^*$. Then, there exist an open disc D_0 centered at z_0 such that $D_0 \cap \gamma^* = \varnothing$, and a determination $\Phi : [a,b] \times D_0 \to \mathbb{C}$ of $\ln(\gamma(t) - z_0)$. Moreover, given $t \in [a,b]$ the function $z \mapsto \Phi(t,z)$ is holomorphic and for each $z \in D_0$ the function $t \mapsto \Phi(t,z)$ has the same differentiability as γ.

Definition 143. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve, $z_0 \notin \gamma^*$ and $\Phi : [a,b] \times D_0 \to \mathbb{C}$ be a determination of $\ln(\gamma(t) - z_0)$. We define the *index* (or *winding number*) of γ with respect to z_0, denoted by $\mathrm{Ind}(\gamma, z_0)$, as:

$$\mathrm{Ind}(\gamma, z_0) := \frac{\Phi(b,z_0) - \Phi(a,z_0)}{2\pi\mathrm{i}}$$ [16]

Proposition 144. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve and $z_0 \notin \gamma^*$. Then:

1. The value $\mathrm{Ind}(\gamma, z_0) \in \mathbb{Z}$ does not depend on the determination of $\ln(\gamma(t) - z_0)$ chosen.
2. $\mathrm{Ind}(\gamma, z_0) \in \mathbb{Z}$.
3. $\mathrm{Ind}(\gamma^-, z_0) = -\mathrm{Ind}(\gamma, z_0)$.
4. The function
$$\begin{aligned} \mathbb{C}\setminus\gamma^* &\longrightarrow \mathbb{Z} \\ z &\longmapsto \mathrm{Ind}(\gamma, z) \end{aligned}$$
is continuous and therefore it is constant on each connected component of $\mathbb{C} \setminus \gamma^*$.
5. If $z \in \mathbb{C}\setminus\gamma^*$ is in the unbounded component of $\mathbb{C}\setminus\gamma^*$, then $\mathrm{Ind}(\gamma, z) = 0$.

Theorem 145. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise curve of class $\mathcal{C}^1$ and $z_0 \notin \gamma^*$. Then:

$$\mathrm{Ind}(\gamma, z_0) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

Theorem 146 (Local Cauchy's integral formula). Let $U \subseteq \mathbb{C}$ be an open convex set, $f \in \mathcal{H}(U)$, $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise curve of class $\mathcal{C}^1$ and $z_0 \notin \gamma^*$. Then:

$$f(z_0) \cdot \mathrm{Ind}(\gamma, z_0) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

[16] Intuitively it can be though as the number of turns made by γ around z_0.

Corollary 147. Let $U \subseteq \mathbb{C}$ be an open convex set, $f \in \mathcal{H}(U)$, $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_{\geq 0}$. Then:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

Lemma 148. Let $U \subseteq \mathbb{C}$ be an open, $(f_n) \in U \subseteq \mathbb{C}$ be a sequence of continuous functions, $f : U \to \mathbb{C}$ be such that (f_n) converge uniformly to f over compact sets in U, and $\gamma : [a,b] \to \mathbb{C}$ be a closed piecewise curve of class $\mathcal{C}^1$. Then:

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$$

Corollary 149. Let $z_0 \in \mathbb{C}$ and $f(z) = \sum a_n (z - z_0)^n$. Then:

$$\int_0^z f(\zeta)\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n+1}$$

Analytic functions

Definition 150. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. We say that f is *analytic* on U if for each $z_0 \in U$, there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence $R_{z_0} > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in a neighbourhood of z.

Theorem 151. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$. Then, $\forall z_0 \in U$ there exists a power series $\sum a_n (z - z_0)^n$ with radius of convergence $R_{z_0} \geq d(z_0, \partial U) > 0$ such that:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

That is, f is analytic.

Corollary 152. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. Then, f is analytic on U if and only if f is holomorphic on U.

Corollary 153 (Local Cauchy's integral formula for derivatives). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$ and $z_0 \in U$, $r \in \mathbb{R}_{>0}$ such that $\overline{D(z_0, r)} \subset U$. Let $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$. Then:

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z$$

Corollary 154. Let $f \in \mathcal{H}(\mathbb{C})$. Then, for all $z_0 \in \mathbb{C}$ there exists a power series $\sum a_n (z - z_0)^n$ with infintie radius of convergence such that:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Corollary 155. Let $f(z) = \sum a_n(z-z_0)^n$ be a power series with radius of convergence $R \in (0,\infty)$. Then, f is analytic on $D(z_0, R)$ and $\forall w_0 \in D(z_0, R)$ we have

$$f(z) = \sum_{n=0}^{\infty} b_n(z-w_0)^n$$

whenever $|z-w_0| < R - |z_0 - w_0|$. Here the coefficients b_n can be determined with the formula $b_n = \frac{f^{(n)}(w_0)}{n!}$.

Some important theorems

Proposition 156 (Cauchy's inequality). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$ and $z_0 \in U$, $r \in \mathbb{R}_{>0}$ such that $\overline{D(z_0, r)} \subset U$. Let $M = \sup\{|f(z)| : |z-z_0| = r\}$. Then:

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$$

Theorem 157 (Liouville's theorem). Let $f \in \mathcal{H}(\mathbb{C})$ and bounded. Then, f is constant.

Corollary 158. Let $f \in \mathcal{H}(\mathbb{C})$ such that $\mathrm{Re}(f(z)) \geq 0$ $\forall z \in \mathbb{C}$. Then, f is constant.

Theorem 159 (Fundamental theorem of algebra). Let $p(z) = a_0 + a_1 z \cdots + a_n z^n \in \mathbb{C}[z]$ with $a_n \neq 0$. Then, $\exists \alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Corollary 160. $\mathbb{C}$ is an algebraically closed field.

Proposition 161 (Cardano-Vieta's formulas). Let $p(z) = a_0 + a_1 z \cdots + a_n z^n \in \mathbb{C}[z]$ with $a_n \neq 0$ and with roots $z_1, \ldots, z_n$. Then, we have the following relations:

$$\begin{aligned} \sum_{i=1}^{n} z_i &= -\frac{a_{n-1}}{a_n} \\ \sum_{1 \leq i < j \leq n} z_i z_j &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ z_1 \cdots z_n &= (-1)^n \frac{a_0}{a_n} \end{aligned}$$

Theorem 162 (Morera's theorem). Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a continuous function such that $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = 0$ for all triangular path γ with $\gamma^* \subset U$. Then, f is analytic.

