Functions of several variables

1. | Topology of $\mathbb{R}^n$

Definition 1. Let M be a set. A *distance* in M is a function $d : M \times M \to \mathbb{R}$ such that $\forall x, y, z \in M$ the following properties are satisfied:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$ (*triangular inequality*)

We define a *metric space* as a pair (M, d) that satisfy the previous properties.

Definition 2. Let V be a real vector space. A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \lambda \in \mathbb{R}$ the following properties are satisfied:

1. $\|\mathbf{u}\| \geq 0$
2. $\|\mathbf{u}\| = 0 \iff \mathbf{u} = 0$
3. $\|\lambda \mathbf{u}\| = |\lambda| \|\mathbf{u}\|$
4. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (*triangular inequality*)

We define a *normed vector space* as a pair $(V, \|\cdot\|)$ that satisfy the previous properties.

Proposition 3. Let $(V, \|\cdot\|)$ be a normed vector space. Then (V, d) is a metric space with associated distance $d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|$, $\forall \mathbf{u}, \mathbf{v} \in V$.

Definition 4. Let V be a real vector space. A *dot product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ the following properties are satisfied:

1. $\langle \alpha \mathbf{u} + \beta \mathbf{w}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{v} \rangle$
   $\langle \mathbf{u}, \alpha \mathbf{v} + \beta \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
3. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$
4. $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = 0$

We define an *Euclidean space* as a pair $(V, \langle \cdot, \cdot \rangle)$ that satisfy the previous properties[1].

Proposition 5. Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidean space. Then $(V, \|\cdot\|)$ is a normed space with associated norm $\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$, $\forall \mathbf{u} \in V$. .

Proposition 6. Let $\langle \cdot, \cdot \rangle_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a map defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle_2 = \sum_{i=1}^{n} u_i v_i$$

$\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, being $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. Then, the pair $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_2)$ is an Euclidean space.

Corollary 7. Consider the norm $\|\cdot\|_2$ and distance d_2 in $\mathbb{R}^n$ defined as follows:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_2} = \sqrt{\sum_{i=1}^{n} u_i^2}$$

$$d_2(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$$

Then, $(\mathbb{R}^n, \|\cdot\|_2)$ is a normed space and $(\mathbb{R}^n, d_2)$ is a metric space.

Proposition 8. Let $(V, \langle \cdot, \cdot \rangle)$ be an Euclidean space with the norm defined as $\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. Then for all $\mathbf{u}, \mathbf{v} \in V$ the following properties are satisfied:

1. $\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\| \|\mathbf{v}\|$ (*Cauchy-Schwarz inequality*)
2. $\|\mathbf{u} - \mathbf{v}\| \geq |\|\mathbf{u}\| - \|\mathbf{v}\||$
3. $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ (*Parallelogram law*)
4. $\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\langle \mathbf{u}, \mathbf{v} \rangle$
5. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}\left(\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right)$
6. On $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_2)$, if $\mathbf{u} = (u_1, \ldots, u_n)$, then:
$$|u_i| \leq \|\mathbf{u}\| \leq \sum_{i=1}^{n} |u_i|$$

Definition 9. Let $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. We define the *norm* of $\mathbf{L}$ as

$$\|\mathbf{L}\| = \sup\{\|\mathbf{L}(x)\| : \|x\| = 1\}$$

Lemma 10. Let $\Phi : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a map defined as $\Phi(\mathbf{L}) = \|\mathbf{L}\|$. Then, Φ is a norm on the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Proposition 11. Let $\mathbf{L} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then:

$$\|\mathbf{L}\| = \inf\{C : \|\mathbf{L}(x)\| \leq C\|x\|\}$$

Corollary 12. Let $\mathbf{L}, \mathbf{M} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be linear maps with associated matrices $\mathbf{L} = (a_{ij})$, $\mathbf{M} = (b_{ij})$ respectively. The following properties are satisfied:

1. $\|\mathbf{L}(x)\| \leq \|\mathbf{L}\| \|x\|$
2. $\|\mathbf{L}\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$
3. $|a_{ij} - b_{ij}| < \varepsilon, \forall i, j \iff \|\mathbf{L} - \mathbf{M}\| < \varepsilon'$

Definition 13. Let (M, d) be a metric space. The *sphere* with center p and radius $r \in \mathbb{R}_{\geq 0}$ is the set $S(p, r) = \{x \in M : d(x, p) = r\}$.

