Mathematical analysis

1. | Numeric series

Series convergence

Definition 1. Let (a_n) be a sequence of real numbers. A *numeric series* is an expression of the form

$$\sum_{n=1}^{\infty} a_n$$

We call a_n *general term of the series* and $S_N = \sum_{n=1}^{N} a_n$, for all $N \in \mathbb{N}$, *N-th partial sum of the series*[1].

Definition 2. We say the series $\sum a_n$ is *convergent* if the sequence of partial sums is convergent, that is, if $S = \lim_{N\to\infty} S_N$ exists and it is finite. In that case, S is called the *sum of the series*. If the previous limit doesn't exist or it is infinite, we say the series is *divergent*[2].

Proposition 3. Let (a_n) be a sequence such that $\sum a_n < \infty$. Then, $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that

$$\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n\right| < \varepsilon$$

if $N \geq n_0$.

Theorem 4 (Cauchy's test). Let (a_n) be a sequence. $\sum a_n < \infty$ if and only if $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ such that

$$\left|\sum_{n=N}^{M} a_n\right| < \varepsilon$$

if $M \geq N \geq n_0$.

Corollary 5. Changing a finite number of terms in a series has no effect on the convergence or divergence of the series.

Corollary 6. If $\sum a_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 7 (Linearity). Let $\sum a_n, \sum b_n$ be two convergent series with sums A and B respectively and let λ be a real number. The series

$$\sum_{n=1}^{\infty} (a_n + \lambda b_n)$$

is convergent and has sum $A + \lambda B$.

Theorem 8 (Associative property). Let $\sum a_n$ be a convergent series with sum A. Suppose (n_k) is a strictly increasing sequence of natural numbers. The series $\sum b_n$, with $b_k = a_{n_{k-1}+1} + \cdots + a_{n_k}$ for all $i \in \mathbb{N}$, is convergent and its sum is A.

Non-negative terms series

Theorem 9. Let $\sum a_n$ be a series of non-negative terms $a_n \geq 0$[3]. The series converges if and only if the sequence (S_N) of partial sums is bounded.

Theorem 10 (Comparison test). Let $(a_n), (b_n) \geq 0$ be two sequences of real numbers. Suppose that exists a constant $C > 0$ and a number $n_0 \in \mathbb{N}$ such that $a_n \leq C b_n$ for all $n \geq n_0$.

1. If $\sum b_n < \infty \implies \sum a_n < \infty$
2. If $\sum a_n = +\infty \implies \sum b_n = +\infty$

Theorem 11 (Limit comparison test). Let (a_n), $(b_n) \geq 0$ be two sequences of real numbers. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists.

1. If $0 < \ell < \infty \implies \sum a_n < \infty \iff \sum b_n < \infty$
2. If $\ell = 0$ and $\sum b_n < \infty \implies \sum a_n < \infty$
3. If $\ell = \infty$ and $\sum a_n < \infty \implies \sum b_n < \infty$

Theorem 12 (Root test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

1. If $\ell < 1 \implies \sum a_n < \infty$
2. If $\ell > 1 \implies \sum a_n = +\infty$

Theorem 13 (Ratio test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists.

1. If $\ell < 1 \implies \sum a_n < \infty$
2. If $\ell > 1 \implies \sum a_n = +\infty$

Theorem 14 (Raabe's test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right)$ exists.

1. If $\ell > 1 \implies \sum a_n < \infty$
2. If $\ell < 1 \implies \sum a_n = +\infty$

Theorem 15 (Condensation test). Let $(a_n) \geq 0$ be a decreasing sequence. Then:

$$\sum a_n < \infty \iff \sum 2^n a_{2^n} < \infty$$

Theorem 16 (Logarithmic test). Let $(a_n) \geq 0$. Suppose that the limit $\ell = \lim_{n\to\infty} \frac{\log \frac{1}{a_n}}{\log n}$ exists.

1. If $\ell > 1 \implies \sum a_n < \infty$

[1] From now on we will write $\sum a_n$ to refer $\sum_{n=1}^{\infty} a_n$.

[2] We will use the notation $\sum a_n < \infty$ or $\sum a_n = +\infty$ to express that the series converges or diverges, respectively.

[3] Obviously the following results are also valid if the series is of non-positive terms or has a finite number of negative or positive terms.

2. If $\ell < 1 \implies \sum a_n = +\infty$

Theorem 17 (Integral test). Let $f : [1, \infty) \to (0, \infty)$ be a decreasing function. Then:

$$\sum f(n) < \infty \iff$$
$$\iff \exists C > 0 \text{ such that } \int_1^n f(x)\,\mathrm{d}x \leq C \;\forall n$$

Alternating series

Definition 18. An *alternating series* is a series of the form $\sum(-1)^n a_n$, with $(a_n) \geq 0$.

Theorem 19 (Leibnitz's test). Let $(a_n) \geq 0$ be a decreasing sequence such that $\lim\limits_{n\to\infty} a_n = 0$. Then, $\sum(-1)^n a_n$ is convergent.

Theorem 20 (Abel's summation formula). Let $(a_n), (b_n)$ be two sequences of real numbers. Then:

$$\sum_{n=N}^{M} a_n(b_{n+1} - b_n) = a_{M+1}b_{M+1} - a_N b_N -$$
$$- \sum_{n=N}^{M} b_{n+1}(a_{n+1} - a_n)$$

Theorem 21 (Dirichlet's test). Let $(a_n), (b_n)$ be two sequences of real numbers such that:

1. $\exists C > 0$ such that $\left|\sum_{n=1}^{N} a_n\right| \leq C$ for all $N \in \mathbb{N}$.
2. (b_n) is monotone and $\lim\limits_{n\to\infty} b_n = 0$.

Then, $\sum a_n b_n$ is convergent.

Theorem 22 (Abel's test). Let $(a_n), (b_n)$ be two sequences of real numbers such that:

1. The series $\sum a_n$ is convergent.
2. (b_n) is monotone and bounded.

Then, $\sum a_n b_n$ is convergent.

Absolute convergence and rearrangement of series

Definition 23. We say a series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Theorem 24. If a series converges absolutely, it converges.

Definition 25. We say a sequence (b_n) is a *rearrangement of the sequence* (a_n) if exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$. A *rearrangement of the series* $\sum a_n$ is the series $\sum a_{\sigma(n)}$ for some bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

Definition 26. Let $x \in \mathbb{R}$. We define the *positive part* of x as

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Analogously, we define the *negative part* of x as

$$x^- = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note that we can write $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Theorem 27. A series $\sum a_n$ is absolutely convergent if and only if positive and negative terms series, $\sum {a_n}^+$ and $\sum {a_n}^-$, converge. In this case,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} {a_n}^+ - \sum_{n=1}^{\infty} {a_n}^-$$

Theorem 28. Let $\sum a_n$ be an absolutely convergent series. Then, for all bijection $\sigma : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum a_{\sigma(n)}$ is absolutely convergent and $\sum a_n = \sum a_{\sigma(n)}$.

Theorem 29 (Riemann's theorem). Let $\sum a_n$ be a convergent series but not absolutely convergent. Then, $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum a_{\sigma(n)}$ converges and $\sum a_{\sigma(n)} = \alpha$.

Theorem 30. A series $\sum a_n$ is absolutely convergent if and only if any rearranged series converges to the same value of $\sum a_n$.

2. | Sequences and series of functions

Sequences of functions

Definition 31. Let $D \subseteq \mathbb{R}$. A set

$$(f_n(x)) = \{f_1(x), f_2(x), \ldots, f_n(x), \ldots\}$$

is a *sequence of real functions* if $f_i : D \to \mathbb{R}$ is a real-valued function. In this case we say the sequence $(f_n(x))$, or simply (f_n), is well-defined on D.