Theorem 163. Let $f \in \mathcal{H}(\mathbb{C})$ be non-constant. Then, $\overline{f(\mathbb{C})} = \mathbb{C}$.

Corollary 164. Let $f = u + \mathrm{i}v \in \mathcal{H}(\mathbb{C})$ be non-constant. Then, $u(\mathbb{C}) = \mathbb{R}$ and $v(\mathbb{C}) = \mathbb{R}$.

Theorem 165 (Weierstraß' theorem). Let $U \subseteq \mathbb{C}$ be an open set, $(f_n) \in \mathcal{H}(U)$ be a sequence of functions such that they converge uniformly to $f : U \to \mathbb{C}$ over compact sets of U. Then, $f \in \mathcal{H}(U)$ and $\forall k \in \mathbb{N}$, $({f_n}^{(k)})$ converge uniformly to $f^{(k)}$ over compact sets.

Corollary 166. Let $U \subseteq \mathbb{C}$ be an open set, $(f_n) \in \mathcal{H}(U)$ be a sequence of functions such that they converge uniformly to 0 over compact sets of U, and $f(z) = \sum_{n=0}^{\infty} f_n(z)$. Then, $f \in \mathcal{H}(U)$ and:

$$f^{(k)}(z) = \sum_{n=0}^{\infty} {f_n}^{(k)}(z)$$

7. | Local properties of holomorphic functions

Zeros of holomorphic functions

Definition 167. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$. We denote by $Z(f)$ the set of all zeros of f.

Theorem 168. Let $U \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{H}(U)$. Then, the following statements are equivalent:

1. $f(z) = 0$.
2. $\exists z_0 \in U$ such that $f^{(n)}(z_0) = 0$ $\forall n \in \mathbb{N} \cup \{0\}$.
3. $Z(f)' \cap U \neq \varnothing$.

Corollary 169. Let $U \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{H}(U)$ such that f is not identically zero. Then, if $z_0 \in U$ is a zero of f, $\exists m \in \mathbb{N}$ and $g \in \mathcal{H}(U)$ such that $g(z_0) \neq 0$ and:

$$f(z) = (z-z_0)^m g(z)$$

The value of m is called *multiplicity* of z_0. In particular, the zeros of f are isolated.

Theorem 170 (Analytic continuation theorem). Let $U \subseteq \mathbb{C}$ be an open connected set and $f, g \in \mathcal{H}(U)$ such that $\{z \in U : f(z) = g(z)\}' \cap U \neq \varnothing$. Then, $f(z) = g(z)$ $\forall z \in U$.

Corollary 171. Let $U \subseteq \mathbb{C}$ be an open connected set, $f \in \mathcal{H}(U)$ be not identically zero. Then, $Z(f)$ is countable.

Maximum and minimum principles

Theorem 172 (Maximum modulus principle). Let $U \subseteq \mathbb{C}$ be an open connected set, $f \in \mathcal{H}(U)$ and $a \in U$ such that $|f(a)| \geq |f(z)|$ $\forall z \in \overline{D(a,r)} \subset U$, for some $r \in \mathbb{R}_{>0}$. Then, f is constant.

Corollary 173 (Minimum modulus principle). Let $U \subseteq \mathbb{C}$ be an open connected set, $f \in \mathcal{H}(U)$ such that it doesn't have zeros on U and $a \in U$ such that $0 < |f(a)| \leq |f(z)|$ $\forall z \in \overline{D(a,r)} \subset U$, for some $r \in \mathbb{R}_{>0}$. Then, f is constant.

Corollary 174. Let $U \subseteq \mathbb{C}$ be an open bounded set and $f \in \mathcal{H}(\overline{U})$. Then:

1. $|f|$ attains its absolute maximum on ∂U, that is:
$$\max\{|f(z)| : z \in \overline{U}\} = \max\{|f(z)| : z \in \partial U\}$$
2. If f doesn't have zeros on $\overline{U}$, $|f|$ attains its absolute minimum on ∂U, that is:
$$\min\{|f(z)| : z \in \overline{U}\} = \min\{|f(z)| : z \in \partial U\}$$

Harmonic functions

Definition 175. Let $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ be an open set and $u : U \to \mathbb{R}$ be a function of class $\mathcal{C}^2$. We say that u is *harmonic* if $\Delta u(z) = 0\ \forall z \in U$[17].

Proposition 176. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$ such that $f = u + \mathrm{i}v$ with $u, v : U \to \mathbb{R}$. Then, u and v are harmonic and $u, v \in \mathcal{C}^\infty(U)$.

Definition 177. Let $U \subseteq \mathbb{R}^n$ be a set. We say that U is a *domain* or *region* if U is open and connected.

Definition 178. Let $S \subseteq \mathbb{R}^n$ be a subset and $s_0 \in S$. We say that S is a *star domain* with respect to s_0 if $\forall s \in S$, the line segment from s_0 to s lies in S.

Definition 179. Let $U \subseteq \mathbb{R}^n$ be a subset. We say that U is *convex* if it is a star domain with respect to $x \in U$, for all $x \in U$.

Proposition 180. Let $U \subseteq \mathbb{C}$ be a star domain with respect to $z_0 = x_0 + \mathrm{i}y_0 \in U$ and $u : U \to \mathbb{R}$ be a harmonic function. Then, there exists a harmonic function $v : U \to \mathbb{R}$ such that $f = u + \mathrm{i}v$ is holomorphic. That function v is:

$$v(x + \mathrm{i}y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x + \mathrm{i}t)\, \mathrm{d}t - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t + \mathrm{i}y_0)\, \mathrm{d}t + C$$

for some constant $C \in \mathbb{R}$.

Theorem 181 (Mean value property). Let $U \subseteq \mathbb{C}$, $u : U \to \mathbb{C}$ be a harmonic function, $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$ such that $\overline{D(a, r)} \subset U$. Then:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + r\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}t$$

8. | General Cauchy theory

Chains and homology

Definition 182. Let $\gamma_1 : [a, b] \to \mathbb{C}$, $\gamma_2 : [c, d] \to \mathbb{C}$ be two paths and $n \in \mathbb{Z}$. We define the path $\gamma_1 + \gamma_2$ as the concatenation of γ_1 and γ_2. That is:

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } \frac{a}{2} \leq t \leq \frac{b}{2} \\ \gamma_2(2t - b) & \text{if } \frac{c+b}{2} < t \leq \frac{d+b}{2} \end{cases}$$

We define the path $n\gamma_1$ as the path:

$$(n\gamma_1)(t) := \begin{cases} \gamma_1 + \overset{(n)}{\cdots} + \gamma_1 & \text{if } n \geq 0 \\ {\gamma_1}^- + \overset{(|n|)}{\cdots} + {\gamma_1}^- & \text{if } n < 0 \end{cases}$$

Definition 183. Let $\gamma_1, \ldots, \gamma_k$ be paths. A *chain of paths* is a linear combination

$$\Gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$$

where $n_i \in \mathbb{Z}\ \forall i = 1, \ldots, k$. We define the *image* of Γ, Γ^*, as:

$$\Gamma^* := \bigcup_{i=1}^{k} {\gamma_i}^*$$

[17] Recall ??.