[1] Sometimes the notation $\mathbf{u} \cdot \mathbf{v}$ is used, instead of $\langle \mathbf{u}, \mathbf{v} \rangle$, to denote the dot product between $\mathbf{u}$ and $\mathbf{v}$.

Definition 14. Let (M,d) be a metric space. The *open ball* with center p and radius $r \in \mathbb{R}_{>0}$ is the set $B(p,r) = \{x \in M : d(x,p) < r\}$.

Definition 15. Let (M,d) be a metric space. The *closed ball* with center p and radius $r \in \mathbb{R}_{\geq 0}$ is the set $\overline{B}(p,r) = \{x \in M : d(x,p) \leq r\}$.

Definition 16. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. A is a *bounded set* if exists a ball containing it.

Definition 17. Let (M,d) be a metric space. A *neighbourhood* of p is a bounded set $E(p) \subset M$ such that $\exists r \in \mathbb{R}_{>0}$ satisfying $B(p,r) \subset E(p)$.

Definition 18. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is an *interior point* of A if $\exists r \in \mathbb{R}_{>0}$ such that $B(p,r) \subset A$. The *interior* of A is the set $\operatorname{Int} A$ containing all interior points of A.

Definition 19. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is an *exterior point* of A if $\exists r \in \mathbb{R}_{>0}$ such that $B(p,r) \cap A = \varnothing$. The *exterior* of A is the set $\operatorname{Ext} A$ containing all exterior points of A.

Definition 20. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is an *adherent point* of A if $\forall r \in \mathbb{R}_{>0}$, $B(p,r) \cap A \neq \varnothing$. The *adherence* of A is the set $\overline{A}$ containing all adherent points of A.

Definition 21. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is a *accumulation point* of A if $\forall r \in \mathbb{R}_{>0}$, $B(p,r) \setminus \{p\} \cap A \neq \varnothing$. The *limit set* of A is the set A' containing all accumulation points of A.

Definition 22. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is an *isolated point* of A if it is an adherent but not accumulation point, that is, if $p \in A$ and $\exists r \in \mathbb{R}_{>0}$ such that $B(p,r) \setminus \{p\} \cap A = \varnothing$.

Definition 23. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. p is a *boundary point* of A if $\forall r \in \mathbb{R}_{>0}$, $B(p,r) \cap A \neq \varnothing$ and $B(p,r) \cap A^c \neq \varnothing$. The *boundary* of A is the set ∂A containing all boundary points of A.

Proposition 24. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. If p is an accumulation point of A, then $\forall r \in \mathbb{R}_{>0}$, $B(p,r)$ has infinity many points of A.

Theorem 25 (Bolzano-Weierstraß theorem). Let $B \subset \mathbb{R}^n$ be a set. If B has infinity many points and it is bounded, then it has least an accumulation point.

Definition 26. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. A is *open* if $\forall p \in A$, $\exists r \in \mathbb{R}_{>0}$ such that $B(p,r) \subset A$.

Definition 27. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. A is *closed* if its complementary A^c is open.

Proposition 28. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. A is closed $\iff A = \overline{A} \iff \partial A \subset A \iff A' \subset A$.

Proposition 29. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M. A is open $\iff A = \operatorname{Int} A$.

Proposition 30. Let (M,d) be a metric space and $A \subseteq M$ be a subset of M.

- $\operatorname{Int} A$ is the biggest open set contained in A. That is, if $B \subset A$ is open, $B \subset \operatorname{Int} A$.
- $\overline{A}$ is the smallest set containing A. That is, if $B \supset A$ is closed, $B \supset \overline{A}$.

Proposition 31.

- The union of open sets is open.
- The intersection of a finite number of open sets is open.
- The union of a finite number of closed sets is closed.
- The intersection of closed sets is closed.

Definition 32. We say that a set A is *connected* if there are no open sets $U, V \neq \varnothing$ such that:

$$A \subseteq U \cup V \quad A \cap U \cap V = \varnothing \quad A \cap U \neq \varnothing \quad A \cap V \neq \varnothing$$

Sequences

Definition 33. Let (M,d) be a metric space. A *sequence* in M is a map

$$\begin{aligned} x : \mathbb{N} &\longrightarrow M \\ n &\longmapsto x(n) \end{aligned}$$

The sequence $x(n)$ is usually represented as (x_n).