Definition 32. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges pointwise* to f on D if $\forall x \in D$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$

Definition 33. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say (f_n) *converges uniformly* to f on D if $\forall \varepsilon > 0$, $\exists n_0 : |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq n_0$ and $\forall x \in D$.

Lemma 34. Let (f_n) be an uniform convergent sequence of functions defined on $D \subseteq \mathbb{R}$ and let f be a function such that (f_n) converges pointwise to f. Then, (f_n) converges uniformly f on D.

Lemma 35. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. (f_n) converges uniformly to f on D if and only if $\lim\limits_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0$.

Corollary 36. A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if there is a sequence (a_n), with $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, and a number $n_0 \in \mathbb{N}$ such that $\sup\{|f_n(x) - f(x)| : x \in D\} \leq a_n$, $\forall n \geq n_0$.

Theorem 37 (Cauchy's test). A sequence of functions (f_n) converges uniformly to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0 \ \exists n_0 : \sup\{|f_n(x) - f_m(x)| : x \in D\} < \varepsilon$ if $n, m \geq n_0$.

Theorem 38. Let (f_n) be a sequence of continuous functions defined on $D \subseteq \mathbb{R}$. If (f_n) converges uniformly to f on D, then f is continuous on D, that is, for any $x_0 \in D$, it satisfies:

$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = \lim_{x\to x_0} f(x)$$

Theorem 39. Let (f_n) be a sequence of functions defined on $I = [a, b] \subseteq \mathbb{R}$. If (f_n) are Riemann-integrable on I and (f_n) converges uniformly to f on I, then f is Riemann-integrable on I and

$$\int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x$$

Theorem 40. Let (f_n) be a sequence of functions defined on $I = (a, b) \subset \mathbb{R}$. If (f_n) are derivable on I, $(f_n'(x))$ converges uniformly on I and $\exists x_0 \in I$ such that $\lim\limits_{n\to\infty} f_n(x_0) \in \mathbb{R}$, then there is a function f such that (f_n) converges uniformly to f on I, f is derivable on I and $(f_n'(x))$ converges uniformly to f' on I.

Series of functions

Definition 41. Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$. The expression

$$\sum_{n=1}^{\infty} f_n(x)$$

is the *series of functions* associated with (f_n).

Definition 42. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges pointwise* on D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges pointwise. If the pointwise limit of (F_N) is $F(x)$, we say F is the *sum of the series in a pointwise sense.*

Definition 43. A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ *converges uniformly* on D if the sequence of partials sums

$$F_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges uniformly. If the uniform limit of (F_N) is $F(x)$, we say F is the *sum of the series in an uniform sense.*

Theorem 44 (Cauchy's test). A series of functions $\sum f_n(x)$ defined on $D \subseteq \mathbb{R}$ converges uniformly if and only if $\forall \varepsilon > 0 \ \exists n_0$ such that

$$\sup\left\{\left|\sum_{n=N}^{M} f_n(x)\right| : x \in D\right\} < \varepsilon$$

if $M \geq N \geq n_0$.

Corollary 45. If $\sum f_n(x)$ is an uniformly convergent series of functions on $D \subseteq \mathbb{R}$, then (f_n) converges uniformly to zero on D.

Theorem 46. If $\sum f_n(x)$ is an uniformly convergent series of continuous functions on $D \subseteq \mathbb{R}$, then its sum function is also continuous on D.

Theorem 47. Let (f_n) be a sequence of functions defined on $I = [a, b] \subseteq \mathbb{R}$. If (f_n) are Riemann-integrable on I and $\sum f_n(x)$ converges uniformly on I, then $\sum f_n(x)$ is Riemann-integrable on I and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_a^b f_n(x)\,\mathrm{d}x$$

Theorem 48. Let (f_n) be a sequence of functions defined on $I = (a, b) \subset \mathbb{R}$. If (f_n) are derivable on I, $\sum f_n'(x)$ converges uniformly on I and $\exists c \in I : \sum f_n(c) < \infty$, then $\sum f_n(x)$ converges uniformly on I, $\sum f_n(x)$ is derivable on I and

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x)$$

Theorem 49 (Weierstraß M-test). Let (f_n) be a sequence of functions defined on $D \subseteq \mathbb{R}$ such that $|f_n(x)| \leq M_n \ \forall x \in D$ and suppose that $\sum M_n$ is a convergent numeric series. Then, $\sum f_n(x)$ converges uniformly on D.

Theorem 50 (Dirichlet's test). Let $(f_n), (g_n)$ be two sequences of functions defined on $D \subseteq \mathbb{R}$. Suppose:

1. $\exists C > 0 : \sup\left\{\left|\sum_{n=1}^{N} f_n(x)\right| : x \in D\right\} \leq C, \ \forall N.$
2. $(g_n(x))$ is a monotone sequence for all $x \in D$ and $\lim\limits_{n\to\infty}\sup\{|g_n(x)| : x \in D\} = 0$.

Then, $\sum f_n(x) g_n(x)$ converges uniformly on D.

Theorem 51 (Abel's test). Let $(f_n), (g_n)$ be two sequences of functions defined on $D \subseteq \mathbb{R}$. Suppose:

1. The series $\sum f_n(x)$ converges uniformly on D.
2. $(g_n(x))$ is a monotone and bounded sequence for all $x \in D$.

Then, $\sum f_n(x) g_n(x)$ converges uniformly on D.

Power series

Definition 52. Let (a_n) be a sequence of real numbers and $x_0 \in \mathbb{R}$. A *power series* centred on x_0 is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Proposition 53. Let $\sum a_n (x - x_0)^n$ be a power series. Suppose there exists an $x_1 \in \mathbb{R}$ such that $\sum a_n (x_1 - x_0)^n < \infty$. Then, $\sum a_n (x - x_0)^n$ converges uniformly on any closed interval $I \subset A = \{x \in \mathbb{R} : |x - x_0| < |x_1 - x_0|\}$.

Theorem 54. Let $\sum a_n(x-x_0)^n$ be a power series and consider

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} \in [0,\infty]$$

Then:

1. If $|x-x_0| < R \implies \sum a_n(x-x_0)^n$ converges absolutely.
2. If $0 \le r < R \implies \sum a_n(x-x_0)^n$ converges uniformly on $[x_0-r, x_0+r]$.
3. If $|x-x_0| > R \implies \sum a_n(x-x_0)^n$ diverges.

The number R is called *radius of convergence of the power series.*

Theorem 55 (Abel's theorem). Let $\sum a_n x^n$ be a power series[4] with radius of convergence R satisfying $\sum a_n R^n < \infty$. Then, the series $\sum a_n x^n$ converges uniformly on $[0,R]$. In particular, if $f(x) = \sum a_n x^n$,

$$\lim_{x\to R^-} f(x) = \sum_{n=0}^{\infty} a_n R^n$$

Corollary 56. Let f be the sum function of a power series $\sum a_n x^n$. Then, f is continuous on the domain of convergence of the series.

Corollary 57. If the series $\sum a_n x^n$ has radius of convergence $R \ne 0$ and f is its sum function, then f is Riemann-integrable on any closed subinterval on the domain of convergence of the series. In particular, for $|x| < R$,

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$ [5]

Corollary 58. Let f be the sum function of the power series $\sum a_n x^n$. Then, f is derivable within the domain of convergence of the series and

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Corollary 59. Any function f defined as a sum of a power series $\sum a_n x^n$ is indefinitely derivable within the domain of convergence of the series and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k}$$

for all $k \in \mathbb{N}\cup\{0\}$. In particular $f^{(k)}(0) = k!a_k$.

Definition 60. A function is *analytic* if it can be expressed locally as a power series.