Definition 184. Let $\Gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$ be a chain of paths. We say that Γ is a *cycle* if the paths γ_i are closed $\forall i = 1, \ldots, k$.

Definition 185. Let $\Gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$ be a cycle and $z_0 \in \mathbb{C} \setminus \gamma^*$. We define the *index* of Γ with respect to z_0 as:

$$\operatorname{Ind}(\Gamma, a) := \sum_{i=1}^{k} n_i \operatorname{Ind}(\gamma_i, a)$$

Definition 186. Let $U \subseteq \mathbb{C}$ be an open set and Γ be a cycle such that $\Gamma^* \subset U$. We say that Γ is *homologous to zero* on U, and we denoted it by $\Gamma \underset{U}{\approx} 0$, if $\operatorname{Ind}(\Gamma, z) = 0$ $\forall z \in \mathbb{C} \setminus U$.

Lemma 187. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$. Consider the function $\varphi : U \times U \to \mathbb{C}$ defined as:

$$\varphi(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases}$$

Then, φ is continuous and $\forall w_0 \in U$ the function $\varphi(\cdot, w_0)$ is holomorphic on U.

Lemma 188. Let $U, V \subseteq \mathbb{C}$ be open sets, $g : U \times V \to \mathbb{C}$ be a continuous function and Γ be a chain of path of class $\mathcal{C}^1$ with $\Gamma^* \subset V$. Suppose that given $w_0 \in V$ we have $g(\cdot, w_0) \in \mathcal{H}(U)$. Then, $F(z) := \int_\Gamma g(z, w)\, \mathrm{d}w$ is holomorphic on U and

$$F'(z) = \int_\Gamma \frac{\partial g}{\partial z}(z, w)\, \mathrm{d}w$$

Theorem 189 (General Cauchy's integral formula). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$, Γ be a piecewise cycle of class $\mathcal{C}^1$ with $\Gamma^* \subset U$ and $z_0 \in U \setminus \Gamma^*$. Suppose that $\Gamma \underset{U}{\approx} 0$. Then:

$$f(z_0) \cdot \operatorname{Ind}(\Gamma, z_0) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z$$

If $\Gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$, the previous formula is equivalent to:

$$f(z_0) \sum_{i=1}^{k} n_i \operatorname{Ind}(\gamma_i, z_0) = \frac{1}{2\pi\mathrm{i}} \sum_{i=1}^{k} n_i \int_{\gamma_i} \frac{f(z)}{z - z_0}\, \mathrm{d}z$$

Theorem 190 (Cauchy's integral theorem). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$ and $\Gamma : [a, b] \to \mathbb{C}$ be a piecewise cycle of class $\mathcal{C}^1$ with $\Gamma^* \subset U$. Suppose that $\Gamma \underset{U}{\approx} 0$. Then:

$$\int_\Gamma f(z)\, \mathrm{d}z = 0$$

Corollary 191 (General Cauchy's integral formula for derivatives). Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$, Γ be a piecewise cycle of class $\mathcal{C}^1$ with $\Gamma^* \subset U$ and $z_0 \in U \setminus \Gamma^*$. Suppose that $\Gamma \underset{U}{\approx} 0$. Then:

$$f^{(n)}(z_0) \cdot \mathrm{Ind}(\Gamma, z_0) = \frac{n!}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z$$

Theorem 192. Let $U \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{H}(U)$. Let $z_1, \ldots, z_n$ be the zeros of f (counting repetitions) and $\Gamma \underset{U}{\approx} 0$ be a piecewise cycle of class $\mathcal{C}^1$ such that $\Gamma^* \cap \{z_1, \ldots, z_n\} = \varnothing$. Then:

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{j=1}^{n} \mathrm{Ind}(\Gamma, z_j) =: \sum \mathrm{Ind}(\Gamma, f^{-1}(0))$$

where the last expression is a new notation that we will sometimes use in order to simplify the lecture.

Corollary 193. Let $U \subseteq \mathbb{C}$ be an open connected set, $f \in \mathcal{H}(U)$, $w \in \mathbb{C}$ and $\Gamma \underset{U}{\approx} 0$ be a piecewise cycle of class $\mathcal{C}^1$ such that $\Gamma^* \cap f^{-1}(w) = \varnothing$. Then:

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'(z)}{f(z) - w} \,\mathrm{d}z = \sum \mathrm{Ind}(\Gamma, f^{-1}(w))$$

Moreover we have that:

$$\mathrm{Ind}(f \circ \Gamma, w) = \sum \mathrm{Ind}(\Gamma, f^{-1}(w))$$

Local behaviour and open mapping theorems

Theorem 194 (Local behaviour of a holomorphic function). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r))$ be a non constant function. Suppose $f(a) = b \in \mathbb{C}$ and so we can write $f(z) - b = (z-a)^m g(z)$ with $m \in \mathbb{N}$ and $g \in \mathcal{H}(D(a,r))$ with $g(a) \neq 0$. Then, there exists $\varepsilon > 0$ and $\delta > 0$ such that $D(a,\varepsilon) \subset D(a,r)$ and all the points of $D(b,\delta) \setminus \{b\}$ have m preimages in $D(a,\varepsilon)$ of multiplicity 1.

Theorem 195 (Open mapping theorem). Let $U \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{H}(U)$ be a non-constant function. Then, f is open.

Corollary 196. Let $U \subseteq \mathbb{C}$ be an open set, $f \in \mathcal{H}(U)$ and $z_0 \in U$ such that $f'(z_0) \neq 0$. Then, f is a local homeomorphism at z_0.