Definition 34. Let (M,d) be a metric space. We say a sequence $(x_n) \subset M$ is *convergent* to $p \in M$ if

$$\forall \varepsilon \in \mathbb{R}_{>0},\ \exists n_0 \in \mathbb{N} : d(x_n, p) < \varepsilon \text{ if } n > n_0$$

Definition 35. Let (M,d) be a metric space. We say a sequence (x_n) is a *Cauchy sequence* if $\forall \varepsilon > 0$ $\exists n_0$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n \geq n_0$.

Definition 36. A metric space (M,d) is *complete* if every Cauchy sequence in M converges in M.

Definition 37. A subset $K \subset \mathbb{R}^n$ is *compact* if it is closed and bounded.

Theorem 38. Let $K \subset \mathbb{R}^n$ be an arbitrary set and $(x_m) \in K$ be a sequence. Then K is compact if and only if there exists a partial sequence (x_{m_k}) and $x \in K$ such that $\lim\limits_{k \to \infty} x_{m_k} = x$.

2. | Continuity

Definition 39 (Graph of a function). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. We define the *graph* of f as the following subset of $\mathbb{R}^{n+1}$:

$$\operatorname{graph}(f) = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in U\}$$

Definition 40. Given a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, we define the *level set* $C_k(f)$ as $C_k(f) = \{x \in \mathbb{R}^n : f(x) = k\}$.

Definition 41. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $p \in U'$. We say $\lim\limits_{x \to p} \mathbf{f}(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|\mathbf{f}(x) - L\| < \varepsilon$ if $\|x - p\| < \delta$.

Proposition 42. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f} = (f_1, \ldots, f_m)$, and $p \in U'$.

1. The limit of $\mathbf{f}$ at point p, if exists, is unique.
2. Suppose $L = (L_1, \ldots, L_m)$. Then:
$$\lim_{x \to p} \mathbf{f}(x) = L \iff \lim_{x \to p} f_j(x) = L_j \quad \forall j = 1, \ldots, m$$

Lemma 43. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $p \in U'$. $\exists \lim_{x\to p} \mathbf{f}(x) = L \iff \forall (x_n) \in \mathbb{R}^n : \lim_{n\to\infty} x_n = p$ and $x_n \neq p$ for all n we have $\lim_{n\to\infty} \mathbf{f}(x_n) = L$.

Definition 44. We say that $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $p \in U'$ if $\lim_{x\to p} \mathbf{f}(x) = \mathbf{f}(p)$. We say that $\mathbf{f}$ is continuous on U, if so it is at each point $p \in U$.

Definition 45. We say that $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *uniformly continuous* on U if $\forall \varepsilon > 0$, $\exists \delta > 0 : \|\mathbf{f}(x) - \mathbf{f}(y)\| < \varepsilon$, $\forall x, y \in U : \|x - y\| < \delta$.

Corollary 46. A uniformly continuous function is continuous.

Theorem 47 (Heine's theorem). Let $\mathbf{f} : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous function and K be a compact set. Then, $\mathbf{f}$ is uniformly continuous on K.

Theorem 48. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be an uniformly continuous function and $(x_n) \in U$ be a Cauchy sequence. Then $(\mathbf{f}(x_n)) \in \mathbb{R}^m$ is a Cauchy sequence.

Theorem 49. Let $\mathbf{f} : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and K be a compact set. Then $\mathbf{f}(K)$ is a compact set.

Theorem 50 (Weierstraß' theorem). Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function and K a compact set. Then f attains a maximum and a minimum on K.

Theorem 51 (Intermediate value theorem). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function and U be a connected set. Then $\forall x, y \in U$ and $\forall c \in [f(x), f(y)]$, $\exists z \in U$ such that $f(z) = c$.

Definition 52. A function $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz continuous* if $\exists k > 0$ such that
$$\|\mathbf{f}(x) - \mathbf{f}(y)\| \leq k\|x - y\|$$
$\forall x, y \in U$. If $0 \leq k < 1$, we say that $\mathbf{f}$ is a *contraction*.

Proposition 53. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function at $p \in U$. Then $\mathbf{f}$ is continuous at p.