Stone-Weierstraß approximation theorem

Definition 61. Let f be a real-valued function. We say f has *compact support*[6] if exists $M > 0$ such that $f(x) = 0$ for all $x \in \mathbb{R}\setminus[-M,M]$.

Definition 62. Let f, g be real-valued functions with compact support. We define the *convolution* of f with g as

$$(f*g)(x) = \int_{\mathbb{R}} f(t)g(x-t)\,\mathrm{d}t = \int_{\mathbb{R}} f(x-t)g(t)\,\mathrm{d}t$$

Remark. The idea behind the convolution is to "blend" one function with the other one. In Fourier Analysis, g represents an input signal and f a *kernel function* for our purpose. This results in a new function that averages both functions.

Definition 63. We say that a sequence of functions (ϕ_ε) with compact support is an *approximation of identity* if

1. $\phi_\varepsilon \ge 0$.
2. $\displaystyle\int_{\mathbb{R}} \phi_\varepsilon = 1$.
3. For all $\delta > 0$, $\phi_\varepsilon(t)$ converges uniformly to zero when $\varepsilon \to 0$ if $|t| > \delta$.

Lemma 64. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Let (ϕ_ε) be an approximation of identity. Then, $(f*\phi_\varepsilon)$ converges uniformly to f on $\mathbb{R}$ as $\varepsilon \to 0$.

Theorem 65 (Weierstraß approximation theorem). Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, there exists polynomials $p_n \in \mathbb{R}[x]$ such that the sequence (p_n) converge uniformly to f on $[a,b]$.

3. | Improper integrals

Locally integrable functions

Definition 66. Let $f : [a,b) \to \mathbb{R}$, with $b \in \mathbb{R}\cup\{\infty\}$. We say f is *locally integrable* on $[a,b)$ if f is Riemann-integrable on $[a,x]$ for all $a \le x < b$.

Definition 67. Let $f : [a,b) \to \mathbb{R}$ be a locally integrable function. If there exists

$$\lim_{x\to b^-} \int_a^x f$$

and it's finite, we say that the *improper integral* of f on $[a,b)$, $\displaystyle\int_a^b f$, is *convergent.*

[4] From now on we will suppose, for simplicity, $x_0 = 0$.
[5] The formula is also valid for $|x| = R$ if the series $\sum a_n R^n$ (or $\sum a_n(-R)^n$) is convergent.
[6] In general, the *support* of a function is the closure of the set of points which are not mapped to zero.

Theorem 68 (Cauchy's test). Let $f : [a, b) \to \mathbb{R}$ be a locally integrable function. The improper integral $\int_a^b f$ is convergent if and only if $\forall \varepsilon > 0\ \exists b_0,\ a < b_0 < b$, such that

$$\left| \int_{x_1}^{x_2} f \right| < \varepsilon$$

if $b_0 < x_1 < x_2 < b$.

Improper integrals of non-negative functions

Theorem 69. Let $f : [a, b) \to \mathbb{R}$ be a locally integrable non-negative function. A necessary and sufficient condition for $\int_a^b f$ to be convergent is that the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

must be bounded for all $x < b$.

Theorem 70 (Comparison test). Let $f, g : [a, b) \to [0, +\infty)$ be two locally integrable non-negative functions. Then:

1. If $\exists C > 0$ such that $f(x) \leq Cg(x)\ \forall x$ on a neighbourhood of b and $\int_a^b g < \infty \implies \int_a^b f < \infty$.
2. Suppose the limit $\ell = \lim_{x \to b} \frac{f(x)}{g(x)}$ exists.
   i) If $\ell \in (0, \infty) \implies \int_a^b f < \infty \iff \int_a^b g < \infty$.
   ii) If $\ell = 0$ and $\int_a^b g < \infty \implies \int_a^b f < \infty$.
   iii) If $\ell = \infty$ and $\int_a^b f < \infty \implies \int_a^b g < \infty$.

Theorem 71 (Integral test). Let $f : [1, \infty) \to (0, \infty)$ be a locally integrable decreasing function. Then:

$$\sum f(n) < \infty \iff \int_1^\infty f(x)\,\mathrm{d}x < \infty$$[7]

Absolute convergence of improper integrals

Definition 72. Let $f : [a, b) \to (0, \infty)$ be a locally integrable function. We say $\int_a^b f$ *converges absolutely* if $\int_a^b |f|$ is convergent.

[7]This is another way of formulating Theorem 17.

Theorem 73 (Dirichlet's test). Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions Suppose:

1. $\exists C > 0$ such that $\left|\int_a^x f(t)\,\mathrm{d}t\right| \leq C$ for all $x \in [a, b)$.
2. g is monotone and $\lim_{x \to b} g(x) = 0$.

Then, $\int_a^b fg$ is convergent.

Theorem 74 (Abel's test). Let $f, g : [a, b) \to \mathbb{R}$ be two locally integrable functions. Suppose:

1. $\int_a^b f$ is convergent.
2. g is monotone and bounded.

Then, $\int_a^b fg$ is convergent.

Differentiation under integral sign

Theorem 75. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Consider the function $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$ defined on $[c, d]$. Then, F is continuous, that is, if $c < y_0 < d$,

$$\lim_{y \to y_0} F(y) = \lim_{y \to y_0} \int_a^b f(x, y)\,\mathrm{d}x = \int_a^b \lim_{y \to y_0} f(x, y)\,\mathrm{d}x = \\ = \int_a^b f(x, y_0)\,\mathrm{d}x = F(y_0)$$

Theorem 76. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a Riemann-integrable function and let $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$. If f is differentiable with respect to y and $\partial f / \partial y$ is continuous on $[a, b] \times [c, d]$, then $F(y)$ is derivable on (c, d) and its derivative is

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\,\mathrm{d}x$$

for all $y \in (c, d)$.

Theorem 77. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b] \times [c, d]$. Let $a, b : [c, d] \to \mathbb{R}$ be to differentiable functions satisfying $a \leq a(y) \leq b(y) \leq b$ for every $y \in [c, d]$. Suppose that $\partial f / \partial y$ is continuous on $\{(x, y) \in \mathbb{R}^2 : a(y) \leq x \leq b(y),\ c \leq y \leq d\}$. Then, $F(y) = \int_{a(y)}^{b(y)} f(x, y)\,\mathrm{d}x$ is derivable on (c, d) and its derivative is

$$F'(y) = b'(y)f(b(y), y) - a'(y)f(a(y), y) + \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y}(x, y)\,\mathrm{d}x$$

for all $y \in (c, d)$.

Theorem 78. Let $f : [a, b) \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b) \times [c, d]$. We consider $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$. Suppose that:

1. $\frac{\partial f}{\partial y}$ is continuous on $[a, b) \times [c, d]$.
2. Given $y_0 \in [c, d]$, $\exists \delta > 0$ such that the integral

   $$\int_a^b \sup\left\{\left|\frac{\partial f}{\partial y}(x, y)\right| : y \in (y_0 - \delta, y_0 + \delta)\right\}\mathrm{d}x$$

   exists and it's finite on $[a, b)$.

Then, $F(y)$ is derivable at y_0 and

$$F'(y_0) = \int_a^b \frac{\partial f}{\partial y}(x, y_0)\,\mathrm{d}x$$

Theorem 79. Let $f : [a, b) \times [c, d] \to \mathbb{R}$ be a continuous function on $[a, b) \times [c, d]$. Let $a, b : [c, d] \to \mathbb{R}$ be two differentiable functions satisfying $a \leq a(y) \leq b(y) \leq b$ for every $y \in [c, d]$. We consider $F(y) = \int_{a(y)}^{b(y)} f(x, y)\,\mathrm{d}x$. Suppose that:

1. $\frac{\partial f}{\partial y}$ is continuous on $\{(x, y) \in \mathbb{R}^2 : a(y) \leq x \leq b(y),\ c \leq y \leq d\}$.
2. Given $y_0 \in [c, d]$, $\exists \delta > 0$ such that the integral

   $$\int_{a(y)}^{b(y)} \sup\left\{\left|\frac{\partial f}{\partial y}(x, y)\right| : y \in (y_0 - \delta, y_0 + \delta)\right\}\mathrm{d}x$$

   exists and it's finite on $[a, b)$.