Corollary 197. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$ be an injective function. Then, $V := f(U) \subseteq \mathbb{C}$ is open and $\forall z \in U$, $f'(z) \neq 0$, $f^{-1} \in \mathcal{H}(V)$ and:

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \qquad \forall w \in V$$

Theorem 198. Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r))$ be a non constant function. Suppose $f(a) = b \in \mathbb{C}$ with multiplicity $m > 1$, that is, we can write $f(z) - b = (z-a)^m g(z)$ with $m > 1$ and $g \in \mathcal{H}(D(a,r))$ with $g(a) \neq 0$. Then, there exists $\varepsilon > 0$ and $\delta > 0$ such that $g(D(a,\varepsilon)) \subset D(g(a), |g(a)|)$ and so $\exists h \in \mathcal{H}(D(a,r))$ such that $h^m = g$ and $f(z) = b + F(z)^m$, where $F(z) = (z-a)h(z)$ and $F'(a) \neq 0$.

Definition 199. Let $U \subseteq \mathbb{C}$ be an open connected set. We say that U is *simply connected* if $\mathbb{C}_\infty \setminus U$ is connected.

Proposition 200. Let $U \subseteq \mathbb{C}$ be an open connected set. Then, U is simply connected is and only if for all closed path γ with $\gamma^* \subset U$ we have $\gamma \underset{U}{\approx} 0$.

Definition 201. Let $U \subseteq \mathbb{C}$ be an open set. We say that U is *simply connected* if each connected component is simply connected.

Theorem 202. Let $U \subseteq \mathbb{C}$ be an open simply connected set and $f \in \mathcal{H}(U)$. Then, f has a primitive.

Theorem 203. Let $U \subseteq \mathbb{C}$ be a simply connected domain, $f \in \mathcal{H}(U)$ be such that $f(z) \neq 0\ \forall z \in U$. Then, there exist a determination of $\log f$. Moreover that determination is unique if we fix its value at a point $z_0 \in U$.

Isolated singularities of holomorphic functions

Definition 204. Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f : D(a,r) \setminus \{a\} \to \mathbb{C}$ be a function. We say that a is an *isolated singularity* of f if $\exists \varepsilon > 0$ such that $f \in \mathcal{H}(D(a,\varepsilon) \setminus \{a\})$.

Definition 205 (Removable singularity). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f : D(a,r) \setminus \{a\} \to \mathbb{C}$ be a function with an isolated singularity at a. We say that a is a *removable singularity* of f if there exists $g \in \mathcal{H}(D(a,R))$ such that $f(z) = g(z)\ \forall z \in D(a,r) \setminus \{a\}$.

Theorem 206 (Riemann's theorem on removable singularities). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$. Then, a is a removable singularity of f if and only if

$$\lim_{z \to a} f(z)(z-a) = 0$$

Definition 207. Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has an isolated singularity at a. We say that a is a *pole* of f if $\lim\limits_{z \to a} |f(z)| = \infty$.

Proposition 208 (Pole). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has a pole at a. Then, we can write f as:

$$f(z) = \frac{A_{-m}}{(z-a)^m} + \cdots + \frac{A_{-1}}{z-a} + \sum_{n=0}^{\infty} A_n (z-a)^n$$

where $A_i \in \mathbb{C}$ for $i \in \{-m, -m+1, \cdots\}$ and $m \in \mathbb{N}$. The value of m is called *order* of the pole a. If $m = 1$, we say that the pole is *simple*. The term A_{-1} is called *residue* of f at a and it is denoted as $\mathrm{Res}(f,a) := A_{-1}$. We denote by $P(f)$ the set of all poles of f.

Definition 209 (Essential singularity). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has an isolated singularity at a. If a is neither a removable singularity nor a pole, we say that a is an *essential singularity* of f.

Definition 210. Let $(z_n)_{n\in\mathbb{Z}} \in \mathbb{C}$ be a sequence. We say that a series of the form of

$$\sum_{n=-\infty}^{+\infty} z_n$$

is *convergent*, *absolutely convergent* or *uniformly convergent* if and only if the series

$$\sum_{n=1}^{\infty} z_{-n} \quad \text{and} \quad \sum_{n=0}^{\infty} z_n$$

are both convergent, absolutely convergent or uniformly convergent, respectively.

Theorem 211 (Laurent series theorem). Let $a \in \mathbb{C}$, $R_1, R_2 \in \mathbb{R}_{>=0} \cup \{\infty\}$ with $R_1 < R_2$ and consider the crown $C = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$. Let $f \in \mathcal{H}(C)$. Then:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \qquad \forall z \in C$$

and the coefficients are given by:

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z$$

where $\gamma_r(t) = a + r\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$ and $r \in (R_1, R_2)$ be any value of that interval. This expression is called *Laurent series* of f. Moreover we have the following properties:

1. If $z \in C$, then Laurent series converges absolutely at z.
2. Given $R_1 < r_1 \leq r_2 < R_2$, the Laurent series converges absolutely and uniformly on the crown $\{z \in \mathbb{C} : r_1 \leq |z - z_0| \leq r_2\}$.

Corollary 212. Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, $f \in \mathcal{H}(U \setminus \{a\})$ be a function with an isolated singularity at a and $f = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$ be its Laurent series centered at a. Then:

1. The following statements are equivalent:
   i) a is a removable singularity.
   ii) $\lim\limits_{z\to a} f(z) = w \in \mathbb{C}$.
   iii) $a_n = 0 \ \forall n \leq -1$.
2. The following statements are equivalent:
   i) a is a pole of order $m \in \mathbb{N}$.
   ii) $\lim\limits_{z\to a} \left| f(z)(z-a)^k \right| = \infty \ \forall k \in \{0, 1, \ldots, m-1\}$.
   iii) $a_n = 0 \ \forall n \leq -m-1$ and $a_{-m} \neq 0$.
3. The following statements are equivalent:
   i) a is an essential singularity.
   ii) $\lim\limits_{z\to a} f(z)$ doesn't exist.
   iii) There exist infinite $n \in \mathbb{N}$ such that $a_{-n} \neq 0$.

[18]That is, all points of A are isolated.

Corollary 213 (Partial fraction decomposition theorem). Let $p, q \in \mathbb{C}[z]$ such that $q(z) = (z-a_1)^{n_1} \cdots (z-a_k)^{n_k}$. Suppose that:

$$\begin{aligned}
\frac{p(z)}{q(z)} &= \frac{A_1^{n_1}}{(z-a_1)^{n_1}} + \cdots + \frac{A_1^1}{z-a_1} + h_1(z) \\
h_1(z) &= \frac{A_2^{n_2}}{(z-a_2)^{n_2}} + \cdots + \frac{A_2^1}{z-a_2} + h_2(z) \\
&\vdots \\
h_k(z) &= \frac{A_k^{n_k}}{(z-a_k)^{n_k}} + \cdots + \frac{A_k^1}{z-a_k} + g(z)
\end{aligned}$$

And define $s_j := \sum_{i=1}^{n_i} \frac{A_j^i}{(z-a_j)^i}$. Then, $g(z) = \frac{p(z)}{q(z)} - \sum_{j=1}^{k} s_j(z)$ is a polynomial.