Definition 54. Let (M, d) be a metric space and $f : M \to \mathbb{R}$ a function. We define the *modulus of continuity* of f as the function $\omega_f : (0, \infty) \to [0, \infty]$ defined as:
$$\omega_f(\delta) := \sup\{|f(x) - f(y)| : d(x, y) < \delta, x, y \in M\}$$

3. | Differential calculus

Differential of a function

Definition 55. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in U$. The function $\mathbf{f}$ is *differentiable* at a if there exists a linear map $\mathbf{Df}(a) : \mathbb{R}^n \to \mathbb{R}^m$ such that:
$$\lim_{x\to a} \frac{\|\mathbf{f}(x) - \mathbf{f}(a) - \mathbf{Df}(a)(x - a)\|}{\|x - a\|} = \lim_{h\to 0} \frac{\|\mathbf{f}(a + h) - \mathbf{f}(a) - \mathbf{Df}(a)(h)\|}{\|h\|} = 0$$
$\mathbf{Df}(a)$[2] is called the *differential* of $\mathbf{f}$ at point a. Furthermore, we say $\mathbf{f}$ is differentiable on $B \subseteq U$ if it is differentiable at each point of B.

Proposition 56. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in U$. $\mathbf{f} = (f_1, \ldots, f_m)$ is differentiable at a if and only if every component function $f_j : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at a.

Definition 57. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in U$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\|\mathbf{v}\| = 1$. The *directional derivative* of f at a in the direction of $\mathbf{v}$ is:
$$D_{\mathbf{v}} f(a) = \lim_{t\to 0} \frac{f(a + t\mathbf{v}) - f(a)}{t}$$

Definition 58. Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$ and $a \in U$. If the following limit exists, we define the *partial derivative* with respect to x_j of f at a as:
$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a + h\mathbf{e}_j) - f(a)}{h}$$ [3]

Definition 59. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in U$. If all partial derivatives of $\mathbf{f}$ at a exist, we call *Jacobian matrix* of $\mathbf{f}$ at a the matrix associated with $\mathbf{Df}(a)$ (with respect to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$):
$$\mathbf{Df}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$
If $n = m$, we define the *Jacobian determinant* as $J\mathbf{f}(a) = \det \mathbf{Df}(a)$.

Definition 60. Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$ and $a \in U$ such that f is differentiable at $a \in U$. The *gradient* of f at a is:
$$\boldsymbol{\nabla} f(a) := \mathbf{D} f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right)$$

Proposition 61. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be a differentiable function at $a \in U$. Then there exists the tangent hyperplane to the graph of f at a and has the equation:
$$x_{n+1} = f(a) + \boldsymbol{\nabla} f(a) \cdot (x - a)$$ [4]

[2] Other commonly used notations for the differential of a function $\mathbf{f}$ at a point a are $\mathbf{Df}_a$, $\mathbf{df}(a)$ or $\mathbf{df}_a$.

[3] Here $\mathbf{e}_j$ is the j-th vector of the canonical basis of $\mathbb{R}^n$, that is, $\mathbf{e}_j = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)$.

[4] In general (not only the case of the graph of a function) the tangent hyperplane to function f at a point a is given by the equation
$$\boldsymbol{\nabla} f(a) \cdot (x - a) = 0$$

Theorem 62. Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$, $a \in U$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\|\mathbf{v}\| = 1$. If f is differentiable at a, the $D_{\mathbf{v}} f(a)$ exists and:

$$D_{\mathbf{v}} f(a) = \boldsymbol{\nabla} f(a) \cdot \mathbf{v}$$

Proposition 63. Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$ be a differentiable function on U and C_k be the level set of value $k \in \mathbb{R}$. Then $\boldsymbol{\nabla} f(a) \perp C_k$ at $a \in C_k$.

Proposition 64. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a differentiable function at $a \in U$ and $\mathbf{v} \in \mathbb{R}^n$. Then:

- $\max\{D_{\mathbf{v}} f(a) : \|\mathbf{v}\| = 1\} = \|\boldsymbol{\nabla} f(a)\|$ and it is attained when $\mathbf{v} = \dfrac{\boldsymbol{\nabla} f(a)}{\|\boldsymbol{\nabla} f(a)\|}$.
- $\min\{D_{\mathbf{v}} f(a) : \|\mathbf{v}\| = 1\} = -\|\boldsymbol{\nabla} f(a)\|$ and it is attained when $\mathbf{v} = -\dfrac{\boldsymbol{\nabla} f(a)}{\|\boldsymbol{\nabla} f(a)\|}$.