Then, $F(y)$ is derivable at y_0 and

$$F'(y_0) = b'(y_0)f(b(y_0), y_0) - a'(y_0)f(a(y_0), y_0) + \int_{a(y_0)}^{b(y_0)} \frac{\partial f}{\partial y}(x, y_0)\,\mathrm{d}x$$

Gamma function

Definition 80. For $x > 0$, *Gamma function* is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

Theorem 81. Gamma function is a generalization of the factorial. In fact, for $x > 0$ we have

$$\Gamma(x + 1) = x\Gamma(x)$$

In particular, $\Gamma(n + 1) = n!$ for all $n \in \mathbb{N}$.

Theorem 82. Gamma function satisfies:

$$\lim_{x\to\infty} \frac{\Gamma(x + 1)}{(x/e)^x\sqrt{2\pi x}} = 1$$

Corollary 83 (Stirling's formula).

$$\lim_{n\to\infty} \frac{n!}{n^n\mathrm{e}^{-n}\sqrt{2\pi n}} = 1$$

4. | Fourier series

Periodic functions

Definition 84. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We say that f is *T-periodic*, or is *periodic with period T*, being $T > 0$, if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

Remark. In general we take T to be the least positive constant satisfying that property.

Lemma 85. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function. Then, $f(x+T') = f(x)$ for all $x \in \mathbb{R}$ if and only if $T' = kT$ for some $k \in \mathbb{Z}$.

Sketch of the proof.

$\Longrightarrow$)

$$f(x + kT) = f(x + (k - 1)T) = \cdots = f(x)$$

$\Longleftarrow$) Assume $T' = kT + \alpha$, $\alpha \in [0, T)$. Then:

$$f(x) = f(x + T') = f(x + \alpha) \quad \forall x \in \mathbb{R}$$

which implies $\alpha = 0$ because otherwise f would be α-periodic with $\alpha < T$.

□

Proposition 86. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function. Then:

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$$

where $a \in \mathbb{R}$. In particular,

$$\int_a^{a+kT} f(x)\,\mathrm{d}x = k\int_0^T f(x)\,\mathrm{d}x$$

Lemma 87. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic continuous function. Then, $|f|$ is bounded.

Sketch of the proof. Use **?? ??** on the interval $[0, T]$ and the periodicity of f. □

Proposition 88. Given a T-periodic function f, there are no power series uniformly convergent to f on $\mathbb{R}$.

Proof. Suppose $\sum a_n x^n$ converges uniformly to f. By Theorem 46, f is continuous and by Theorem 87, $|f|$ is bounded. Therefore

$$\sup_{x\in\mathbb{R}}\left|\sum_{n=1}^{N} a_n x^n - f(x)\right|$$

cannot be arbitrarily small as $N\to\infty$ because $\sum_{n=1}^{N} a_n x^n$ is a polynomial, and therefore, unbounded. □

Orthogonal systems

Definition 89. Let $f:\mathbb{R}\to\mathbb{C}$ be a function. We say that $f\in L^p(I)$, $p\geq 1$, if:

$$\|f\|_p := \left(\int_I |f(t)|^p \,\mathrm{d}t\right)^{1/p} < \infty$$

Definition 90. Let $f,g:[a,b]\to\mathbb{C}$ be Riemann-integrable functions. We define the *inner product* of f and g as

$$\langle f,g\rangle := \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x$$

where $\overline{g}$ denotes the complex conjugate of g. The norm associated with this inner product is the L^2 norm:

$$\|f\|_2 = \langle f,f\rangle^{1/2} = \left(\int_a^b |f(x)|^2\,\mathrm{d}x\right)^{1/2} = \|f\|_2$$

The *distance* between f and g is:

$$d(f,g) := \|f-g\|_2$$

Proposition 91. Let $f,g:[a,b]\to\mathbb{C}$ be Riemann-integrable functions and let $\alpha\in\mathbb{C}$. Then, we have:

1. $\langle f,f\rangle\geq 0$.
2. $\langle f+h,g\rangle = \langle f,g\rangle+\langle h,g\rangle$ and $\langle f,g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$.
3. $\langle f,g\rangle = \overline{\langle g,f\rangle}$.
4. $\langle \alpha f,g\rangle = \alpha\langle f,g\rangle$ and $\langle f,\alpha g\rangle = \overline{\alpha}\langle f,g\rangle$.

Sketch of the proof. They follow from the linearity of the integral. For Item 91-3, write $f=\operatorname{Re} f+\mathrm{i}\operatorname{Im} f$ and $g=\operatorname{Re} g+\mathrm{i}\operatorname{Im} g$ and expand the products of both sides of the equation. □

Theorem 92 (Cauchy-Schwarz inequality). Let $f,g:[a,b]\to\mathbb{C}$ be Riemann-integrable functions. Then:

$$|\langle f,g\rangle|\leq \|f\|_2\cdot\|g\|_2$$

which can be written as:

$$\int_a^b f\overline{g}\leq\left(\int_a^b |f|^2\right)^{1/2}\left(\int_a^b |g|^2\right)^{1/2}$$

Proof. First suppose that $\|f\|_2=\|g\|_2=1$. Then:

$$|\langle f,g\rangle|\leq\int_a^b |fg|\leq\int_a^b\frac{|f|^2+|g|^2}{2}=1$$

because $|ab|\leq\frac{a^2+b^2}{2}$ $\forall a,b\in\mathbb{R}$.
For the general case, note that $\frac{f}{\|f\|_2}$ and $\frac{g}{\|g\|_2}$ have norm 1 and so:

$$\left|\left\langle\frac{f}{\|f\|_2},\frac{g}{\|g\|_2}\right\rangle\right|\leq 1\implies|\langle f,g\rangle|\leq\|f\|_2\|g\|_2$$

□

Theorem 93 (Minkowski inequality). Let $f,g:[a,b]\to\mathbb{C}$ be Riemann-integrable functions. Then:

$$\|f+g\|_2\leq\|f\|_2+\|g\|_2$$

Proof. Using 92 Cauchy-Schwarz inequality we have:

$$\begin{aligned}\|f+g\|_2{}^2 &= \|f\|_2{}^2+\|g\|_2{}^2+2\langle f,g\rangle\\ &\leq\|f\|_2{}^2+\|g\|_2{}^2+\|f\|_2\cdot\|g\|_2\\ &=(\|f\|_2+\|g\|_2)^2\end{aligned}$$

□

Definition 94. Let $f,g:[a,b]\to\mathbb{C}$ be Riemann-integrable functions with $f\neq g$. We say f and g are *orthogonal* if $\langle f,g\rangle=0$. We say f and g are *orthonormal* if they are orthogonal and $\|f\|_2=\|g\|_2=1$.

Definition 95. Let $S=\{\phi_0,\phi_1,\ldots\}$ be a collection of Riemann-integrable functions on $[a,b]$. We say S is an *orthonormal system* if $\|\phi_n\|_2=1$ $\forall n$ and $\langle\phi_n,\phi_m\rangle=0$ $\forall n\neq m$.