Residues theorem and applications

Theorem 214 (Residues theorem). Let $U \subseteq \mathbb{C}$ be an open set, $A \subset U$ such that $A' \cap U = \varnothing$[18], $f \in \mathcal{H}(U \setminus A)$ and $\Gamma \underset{U}{\approx} 0$ be a piecewise cycle of class $\mathcal{C}^1$ such that $\Gamma^* \subset U \setminus A$. Then:

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma} f(z) \,\mathrm{d}z = \sum_{a\in A} \mathrm{Res}(f, a) \,\mathrm{Ind}(\Gamma, a)$$

Proposition 215. Let $U \subseteq \mathbb{C}$ be an open set, $a \in U$, $f \in \mathcal{H}(U \setminus \{a\})$ be a function with a pole of order m at a. Then:

$$\mathrm{Res}(f, a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z-a)^m f(z)\right]$$

In particular, if $f(z) = \frac{g(z)}{h(z)}$ with $g, h \in \mathcal{H}(U)$ and the pole is simple, then:

$$\mathrm{Res}(f, a) = \frac{g(a)}{h'(a)}$$

Proposition 216 (Calculation of integrals). For the following cases use the function and cycle given to find the integrals using the Residues theorem.

1. $\displaystyle\int_0^{2\pi} g(\cos x, \sin x) \,\mathrm{d}x$, $g \in \mathbb{R}(x, y)$.
   i) Function: $f(z) = g\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2\mathrm{i}}\right)$
   ii) Cycle: $\Gamma(t) = \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$
2. $\displaystyle\int_{-\infty}^{+\infty} g(x) \,\mathrm{d}x$, $g = \frac{p}{q} \in \mathbb{C}(x)$ such that $q(x) \neq 0 \ \forall x \in \mathbb{R}$ and $\deg q \geq \deg p + 2$.
   i) Function: $f(z) = g(z)$
   ii) Cycle: $\Gamma(t) = \gamma_1(t) + \gamma_2(t)$, where:
   $$\begin{cases} \gamma_1(t) = t & t \in [-R, R] \\ \gamma_2(t) = R\mathrm{e}^{\mathrm{i}t} & t \in [0, \pi] \end{cases}$$
   and $R \in \mathbb{R}_{>0}$.

3. $\displaystyle\int_0^{+\infty} \frac{g(x)}{x^\alpha}\,\mathrm{d}x$, $g \in \mathbb{R}(x)$ and $\alpha \in \mathbb{R}$ be such that the integral converges.

   i) Function: $f(z) = g(z)z^{-\alpha}$

   ii) Cycle: $\Gamma(t) = \sum_{i=1}^4 \gamma_i(t)$, where:

$$\begin{cases} \gamma_1(t) = t & t \in [\varepsilon, R] \\ \gamma_2(t) = R\mathrm{e}^{\mathrm{i}t} & t \in [0, 2\pi - \delta] \\ \gamma_3(t) = t\mathrm{e}^{-\mathrm{i}\delta} & t \in [R, \varepsilon] \\ \gamma_4(t) = \varepsilon\mathrm{e}^{\mathrm{i}t} & t \in [2\pi - \delta, 0] \end{cases}$$

   and $\varepsilon, \delta, R \in \mathbb{R}_{>0}$.

4. $\displaystyle\int_0^{+\infty} g(x)\log x\,\mathrm{d}x$, $g \in \mathbb{R}(x)$ such that it doesn't have singularities on $\mathbb{R}$ and $\lim_{x\to 0} xg(x) = 0$.

   i) Function: $f(z) = g(z)(\log z)^2$

   ii) Cycle: $\Gamma(t) = \sum_{i=1}^4 \gamma_i(t)$, where:

$$\begin{cases} \gamma_1(t) = t & t \in [\varepsilon, R] \\ \gamma_2(t) = R\mathrm{e}^{\mathrm{i}t} & t \in [0, 2\pi - \delta] \\ \gamma_3(t) = t\mathrm{e}^{-\mathrm{i}\delta} & t \in [R, \varepsilon] \\ \gamma_4(t) = \varepsilon\mathrm{e}^{\mathrm{i}t} & t \in [2\pi - \delta, 0] \end{cases}$$

   and $\varepsilon, \delta, R \in \mathbb{R}_{>0}$.

5. $\displaystyle\int_0^{+\infty} g(x)\sin x\,\mathrm{d}x$ or $\displaystyle\int_0^{+\infty} g(x)\sin x\,\mathrm{d}x$, $g(z) \in \mathcal{H}(U)$ with certain restrictions.

   i) Function: $f(z) = g(z)\mathrm{e}^{\mathrm{i}z}$

   ii) Cycle: $\Gamma(t) = \gamma_1(t) + \gamma_2(t)$, where:

$$\begin{cases} \gamma_1(t) = t & t \in [-R, R] \\ \gamma_2(t) = R\mathrm{e}^{\mathrm{i}t} & t \in [0, \pi] \end{cases}$$

   and $R \in \mathbb{R}_{>0}$.

Theorem 217. Let $f \in \mathbb{C}(z)$ be a rational function such that $\lim_{|z|\to\infty} f(z) = 0$ and P be the set of poles of f.
If $P \cap \mathbb{Z} = \varnothing$, then:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\sum_{p\in P} \operatorname{Res}(\pi\cot(\pi z)f(z), p)$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} (-1)^n f(n) = -\sum_{p\in P} \operatorname{Res}(\pi\csc(\pi z)f(z), p)$$

If $P \cap (\mathbb{Z} + \frac{1}{2}) = \varnothing$, then:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f\left(n + \frac{1}{2}\right) = -\sum_{p\in P} \operatorname{Res}(\pi\cot(\pi z)f(z), p)$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} (-1)^n f\left(n + \frac{1}{2}\right) = -\sum_{p\in P} \operatorname{Res}(\pi\csc(\pi z)f(z), p)$$

Theorem 218 (Casorati-Weierstraß theorem). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has an essential singularity at a. Then, $\forall\delta > 0$:

$$\overline{f(D(a,\delta) \setminus \{a\})} = \mathbb{C}$$

Corollary 219. Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$, $w \in \mathbb{C}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has an essential singularity at a. Then, there exists a sequence $(z_n) \in \mathbb{C} \setminus \{a\}$ such that $\lim_{n\to\infty} z_n = a$ and $\lim_{n\to\infty} f(z_n) = w$.