Theorem 65. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function at $a \in U$. Then $\mathbf{f}$ is locally Lipschitz continuous at a.

Theorem 66. Let $\mathbf{f}, \mathbf{g} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be two differentiable functions at a point $a \in U$ and let $c \in \mathbb{R}$. Then:

1. $\mathbf{f} + \mathbf{g}$ is differentiable at a and:
$$\mathbf{D}(\mathbf{f} + \mathbf{g})(a) = \mathbf{Df}(a) + \mathbf{Dg}(a)$$
2. $c\mathbf{f}$ is differentiable at a and:
$$\mathbf{D}(c\mathbf{f})(a) = c\mathbf{Df}(a)$$
3. If $m = 1$, then $(fg)(x) = f(x)g(x)$ is differentiable at a and:
$$D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$$
4. If $m = 1$ and $g(a) \neq 0$, then $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ is differentiable at a and:
$$D\left(\frac{f}{g}\right)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$$

Theorem 67 (Chain rule). Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. Let $\mathbf{f} : U \to \mathbb{R}^m$ and $\mathbf{g} : V \to \mathbb{R}^p$. Suppose that $\mathbf{f}(U) \subset V$, $\mathbf{f}$ is differentiable at $a \in U$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(a)$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at a and:

$$\mathbf{D}(\mathbf{g} \circ \mathbf{f})(a) = \mathbf{Dg}(\mathbf{f}(a)) \circ \mathbf{Df}(a)$$

Definition 68. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^m$. We say that $\mathbf{f}$ is a function of *class* $\mathcal{C}^k(U)$, $k \in \mathbb{N}$, if all partial derivatives of order k exists and are continuous on U. We say that $\mathbf{f}$ is function of *class* $\mathcal{C}^\infty(U)$ if it is of class $\mathcal{C}^k(U)$, $\forall k \in \mathbb{N}$.

Theorem 69 (Differentiability criterion). Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f}(x) = (f_1(x), \ldots, f_m(x))$. If all partial derivatives $\dfrac{\partial f_i(x)}{\partial x_j}$ exist in a neighbourhood of $a \in U$ and are continuous at a, then $\mathbf{f}$ is differentiable at $a \in U$.

Proposition 70. Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $A \subseteq U$. If all partial derivatives of $\mathbf{f}$ exist on A and are bounded functions on A, then $\mathbf{f}$ is uniformly continuous on A.

Theorem 71 (Mean value theorem). Let $f : B \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ in an open connected set $B \subseteq \mathbb{R}^n$ and $x, y \in B$. Then:

$$f(x) - f(y) = \boldsymbol{\nabla} f(z) \cdot (x - y)$$

for some $z \in [x, y]$.

Theorem 72 (Mean value theorem for vector-valued functions). Let $\mathbf{f} : B \to \mathbb{R}^m$ be a function of class $\mathcal{C}^1$ in an open connected set $B \subseteq \mathbb{R}^n$ and $x, y \in B$. Then:

$$\|\mathbf{f}(x) - \mathbf{f}(y)\| \leq \|\mathbf{Df}(z)\|\|x - y\|$$

for some $z \in [x, y]$.

Higher order derivatives

Definition 73. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$. We denote the *partial derivative of order* k of f with respect to the variables $x_{i_1}, \ldots, x_{i_k}$ at a point $a \in U$ as:

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}}(a)$$

Definition 74. Let $U \subseteq \mathbb{R}^n$ be an open set. If $f : U \to \mathbb{R}$ has second order partial derivatives at $a \in U$, we define the *Hessian matrix* of f at a point a as:

$$\mathbf{H}f(a) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial {x_1}^2}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \cdots & \dfrac{\partial^2 f}{\partial {x_n}^2}(a) \end{pmatrix}$$ [5]