Proposition 96. Let $T>0$ and:

$$S_1=\left\{\frac{1}{\sqrt{T}}\mathrm{e}^{\frac{2\pi\mathrm{i}nx}{T}}:n\in\mathbb{Z}\right\}$$

$$S_2=\left\{\frac{1}{\sqrt{T}},\frac{\cos\left(\frac{2\pi nx}{T}\right)}{\sqrt{T/2}},\frac{\sin\left(\frac{2\pi mx}{T}\right)}{\sqrt{T/2}}:n,m\in\mathbb{N}\right\}$$

Then, S_1 and S_2 orthonormal systems on $[-T/2,T/2]$.

Definition 97. A collection of functions $S=\{\phi_0,\phi_1,\ldots,\phi_n\}$ is *linearly dependent* on $[a,b]$ if there exist $c_0,c_1,\ldots,c_n\in\mathbb{R}$ not all zero, such that

$$c_0\phi_0+c_1\phi_1+\cdots+c_n\phi_n=0,\quad\forall x\in[a,b]$$

Otherwise we say S is *linearly independent*. If the collection S has an infinity number of functions, we say S is linearly independent on $[a,b]$ if any finite subset of S is linearly independent on $[a,b]$.

Theorem 98. Let $S=\{\phi_0,\phi_1,\ldots\}$ be an orthonormal system on $[a,b]$. Suppose that $\sum c_n\phi_n(x)$ converges uniformly to a function f on $[a,b]$. Then, f is Riemann-integrable on $[a,b]$ and, moreover:

$$c_n=\langle f,\phi_n\rangle=\int_a^b f(x)\overline{\phi_n(x)}\,\mathrm{d}x\,,\quad\forall n\geq 0$$

Proof. Using Theorem 47 we have that f is Riemann-integrable and that $\forall m \in \mathbb{N}$:

$$\langle f, \phi_m \rangle = \sum_{n=0}^{\infty} c_n \langle \phi_n, \phi_m \rangle = c_n$$

by the orthonormality of S. $\square$

Fourier coefficients and Fourier series

Definition 99. Let $S = \left\{ \frac{1}{\sqrt{T}} \mathrm{e}^{\frac{2\pi i n x}{T}}, n \in \mathbb{Z} \right\}$ be an orthonormal system on $[-T/2, T/2]$ and let $f \in L^1([-T/2, T/2])$[8] be a T-periodic function[9]. We define the *n-th Fourier coefficient* of f as

$$\widehat{f}(n) = \frac{1}{T} \left\langle f, \mathrm{e}^{\frac{2\pi i n x}{T}} \right\rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}x$$

for all $n \in \mathbb{Z}$.

Proposition 100. Let $f, g \in L^1([-T/2, T/2])$. The following properties are satisfied:

1. For all $\lambda, \mu \in \mathbb{C}$:
$$(\widehat{\lambda f + \mu g})(n) = \lambda \widehat{f}(n) + \mu \widehat{g}(n)$$
2. Let $\tau \in \mathbb{R}$. We define $f_\tau(x) = f(x - \tau)$. Then:
$$\widehat{f_\tau}(n) = \mathrm{e}^{-\frac{2\pi i n \tau}{T}} \widehat{f}(n)$$
3. If f is even, then $\widehat{f}(n) = \widehat{f}(-n)$, $\forall n \in \mathbb{Z}$.
If f is odd, then $\widehat{f}(n) = -\widehat{f}(-n)$, $\forall n \in \mathbb{Z}$.
4. If $f \in \mathcal{C}^k$ such that $f^{(r)}(-T/2) = f^{(r)}(T/2)$ $\forall r = 0, \ldots, k-1$, then
$$\widehat{f^{(k)}}(n) = \left(\frac{2\pi i n}{T} \right)^k \widehat{f}(n)$$
5. $(\widehat{f * g})(n) = \widehat{f}(n) \widehat{g}(n)$.

Proof.

1. It follows from the linearity of the integral.
2. $$\begin{aligned}\widehat{f_\tau}(n) &= \frac{1}{T} \int_{-T/2}^{T/2} f(x - \tau) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}x \\ &= \frac{1}{T} \int_{-T/2-\tau}^{T/2-\tau} f(u) \mathrm{e}^{-\frac{2\pi i n (u+\tau)}{T}} \, \mathrm{d}x \\ &= \mathrm{e}^{-\frac{2\pi i n \tau}{T}} \widehat{f}(n)\end{aligned}$$
where we have done the change of variable $u = x - \tau$ and we have used Theorem 86.
3. Make the change of variable $u = -x$.
4. We will use induction. The case $k = 0$ is clear. For the other ones:
$$\begin{aligned}\widehat{f^{(k)}}(n) &= \frac{1}{T} \int_{-T/2}^{T/2} f^{(k)}(x) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}x \\ &= \frac{2\pi i n}{T} \int_{-T/2}^{T/2} f^{(k-1)}(x) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}x \\ &= \left(\frac{2\pi i n}{T} \right) \widehat{f^{(k-1)}}(n) \\ &= \left(\frac{2\pi i n}{T} \right)^k \widehat{f}(n)\end{aligned}$$
where we have used integration by parts.
5. Using ?? ?? we have that:
$$\begin{aligned}(\widehat{f * g})(n) &= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} f(t) g(x - t) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_{-T/2}^{T/2} f(t) \left(\int_{-T/2}^{T/2} g(x - t) \mathrm{e}^{-\frac{2\pi i n x}{T}} \, \mathrm{d}x \right) \mathrm{d}t \\ &= \int_{-T/2}^{T/2} f(t) \mathrm{e}^{-\frac{2\pi i n t}{T}} \widehat{g}(n) \, \mathrm{d}t \\ &= \widehat{f}(n) \widehat{g}(n)\end{aligned}$$
where we have used Item 100-2.

$\square$

Definition 101. Let $f \in L^1([-T/2, T/2])$. We define the *Fourier series* of f as:

$$Sf(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \mathrm{e}^{\frac{2\pi i n x}{T}}$$

Definition 102. Let $f \in L^1([-T/2, T/2])$ and Sf be the Fourier series of f. We define *N-th partial sum of Sf* as:

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) \mathrm{e}^{\frac{2\pi i n x}{T}}$$

Proposition 103. Let $f \in L^1([-T/2, T/2])$. Then:

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T} \right) + b_n \sin\left(\frac{2\pi n x}{T} \right)$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2\pi n x}{T} \right) \mathrm{d}x \,,$$

[8]Saying that $f \in L^1([-T/2, T/2])$ is equivalent to say that f is integrable on $[-T/2, T/2]$.
[9]From now on, we will work only with functions defined on $[-T/2, T/2]$ and extended periodically on $\mathbb{R}$.

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi n x}{T}\right) \mathrm{d}x\,,$$

for $n \geq 0$[10]. In particular, if f is even we have:

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right)$$

and if f is odd we have:

$$Sf(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

Sketch of the proof. Remember that:

$$\mathrm{e}^{\frac{2\pi \mathrm{i} n x}{T}} = \cos\left(\frac{2\pi n x}{T}\right) + \mathrm{i}\sin\left(\frac{2\pi n x}{T}\right)$$

□

Definition 104. Let $f : (0, L) \to \mathbb{C}$ be a function. We define the *even extension* of f as

$$f_{\mathrm{e}}(x) = \begin{cases} f(x) & \text{if } x \in (0, L) \\ f(-x) & \text{if } x \in (-L, 0) \end{cases}$$

Analogously, we define the *odd extension* of f as

$$f_{\mathrm{o}}(x) = \begin{cases} f(x) & \text{if } x \in (0, L) \\ -f(-x) & \text{if } x \in (-L, 0) \end{cases}$$

Proposition 105. Let $f \in L^1([0, T/2])$. If we make the even extension of f[11], then

$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right)$$

where $a_n = \frac{4}{T}\int_{0}^{T/2} f(x)\cos\left(\frac{2\pi n x}{T}\right) \mathrm{d}x$ for $n \geq 0$. If we make the odd extension of f, then

$$Sf(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

where $b_n = \frac{4}{T}\int_{0}^{T/2} f(x)\sin\left(\frac{2\pi n x}{T}\right) \mathrm{d}x$ for $n \geq 1$.