Theorem 220 (Little Picard's theorem). Let $f \in \mathcal{H}(\mathbb{C})$ be a non-constant function. Then, $\operatorname{im} f$ is either the whole complex plane or the plane minus a single point.

Theorem 221 (Great Picard's theorem). Let $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$, $w \in \mathbb{C}$ and $f \in \mathcal{H}(D(a,r) \setminus \{a\})$ be such that it has an essential singularity at a. Then, $\exists\alpha \in \mathbb{C}$ such that $\forall\delta > 0$:

$$f(D(a,\delta) \setminus \{a\}) \supseteq \mathbb{C} \setminus \{\alpha\}$$

Definition 222. Let $U \subseteq \mathbb{C}$ be an open set, $A \subset U$ such that $A' \cap U = \varnothing$ and $f \in \mathcal{H}(U \setminus A)$ be such that for each $a \in A$, f has a pole at a. In these conditions, we say that f is *meromorphic* and we denote the set of all meromorphic on U by $\mathfrak{m}(U)$.

Theorem 223 (Argument principle). Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathfrak{m}(U)$ be not identically zero. We denote by $Z(f)$ the set of zeros of f and by $P(f)$ the set of its poles. Let $\Gamma \underset{U}{\approx} 0$ be a piecewise cycle of class $\mathcal{C}^1$ such that $\Gamma^* \subset U \setminus (Z(f) \cup P(f))$. Then:

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{z\in Z(f)} \operatorname{Ind}(\Gamma, z) - \sum_{p\in P(f)} \operatorname{Ind}(\Gamma, p)$$

Theorem 224 (Generalized argument principle). Let $U \subseteq \mathbb{C}$ be an open set $w \in U$, $f \in \mathfrak{m}(U)$ be not identically w and $g \in \mathcal{H}(U)$. We denote by $Z(f - w)$ the set of zeros of the function $f(z) - w$ and by $P(f - w)$ the set of its poles. Let $\Gamma \underset{U}{\approx} 0$ be a piecewise cycle of class $\mathcal{C}^1$ such that $\Gamma^* \subset U \setminus (Z(f - w) \cup P(f - w))$. Then:

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}} \int_\Gamma g(\zeta)\frac{f'(\zeta)}{f(\zeta) - w}\,\mathrm{d}\zeta = \\ &\quad = \sum_{z\in Z(f-w)} g(z)\operatorname{Ind}(\Gamma, z) - \sum_{p\in P(f-w)} g(p)\operatorname{Ind}(\Gamma, p) \end{aligned}$$

Corollary 225. Let $U \subseteq \mathbb{C}$ be an open set $w \in U$, $f \in \mathfrak{m}(U)$ be not identically w and $g \in \mathcal{H}(U)$. Let $\gamma \underset{U}{\approx} 0$ be a piecewise curve defined on $[a, b]$ of class $\mathcal{C}^1$ such that $\gamma^* \subset U \setminus (Z(f - w) \cup P(f - w))$. Then:

$$\begin{aligned} \operatorname{Ind}(f \circ \gamma, w) &= \frac{\arg(f(\gamma(b)) - w) - \arg(f(\gamma(a)) - w)}{2\pi} \\ &= \sum_{z\in Z(f-w)} \operatorname{Ind}(\gamma, z) - \sum_{p\in P(f-w)} \operatorname{Ind}(\gamma, p) \end{aligned}$$

Theorem 226 (Rouché's theorem). Let $U \subseteq \mathbb{C}$ be an open set, $f, g \in \mathfrak{m}(U)$ and $\gamma \underset{U}{\approx} 0$ be a piecewise path of class $\mathcal{C}^1$ such that $\gamma^* \subset U \setminus (Z(f) \cup P(f) \cup Z(g) \cup P(g))$ and $\operatorname{Ind}(\gamma, z) \in \{0, 1\}$ $\forall z \in U$. Suppose that:

$$|f(z) + g(z)| < |f(z)| + |g(z)| \quad \forall z \in \gamma^*$$

Then if Z_f and Z_g denote the number of zeros of the respective functions on $\overline{D(\gamma)}$ and P_f and P_g denote the number of poles of the respective functions on $\overline{D(\gamma)}$, we have:

$$Z_f - P_f = Z_g - P_g$$

Definition 227. Let $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{N}$. We define the number $\binom{\alpha}{n}$ as:

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}$$

For the case $n = 0$, we define $\binom{\alpha}{0} := 1$.

Proposition 228. Let $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{N}$. Then:

$$\binom{\alpha}{n} = \binom{\alpha - 1}{n - 1} + \binom{\alpha - 1}{n}$$

Theorem 229 (Binomial theorem). Let $\alpha \in \mathbb{C}^*$. Then, $\forall z \in \mathbb{C}$, with $|z| < 1$ we have:

$$(1 + z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

Theorem 230 (Hurwitz's theorem). Let $U \subseteq \mathbb{C}$ be a domain, $(f_n) \in \mathcal{H}(U)$ be a sequence of functions such that they converge uniformly to $f : U \to \mathbb{C}$ over compact sets of U. Suppose $f(z) \neq 0$ $\forall z \in \partial D(a, r)$, with $D(a, r) \subseteq U$. Then, $\exists m \in \mathbb{N}$ such that $\forall n \geq m$, f_n and f have the same number of zeros on $D(a, r)$.

Corollary 231. Let $U \subseteq \mathbb{C}$ be a domain, $(f_n) \in \mathcal{H}(U)$ be a sequence of functions such that they converge uniformly to $f : U \to \mathbb{C}$ over compact sets of U. Suppose $f_n(z) \neq 0$ $\forall n \in \mathbb{N}$ and $\forall z \in U$. Then, either $f = 0$ or $f(z) \neq 0$ $\forall z \in U$.

Corollary 232. Let $U \subseteq \mathbb{C}$ be a domain, $(f_n) \in \mathcal{H}(U)$ be a sequence of injective functions such that they converge uniformly to $f : U \to \mathbb{C}$ over compact sets of U. Then, either f is injective or $f = \text{const}$.

9. | Conformal representation

Introduction

Definition 233. Let $U \subseteq \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$. We say that f is *conformal* at z_0 if it preserves the oriented angle between curves that intersect at z_0. We say that f is *conformal* if it is conformal at each point $z \in U$.

Definition 234. Let $U, V \subseteq \mathbb{C}$ be open sets and $f : U \to V$ be a function. We say that f is a *conformal representation* between U and V if f is bijective, holomorphic and its inverse is also holomorphic.

Theorem 235. Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$. Then, f is conformal at the points $z \in U$ such that $f'(z) \neq 0$.

Möbius transformations

Definition 236 (Möbius transformations). A *Möbius transformation*[19] of the complex plane is a rational function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form:

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The special values $f(\infty)$ and $f\left(-\frac{d}{c}\right)$ are defined conveniently as $f(\infty) = \frac{a}{c}$ and $f\left(-\frac{d}{c}\right) = \infty$. The set of all Möbius transformations is denoted by $\mathcal{M}$.

Proposition 237. Let $f, g \in \mathcal{M}$ such that

$$f(z) = \frac{az + b}{cz + d} \quad \text{and} \quad g(z) = \frac{a'z + b'}{c'z + d'}$$

Then, $f(z) = g(z) \iff \exists \lambda \in \mathbb{C}^*$ such that $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$ and $d' = \lambda d$.

Proposition 238. Let $f \in \mathcal{M}$. Then, f is a homeomorphism of $\mathbb{C}_\infty$ and moreover if $\circ$ denotes the composition of Möbius transformations, $(\mathcal{M}, \circ)$ is a non-abelian group. Moreover we have the following group morphism:

$$\begin{aligned} \mathrm{GL}_2(\mathbb{C}) &\longrightarrow \mathcal{M} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{az + b}{cz + d} \end{aligned}$$

Definition 239. Let $f \in \mathcal{M}$ and $a, \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$. We say that f is

- a *translation* if $f(z) = z + a$.
- a *rotation* if $f(z) = \mathrm{e}^{\mathrm{i}\theta} z$.
- a *dilatation* if $a \neq 0$ and $f(z) = az$.
- an *inversion* if $f(z) = \frac{1}{z}$.

Theorem 240. Let $f \in \mathcal{M}$. Then, f is a composition of a finite number of translations and dilatations and an inversion.

Proposition 241. Let $f \in \mathcal{M}$. Then, f is a bijection between $\mathbb{C}_\infty$ and $\mathbb{C}_\infty$.

Proposition 242. Let U be a domain and $f \in \mathcal{M}$ be such that f is holomorphic on U. Then, $f|_U$ is conformal.

Proposition 243. Let $f = \frac{az+b}{cz+d} \in \mathcal{M}$. Then, $z \in \mathbb{C}_\infty$ is a fixed point for f if and only if:

$$c^2 z^2 + (d - a)z - b = 0$$

Hence, f can have at most two fixed points.

Corollary 244. Let $f, g \in \mathcal{M}$ such that they coincide in three points. Then, $f = g$.

[19] Also called linear fractional transformation, homography or homographic transformation.

Cross ratio

Definition 245. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points and $z \in \mathbb{C}_\infty$. We define the *cross ratio* of z, z_2, z_3 and z_4, denoted by (z, z_2, z_3, z_4) as the image under the unique Möbius transformation $f \in \mathcal{M}$ such that $f(z_2) = 1$, $f(z_3) = 0$ and $f(z_4) = \infty$.

Proposition 246. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points and $z \in \mathbb{C}_\infty$. Then:

$$(z, z_2, z_3, z_4) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$$

Corollary 247. Let $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and $w_1, w_2, w_3 \in \mathbb{C}_\infty$ be two triplets of distinct points. Then, $\exists! f \in \mathcal{M}$ such that:

$$f(z_1) = w_1 \quad f(z_2) = w_2 \quad f(z_3) = w_3$$

Theorem 248. Let $z_1, z_2, z_3 \in \mathbb{C}_\infty$ be distinct points and $f \in \mathcal{M}$. Then:

$$(z, z_2, z_3, z_4) = (f(z), f(z_2), f(z_3), f(z_4))$$

Circles in $\mathbb{C}_\infty$

Definition 249. We define a *circle* in $\mathbb{C}_\infty$ as a circle in $\mathbb{C}$ or the set $r \cup \{\infty\}$, where r is a line in $\mathbb{C}$[20].

Proposition 250. Given three points of $\mathbb{C}_\infty$, there exists a unique circle that passes through them.

Proposition 251. Let $f \in \mathcal{M}$. Then, $f(\mathbb{R}_\infty)$ is a circle of $\mathbb{C}_\infty$.

Proposition 252. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points. Then, these points lie on a circle if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$

Theorem 253. Let $f \in \mathcal{M}$ and $C \subset \mathbb{C}_\infty$ be a circle. Then, $f(C) \subset \mathbb{C}_\infty$ is also a circle.

Orientation and symmetry

Definition 254. Let $C \subset \mathbb{C}_\infty$ be a circle and $z_2, z_3, z_4 \in C$. An *orientation* of C is an ordered triplet (z_2, z_3, z_4)[21]. This determines a partition of the plane into three sets:

- *Right side* of C: $\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$
- *Left side* of C: $\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) < 0\}$
- *Center* of C: $\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) = 0\}$

Theorem 255 (Orientation principle). Let $C \subset \mathbb{C}_\infty$ be a circle, (z_1, z_2, z_3) be an orientation of C, and $f \in \mathcal{M}$. Then, $C' := f(C)$ is a circle and f carries the right/left side of C to the right/left side of C'.

Definition 256. Let $C \subset \mathbb{C}_\infty$ be a circle, $z_2, z_3, z_4 \in C$ and $z, z^* \in \mathbb{C}_\infty$. We say that z and z^* are symmetric with respect to C if:

$$(z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)}$$[22]

The function

$$\begin{aligned} R_C : \mathbb{C}_\infty &\longrightarrow \mathbb{C}_\infty \\ z &\longmapsto z^* \end{aligned}$$

is called *reflection* with respect to C.

Theorem 257 (Symmetry principle). Let $C \subset \mathbb{C}_\infty$ be a circle, $f \in \mathcal{M}$ and $z, z^* \in \mathbb{C}_\infty$. Then, if z, z^* are symmetric with respect to C, then $f(z)$ and $f(z^*)$ are symmetric with respect to $f(C)$.

Proposition 258. Let $C \subset \mathbb{C}$ be a circle of center a and radius R. Then:

$$R_C(z) = a + \frac{R^2}{|z - a|^2}(z - a)$$

Corollary 259. Let $C_1, C_2 \subset \mathbb{C}_\infty$ be two circles. Then, there exists $f \in \mathcal{M}$ such that $f(C_1) = C_2$. Moreover if we fix the image of three points of C_1, the transformation f is unique. Furthermore, $f(\operatorname{Int} C_1)$ maps to either $\operatorname{Int} C_2$ or $\operatorname{Ext} C_2$ and the same happens with $\operatorname{Ext} C_2$.