Pointwise convergence

Definition 106 (Dirichlet kernel). We define the *Dirichlet kernel* of order $N \in \mathbb{N}$ as:

$$D_N(t) = \sum_{n=-N}^{N} \mathrm{e}^{\frac{2\pi \mathrm{i} n t}{T}}$$

Lemma 107. Let $N \in \mathbb{N}$. Then, $\forall t \in (0, T)$ we have:

$$D_N(t) = \frac{\sin\left(\frac{(2N+1)\pi t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)}$$

Proof. Using the geometric sum formula we have:

$$\begin{aligned} D_N(t) &= \frac{\mathrm{e}^{-\frac{2\pi \mathrm{i} N t}{T}} - \mathrm{e}^{\frac{2\pi \mathrm{i}(N+1)t}{T}}}{1 - \mathrm{e}^{\frac{2\pi \mathrm{i} t}{T}}} \\ &= \frac{\mathrm{e}^{-\frac{\pi \mathrm{i}(2N+1)t}{T}} - \mathrm{e}^{\frac{\pi \mathrm{i}(2N+1)t}{T}}}{\mathrm{e}^{-\frac{\pi \mathrm{i} t}{T}} - \mathrm{e}^{\frac{\pi \mathrm{i} t}{T}}} \\ &= \frac{\sin\left(\frac{(2N+1)\pi t}{T}\right)}{\sin\left(\frac{\pi t}{T}\right)} \end{aligned}$$

□

Proposition 108. The Dirichlet kernel has the following properties:

1. D_N is a T-periodic and even function.
2. $\frac{1}{T}\int_{-T/2}^{T/2} D_N(t)\,\mathrm{d}t = 1$, $\forall N \in \mathbb{N}$.

Sketch of the proof.

1. Use the characterization of Theorem 107.
2. Note that if $n \neq 0$, $\int_{-T/2}^{T/2} \mathrm{e}^{\frac{2\pi \mathrm{i} n t}{T}}\,\mathrm{d}t = 0$

□

Proposition 109. Let $f \in L^1([-T/2, T/2])$. Then:

$$\begin{aligned} S_N f(x) &= \frac{1}{T}(f * D_N)(x) \\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(x-t) D_N(t)\,\mathrm{d}t \\ &= \frac{1}{T}\int_{0}^{T/2} [f(x+t) + f(x-t)] D_N(t)\,\mathrm{d}t \end{aligned}$$

Sketch of the proof. The first equality follows from expandind the Fourier coefficients inside $S_N f$. The second one, making the change of variables $u = x - t$ and noting that both integrant functions are T-periodic. For the last one, make the change of variables $u = -t$ and use that $D_N(t)$ is even. □

[10]The relation between a_n, b_n and $\widehat{f}(n)$ is given by:

$$a_n = \widehat{f}(n) + \widehat{f}(-n) \quad \text{and} \quad b_n = \mathrm{i}\left[\widehat{f}(n) - \widehat{f}(-n)\right], \quad \forall n \in \mathbb{N} \cup \{0\}$$

[11]For simplicity, when we have a function f and make its even or odd extension, we will still call its even or odd extension f instead of $\tilde{f}$ or $\hat{f}$.

Lemma 110 **(Riemann-Lebesgue lemma).** Let $f \in L^1([-T/2, T/2])$ and $\lambda \in \mathbb{R}$. Then:

$$\lim_{\lambda\to\infty} \int_{-T/2}^{T/2} f(t)\sin(\lambda t)\,\mathrm{d}t = \lim_{\lambda\to\infty} \int_{-T/2}^{T/2} f(t)\cos(\lambda t)\,\mathrm{d}t = 0$$

In particular, $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

Proof. We first proof the statement for indicators functions $f(x) = \mathbf{1}_{[a,b]}(x)$, $[a,b] \subseteq [-T/2, T/2]$. We have that:

$$\lim_{\lambda\to\infty} \int_{-T/2}^{T/2} \mathbf{1}_{[a,b]}(t)\sin(\lambda t)\,\mathrm{d}t = \lim_{\lambda\to\infty} \frac{\cos(\lambda a) - \cos(\lambda b)}{\lambda} = 0$$

From the linearity of the integral the statement remains true for f being linear combination of indicator functions. Finally note that taking an upper sum g_ε of f (see **??**) such that $\int_{-T/2}^{T/2} |f - g_\varepsilon| < \varepsilon$, we have:

$$\begin{aligned}\left|\int_{-T/2}^{T/2} f(t)\sin(\lambda t)\,\mathrm{d}t\right| &\le \int_{-T/2}^{T/2} |f(t) - g_\varepsilon(t)|\,\mathrm{d}t + \\ &+ \left|\int_{-T/2}^{T/2} g_\varepsilon(t)\sin(\lambda t)\,\mathrm{d}t\right| \xrightarrow[\varepsilon\to 0]{\lambda\to\infty} 0\end{aligned}$$

The same proof applies for the $\cos(\lambda t)$. □

Theorem 111 **(Dini's theorem).** Let $f \in L^1([-T/2, T/2])$, $x_0 \in (-T/2, T/2)$ and $\ell \in \mathbb{R}$ such that

$$\int_0^\delta \frac{|f(x_0+t) + f(x_0-t) - 2\ell|}{t}\,\mathrm{d}t < \infty$$

for some $\delta > 0$. Then, $\lim_{N\to\infty} S_N f(x_0) = \ell$.

Proof. Note that $-\ell = -2\ell \int_0^{T/2} D_N(t)\,\mathrm{d}t$. So:

$$\begin{aligned}S_N f(x_0) - \ell &= \frac{1}{T}\int_0^{T/2} [f(x_0+t) + f(x_0-t) - 2\ell] D_N(t)\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^{T/2} \frac{f(x_0+t) + f(x_0-t) - 2\ell}{t} \frac{t}{\sin\left(\frac{\pi t}{T}\right)} \cdot \\ &\qquad \cdot \sin\left(\frac{(2N+1)\pi t}{T}\right)\mathrm{d}t\end{aligned}$$

Since the first terms form an integrable function, we can use now the 110 Riemann-Lebesgue lemma. □

Corollary 112. Let $f \in L^1([-T/2, T/2])$ be a function left and right differentiable at x_0, that is, there exists the following limits

$$f'({x_0}^+) = \lim_{t\to 0^+} \frac{f(x_0+t) - f({x_0}^+)}{t}$$

$$f'({x_0}^-) = \lim_{t\to 0^-} \frac{f(x_0+t) - f({x_0}^-)}{t}$$

(supposing the existence of left- and right-sided limits). Then:

$$\lim_{N\to\infty} S_N f(x_0) = \frac{f({x_0}^+) + f({x_0}^-)}{2}$$

Sketch of the proof. Use 111 Dini's theorem with $\ell = \frac{f({x_0}^+) + f({x_0}^-)}{2}$. □

Theorem 113 **(Lipschitz's theorem).** Let $f \in L^1([-T/2, T/2])$ such that at a point $x_0 \in (-T/2, T/2)$ it satisfies

$$|f(x_0+t) - f(x_0)| \le k|t|$$

for some constant $k \in \mathbb{R}$ and for $|t| < \delta$. Then, $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Sketch of the proof. Note that

$$S_N f(x_0) - f(x_0) = \frac{1}{T}\int_{-T/2}^{T/2} [f(x_0+t) - f(x_0)] D_N(t)\,\mathrm{d}t$$

and proceed as in the proof of 111 Dini's theorem. □

Remark. Note that only the continuity is not sufficient to ensure the pointwise convergence of $S_N f$ towards f.