Automorphisms of the unit disk

Definition 260. We denote by $\mathbb{D}$ the unit disk $D(0, 1)$.

Lemma 261 (Schwarz lemma). Let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$ and $|f(z)| \leq 1$. Then:

1. $|f'(0)| \leq 1$
2. $|f(z)| \leq z \ \forall z \in \mathbb{D}$

Moreover if $\exists w \in \mathbb{D}$ such that the equality holds in either of the inequalities of above, then $f(z) = \mathrm{e}^{\mathrm{i}\theta} z$, for some $\theta \in \mathbb{R}$.

Definition 262. Let $a \in \mathbb{D}$. We define the Möbius transformation φ_a as:

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

Proposition 263. Let $a \in \mathbb{D}$. Then:

1. φ_a is a bijection from $\overline{\mathbb{D}}$ to itself.
2. $\varphi_a(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$
3. $\varphi_a(\partial \mathbb{D}) = \partial \mathbb{D}$

Lemma 264 (Schwarz-Pick lemma). Let $f \in \mathcal{H}(\mathbb{D})$ such that $f(0) = 0$ and $|f(z)| \leq 1$. Then $\forall a, z \in \mathbb{D}$ we have:

$$\left| \frac{f(z) - f(a)}{1 - \overline{f(a)} f(z)} \right| \leq \left| \frac{z - a}{1 - \bar{a}z} \right|$$

And therefore:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

Theorem 265. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic and bijective function and $a \in \mathbb{D}$ such that $f(a) = 0$. Then, $\exists \theta \in \mathbb{R}$ such that $f(z) = \mathrm{e}^{\mathrm{i}\theta} \varphi_a(z)$.

[20] Note that with this definition $\mathbb{R}_\infty$ is a circle.

[21] The orientation is determined by going from z_2 to z_3 without passing through z_4, by going form z_3 to z_4 without passing through z_2 and by going form z_4 to z_2 without passing through z_3, all these travels always on C.

[22] It can be seen that this definition does not depend on the triplet (z_2, z_3, z_4) chosen.

10. | Space of holomorphic functions

Definition 266. Let $U \subseteq \mathbb{C}$ be an open set. We denote by $\mathcal{C}(U, \mathbb{C})$ the set of all continuous functions $f : U \to \mathbb{C}$

Definition 267. Let $U \subseteq \mathbb{C}$ be an open set. A sequence $(K_n) \subseteq U$ of compact sets is *surjective* if $K_n \subseteq \operatorname{Int}(K_{n+1})$ $\forall n \in \mathbb{N}$ and $U = \bigcup_{n\in\mathbb{N}} K_n$.

Proposition 268. Let $U \subseteq \mathbb{C}$ be an open set. Then, there exists a sequence (K_n) of compact sets such that:

1. It is surjective.
2. For each compact set $K \subset U$, $\exists n \in \mathbb{N}$ such that $K \subset K_n$.
3. Each connected component of $\mathbb{C}_\infty \setminus K_n$ contains a connected component of $\mathbb{C}_\infty \setminus U$.

Definition 269. Let $U \subseteq \mathbb{C}$ be an open set, $K \subset U$ be a compact set and $f \in \mathcal{C}(U, \mathbb{C})$. We define:

$$\|f\|_K := \sup\{|f(z)| : z \in K\} = \max\{|f(z)| : z \in K\}$$

Theorem 270. Let (K_n) be a surjective sequence of compact sets and $d : \mathcal{C}(U, \mathbb{C}) \times \mathcal{C}(U, \mathbb{C}) \to \mathbb{R}$ be the function defined by:

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{\|f - g\|_{K_n}, 1\} \qquad \forall f, g \in \mathcal{C}(U, \mathbb{C})$$

Then:

1. d is a distance.
2. If $(f_n), f \in \mathcal{C}(U, \mathbb{C})$, then:
$$\lim_{n\to\infty} d(f_n, f) = 0 \iff (f_n) \text{ converges uniformly to } f \text{ over compact sets}$$
3. $(\mathcal{C}(U, \mathbb{C}), d)$ is complete.

Proposition 271. Consider the metric $(\mathcal{C}(U, \mathbb{C}), d)$ defined above. Then, $\mathcal{H}(U) \subset \mathcal{C}(U, \mathbb{C})$ is closed.

Definition 272. Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. We say that $\mathcal{F}$ is *relatively compact* (or *normal*) if for any sequence $(f_n) \in \mathcal{F}$, (f_n) has a subsequence uniformly convergent over compact sets of U. Equivalently $\mathcal{F}$ is relatively compact if $\overline{\mathcal{F}}$ is compact.

Definition 273. Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. We say that $\mathcal{F}$ is *equicontinuous* at $z_0 \in U$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall z \in X$ with $|z - z_0| < \delta$ we have:

$$|f(z) - f(z_0)| < \varepsilon \quad \forall f \in \mathcal{F}$$

Theorem 274 (Arzelà-Ascoli theorem). Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. Then, $\mathcal{F}$ is normal if and only if:

1. The set $\bigcup_{f\in\mathcal{F}}\{f(z)\}$ is relatively compact $\forall z \in U$.
2. $\mathcal{F}$ is equicontinuous at z $\forall z \in U$.

Definition 275. Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. We say that $\mathcal{F}$ is *locally bounded* if $\forall a \in U$ $\exists r > 0$ and $M > 0$ such that $|f(z)| \leq M$ $\forall z \in \overline{D(a, r)}$ $\forall f \in \mathcal{F}$.

Definition 276. Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. We say that $\mathcal{F}$ is *bounded by compact sets* if for each compact set $K \subset U$ $\exists M_K > 0$ such that $\forall f \in \mathcal{F}$ we have $\|f\|_K \leq M_K$.

Lemma 277. Let $\mathcal{F} \subseteq \mathcal{C}(U, \mathbb{C})$. $\mathcal{F}$ is locally bounded $\iff$ $\mathcal{F}$ is bounded by compact sets.

Theorem 278 (Montel's theorem). Let $\mathcal{F} \subseteq \mathcal{H}(U)$. Then, $\mathcal{F}$ is normal $\iff$ $\mathcal{F}$ is locally bounded.

Theorem 279 (Riemann conformal representation theorem). Let $U \subset \mathbb{C}$ be a simply connected domain such that $U \neq \varnothing, \mathbb{C}$ and $a \in U$. Then, $\exists! f \in \mathcal{H}(U)$ such that:

1. $f(a) = 0$, $f'(a) \in \mathbb{R}_{>0}$.
2. $f : U \to \mathbb{D}$ is a bijection.