Uniform convergence

Definition 114. Let $\sum a_n$ be a series with partial sums S_k. The series $\sum a_n$ is called *Cesàro summable* with sum S if

$$\lim_{N\to\infty} \frac{S_1 + \cdots + S_N}{N} = S$$

Remark. Note that if $(a_n) \in \mathbb{R}$ has limit ℓ, by **??** **??** we have that (a_n) is Cesàro summable and $\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \ell$. The other inclusion is false though. For example by taking $a_n = (-1)^n$.

Definition 115 **(Fejér kernel).** We define the *Fejér kernel* of order N as

$$F_N(t) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(t)$$

being $D_k(t)$ the Dirichlet kernel of order k, $0 \le k \le N$.

Lemma 116. Let $N \in \mathbb{N}$. Then, $\forall t \in (0, T)$ we have:

$$F_N(t) = \frac{1}{N+1} \frac{\sin^2\left(\frac{(N+1)\pi t}{T}\right)}{\sin^2\left(\frac{\pi t}{T}\right)}$$

Proof. Multiplying the expression of Theorem 107 by $\sin\left(\frac{\pi t}{T}\right)$ and using the trigonometry identity $\sin(x)\sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2}$ we have that F_N is a telescopic sum that simplifies to:

$$F_N(t) = \frac{1 - \cos\left(\frac{2(N+1)\pi t}{T}\right)}{2(N+1)\sin^2\left(\frac{\pi t}{T}\right)} = \frac{1}{N+1}\frac{\sin^2\left(\frac{(N+1)\pi t}{T}\right)}{\sin^2\left(\frac{\pi t}{T}\right)}$$

□

Proposition 117. The Fejér kernel has the following properties:

1. F_N is a T-periodic, even and non-negative function.
2. $\displaystyle\frac{1}{T}\int_{-T/2}^{T/2} F_N(t)\,\mathrm{d}t = 1 \quad \forall N.$
3. $\forall\delta > 0$, $\displaystyle\lim_{N\to\infty} \sup\{|F_N(t)| : \delta \le |t| \le T/2\} = 0$

Proof. The first two properties are consequence of the definition of Fejér kernel and the reexpression of Theorem 116. For the last one, note that:

$$|F_N(t)| \le \frac{1}{N+1}\frac{1}{\sin^2\left(\frac{\pi\delta}{T}\right)} \overset{N\to\infty}{\longrightarrow} 0$$

□

Definition 118. Let $f \in L^1([-T/2, T/2])$. We define the *Fejér means* $\sigma_N f$, for all $N \in \mathbb{N}$, as:

$$\sigma_N f(x) = \frac{S_0 f(x) + \cdots + S_N f(x)}{N+1} \tag{1}$$

Proposition 119. Let $f \in L^1([-T/2, T/2])$. Then:

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{T}(f * F_N)(x)\\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(x-t)F_N(t)\,\mathrm{d}t\\ &= \frac{1}{T}\int_0^{T/2} [f(x+t) + f(x-t)]F_N(t)\,\mathrm{d}t\end{aligned}$$

Proof. Consequence of Theorem 109 and the linearity of the convolution. □

Theorem 120 (Fejér's theorem). Let $f \in L^1([-T/2, T/2])$ be a function having left- and right-sided limits at point x_0. Then:

$$\lim_{N\to\infty} \sigma_N f(x_0) = \frac{f({x_0}^+) + f({x_0}^-)}{2}$$

In particular, if f is continuous at x_0, $\lim_{N\to\infty} \sigma_N f(x_0) = f(x_0)$.

Sketch of the proof. Let $\delta > 0$ be small enough. Then:

$$\begin{aligned}&\left|\sigma_N f(x_0) - \frac{f({x_0}^+) + f({x_0}^-)}{2}\right| =\\ &= \left|\int_0^{T/2} [f(x+t) - f({x_0}^+) + f(x-t) - f({x_0}^-)]F_N(t)\,\mathrm{d}t\right|\\ &\le \int_0^{T/2} \left|f(x+t) - f({x_0}^+)\right|F_N(t)\,\mathrm{d}t +\\ &\quad + \int_0^{T/2} \left|f(x-t) - f({x_0}^-)\right|F_N(t)\,\mathrm{d}t\end{aligned}$$

In order to bound the to intervals, divide the interval $[0, T/2] = [0, \delta] \cup [\delta, T/2]$. The first part is bounded by the right- (or left-) sided limit at x_0, and the second one is due to the uniform convergence (see Theorem 117). □

Theorem 121 (Fejér's theorem). Let f be a continuous function on $[-T/2, T/2]$. Then, $\sigma_N f$ converges uniformly to f on $[-T/2, T/2]$.

Proof. Let $\delta > 0$ be small enough. Then:

$$\begin{aligned}|\sigma_N f(x) - f(x)| &= \left|\int_0^{T/2} [f(x-t) - f(x)]F_N(t)\,\mathrm{d}t\right|\\ &\le \int_0^{\delta} |f(x-t) - f(x)|F_N(t)\,\mathrm{d}t +\\ &\quad + \int_\delta^{T/2} |f(x-t) - f(x)|F_N(t)\,\mathrm{d}t\end{aligned}$$

To bound the first integral use the uniform continuity of f in $[0, \delta]$ and for the second one use Theorem 117. □

Corollary 122. Let f be a continuous function on $[-T/2, T/2]$. Then, there exists a sequence of trigonometric polynomials that converge uniformly to f on $[-T/2, T/2]$. In fact:

$$\sigma_N f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k)\mathrm{e}^{\frac{2\pi\mathrm{i}kx}{T}}$$

Sketch of the proof. Observe that the term $\widehat{f}(k)\mathrm{e}^{\frac{2\pi\mathrm{i}kx}{T}}$ appears $N + 1 - |k|$ times on the numerator of the fraction of Eq. (1). Hence

$$\sigma_N f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) \widehat{f}(k)\mathrm{e}^{\frac{2\pi\mathrm{i}kx}{T}}$$

which is a trigonometric polynomial. □

Corollary 123. Let f and g be continuous functions on $[-T/2, T/2]$ such that $Sf(x) = Sg(x)$. Then, $f = g$.

Proof. $h := f - g$ satisfies that $\widehat{h}(n) = 0\ \forall n \in \mathbb{Z}$. So $\sigma_N h = 0\ \forall N \in \mathbb{N}$. 121 Fejér's theorem implies $h = 0$. □

Convergence in norm

Definition 124. We say a sequence (f_N) *converge in norm* L^p to f if $\lim_{N\to\infty} \|f_N - f\|_p = 0$.

Theorem 125. Let $f \in L^2([-T/2, T/2])$. Then:

$$\lim_{N\to\infty} \|\sigma_N f - f\|_2 = 0$$

Sketch of the proof. Use ?? ?? and the scheme of the proof of 121 Fejér's theorem. □

Corollary 126. Let $f \in L^1([-T/2, T/2])$. Then:

$$\lim_{N\to\infty} \|\sigma_N f - f\|_1 = 0$$

Sketch of the proof. Note that $\|\sigma_N f - f\|_1 \leq \|\sigma_N f - f\|_2$ by the 92 Cauchy-Schwarz inequality. □

Corollary 127. Let $f, g \in L^1([-T/2, T/2])$ be functions such that $Sf(x) = Sg(x)$. Then, $\lim_{N\to\infty} \|g - f\|_1 = 0$.

Proof. $h := f - g$ satisfies that $\widehat{h}(n) = 0\ \forall n \in \mathbb{Z}$. So $\sigma_N h = 0\ \forall N \in \mathbb{N}$. Thus:

$$\lim_{N\to\infty} \|h\|_1 = \lim_{N\to\infty} \|\sigma_N h - h\|_1 = 0$$

□

Theorem 128. $S_N f$ is the trigonometric polynomial of degree N that best approximates f in norm L^2.

Proof. Let $P(x) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{T}}$ be a trigonometric polynomial. Expanding the norm $\|f - P\|^2$ we have:

$$\|f - P\|_2{}^2 = \|f\|_2{}^2 + \|P\|_2{}^2 - 2\operatorname{Re}\left(\int_{-T/2}^{T/2} f(x)\overline{P(x)}\,\mathrm{d}x\right)$$

One the one hand:

$$\begin{aligned}\|P\|_2{}^2 &= \int_{-T/2}^{T/2} \left(\sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{T}}\right)\left(\sum_{m=-N}^{N} \overline{c_m} \mathrm{e}^{-\frac{2\pi i m x}{T}}\right) \mathrm{d}x \\ &= T \sum_{n=-N}^{N} |c_n|^2\end{aligned}$$

by the orthogonality of the system. On the other hand:

$$\int_{-T/2}^{T/2} f(x)\overline{P(x)}\,\mathrm{d}x = T \sum_{n=-N}^{N} \overline{c_n}\widehat{f}(n)$$

Finally using that $|z - w|^2 - |z|^2 = |w|^2 - 2\operatorname{Re}(z\overline{w})$, $z, w \in \mathbb{C}$, we have:

$$\|f - P\|_2{}^2 = \|f\|_2{}^2 + T \sum_{n=-N}^{N} \left|c_n - \widehat{f}(n)\right|^2 - T \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 \tag{2}$$

which is minimum if $c_n = \widehat{f}(n)\ \forall n = -N, \ldots, N$. That is, $P = S_N f$. □

Corollary 129 (Bessel's inequality). Let $f \in L^2([-T/2, T/2])$. Then:

$$T \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 \leq \|f\|_2{}^2$$

$$\frac{T}{2}\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{N} |a_n|^2 + |b_n|^2\right) \leq \|f\|_2{}^2$$

for all $N \in \mathbb{N}$.

Sketch of the proof. If follows from Eq. (2) with $P = S_N f$. □

Corollary 130. Let $f \in L^2([-T/2, T/2])$. Then, $\lim_{N\to\infty} \|S_N f - f\|_2 = 0$.

Theorem 131 (Parseval's identity). Let $f, g \in L^2([-T/2, T/2])$. Then:

$$\langle f, g\rangle = T \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}$$

In particular, if $f = g$:

$$T \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 = \|f\|_2{}^2$$

$$\frac{T}{2}\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2\right) = \|f\|_2{}^2$$

Proof. Note that $\langle f, g\rangle = \lim_{N\to\infty} \langle f, S_N g\rangle$. Indeed:

$$|\langle f, g\rangle - \langle f, S_N g\rangle| = |\langle f, g - S_N g\rangle| \leq \|f\|_2 \|g - S_N g\|_2$$

where we have applied 92 Cauchy-Schwarz inequality. Now use Theorem 130 to conclude that the right side of the equation tends to 0 as $N \to \infty$. Thus:

$$\begin{aligned}\langle f, g\rangle &= \lim_{N\to\infty} \langle f, S_N g\rangle \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \overline{\widehat{g}(n)}\left\langle f, \mathrm{e}^{\frac{2\pi i n x}{T}}\right\rangle \\ &= T \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}\end{aligned}$$

□

Applications of Fourier series

Theorem 132 (Wirtinger's inequality). Let f be a function such that $f(0) = f(T)$, $f' \in L^2([0, T])$ and $\int_0^T f(t)\,\mathrm{d}t = 0$. Then:

$$\int_0^T |f(x)|^2\,\mathrm{d}x \leq \frac{T^2}{4\pi^2}\int_0^T |f'(x)|^2\,\mathrm{d}x$$

And the inequality holds if and only if

$$f(x) = A\cos\left(\frac{2\pi x}{T}\right) + B\sin\left(\frac{2\pi x}{T}\right)$$

Proof. The continuity of f and $f(0) = f(T)$, implies that $f \in L^2([0, T])$ and that $\widehat{f'}(n) = \frac{2\pi i n}{T}\widehat{f}(n)$. Moreover, note that $\widehat{f'}(0) = 0$ and by hypothesis $\widehat{f}(0) = 0$. By 131 Parseval's identity we have:

$$\begin{aligned}\int_0^T |f(x)|^2\,\mathrm{d}x = T\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 &\leq T \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^2 \left|\widehat{f}(n)\right|^2 = \\ &= T \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{T^2}{4\pi^2}\left|\widehat{f'}(n)\right|^2 = \frac{T^2}{4\pi^2}\int_0^T |f'(x)|^2\,\mathrm{d}x\end{aligned}$$

The equality holds if and only if $\left|\widehat{f}(n)\right|^2 (n^2 - 1) = 0$ $\forall n \in \mathbb{Z} \setminus \{0\}$. That is, if and only if

$$f(x) = c_{-1}\mathrm{e}^{\frac{-2\pi i x}{T}} + c_1 \mathrm{e}^{\frac{2\pi i x}{T}}$$

□

Theorem 133 (Wirtinger's inequality). Let $f \in \mathcal{C}^1([a,b])$ with $f(a)=f(b)=0$. Then:

$$\int_a^b |f(x)|^2 \, \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \, \mathrm{d}x$$

with equality if and only if

$$f(x) = A \sin\left(\frac{\pi}{b-a}(x-a)\right)$$

Sketch of the proof. Let $\tilde{f} : [a, 2b-a] \to \mathbb{R}$ be the odd extension of f centered at b:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [a,b] \\ -f(2b-x) & \text{if } x \in [b, 2b-a] \end{cases}$$

Now use 132 Wirtinger's inequality to the function $\tilde{f}$. □

Theorem 134 (Isoperimetric inequality). Let c be a planar simple and closed curve of class $\mathcal{C}^1$ whose length is ℓ. If A_c is the area enclosed by c, then

$$A_c \leq \frac{\ell^2}{4\pi}$$

and the equality holds if and only if c is a circle.

Proof. Let $\gamma(s) = (x(s), y(s))$ be the arc-length parametrization of c (see **??**). Thus:

$$s = \int_0^s \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t$$

which implies $x'(s)^2 + y'(s)^2 = 1$, by the **?? ??**. Now, by **??** we have that:

$$A_c = \int_0^\ell x(s) y'(s) \, \mathrm{d}s \leq \frac{2\pi}{\ell} \int_0^\ell \frac{x(s)^2 + \frac{\ell^2}{4\pi^2} y'(s)^2}{2} \, \mathrm{d}s =$$

$$= \frac{2\pi}{\ell} \int_0^\ell \left(\frac{\ell^2}{8\pi^2} + \frac{x(s)^2 - \frac{\ell^2}{4\pi^2} x'(s)^2}{2} \right) \mathrm{d}s \leq \frac{\ell^2}{4\pi}$$

by the 132 Wirtinger's inequality (with a translation we can suppose $x(0) = x(\ell) = 0$ and $\int_0^\ell x(s) \, \mathrm{d}s = 0$). Clearly if c is a circle, the equality is hold. Moreover if we have equality, by 132 Wirtinger's inequality we have that $x(s) = A \cos\left(\frac{2\pi s}{\ell}\right) + B \sin\left(\frac{2\pi s}{\ell}\right)$ and since $2ab = a^2 + b^2$ implies $b = a$, we have that $\frac{\ell}{2\pi} y'(s) = x(s)$. So $y(s) = A \sin\left(\frac{2\pi s}{\ell}\right) - B \cos\left(\frac{2\pi s}{\ell}\right) + C$ and therefore c is a circle. □