Theorem 75 (Schwarz's theorem). Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$. If f has mixed partial derivatives of order k and are continuous functions on $A \subseteq U$, then for any permutation $\sigma \in \mathrm{S}_k$ we have:

$$\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_1}}(a) = \frac{\partial^k f}{\partial x_{\sigma(i_k)} \cdots \partial x_{\sigma(i_1)}}(a) \qquad \forall a \in A$$

Inverse and implicit function theorems

Lemma 76. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^m$ with $\mathbf{f} \in \mathcal{C}^1(U)$. Given $a \in U$ and $\varepsilon > 0$, $\exists B(a, r) \subset U$ such that:

$$\|\mathbf{f}(x) - \mathbf{f}(y)\| \leq (\|\mathbf{Df}(a)\| + \varepsilon)\|x - y\| \quad \forall x, y \in B(a, r)$$

Lemma 77. Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^n$ with $\mathbf{f} \in \mathcal{C}^1(U)$. Suppose that for some $a \in U$, $J\mathbf{f}(a) \neq 0$. Then $\exists B(a, r) \subset U$ and $c > 0$ such that:

$$\|\mathbf{f}(y) - \mathbf{f}(x)\| \geq c\|x - y\|, \quad \forall x, y \in B(a, r)$$

In particular, $\mathbf{f}$ is injective on $B(a, r)$.

[5] Note that we can think $\mathbf{H}f(a)$ to be the associated matrix of a bilinear form $\mathbf{H}f(a)$.

Theorem 78 (Inverse function theorem). Let $U \subseteq \mathbb{R}^n$ be an open set, $\mathbf{f} : U \to \mathbb{R}^n$ with $\mathbf{f} \in \mathcal{C}^1(U)$ and $a \in U$ such that $J\mathbf{f}(a) \neq 0$. Then $\exists B := B(a,r) \subset U$ such that:

1. $\mathbf{f}$ is injective on B.
2. $\mathbf{f}(B) = V$ is an open set of $\mathbb{R}^n$.
3. $\mathbf{f}^{-1} : V \to B$ is of class $\mathcal{C}^1$ on V.

Moreover, it is satisfied that $\mathbf{Df}^{-1}(\mathbf{f}(a)) = \mathbf{Df}(a)^{-1}$

Definition 79. A function $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a *diffeomorphism* of class $\mathcal{C}^k$ if it is bijective and both $\mathbf{f}$ and $\mathbf{f}^{-1}$ are of class $\mathcal{C}^k$[6].

Theorem 80 (Implicit function theorem). Let $U \subseteq \mathbb{R}^{n+m}$ be an open set, $\mathbf{f} : U \to \mathbb{R}^m$ with $\mathbf{f} \in \mathcal{C}^1(U)$ and $(a,b) = (a_1,\dots,a_n,b_1,\dots,b_m) \in U$ such that $\mathbf{f}(a,b) = 0$. If $\mathbf{Df}(x) = (\mathbf{Df}_1(x)|\mathbf{Df}_2(x))$ with $\mathbf{Df}_1(x) \in \mathcal{M}_{m\times n}(\mathbb{R})$, $\mathbf{Df}_2(x) \in \mathcal{M}_m(\mathbb{R})$ and $\det \mathbf{Df}_2(x) \neq 0$ (i.e. rang $\mathbf{Df}(a,b) = m$), then there exists an open set $W \subseteq \mathbb{R}^n$ and a function $\mathbf{g} : W \to \mathbb{R}^m$ such that $a \in W$, $\mathbf{g} \in \mathcal{C}^1(W)$ and:

$$\mathbf{g}(a) = b \quad \text{and} \quad \mathbf{f}(x, \mathbf{g}(x)) = 0 \quad \forall x \in W$$

Moreover, is is satisfied that:

$$\mathbf{Dg}(a) = -\mathbf{Df}_2(a, \mathbf{g}(a))^{-1} \circ \mathbf{Df}_1(a, \mathbf{g}(a))$$

Taylor's polynomial and maxima and minima

Theorem 81 (Taylor's theorem). Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}$, $a \in U$ and $f \in \mathcal{C}^{k+1}(U)$. Then:

$$f(x) = f(a) + \\ + \sum_{m=1}^{k} \frac{1}{m!} \left(\sum_{i_m,\dots,i_1=1}^{n} \frac{\partial^m f}{\partial x_{i_m} \cdots \partial x_{i_1}}(a) \prod_{j=1}^{m} (x_{i_j} - a_{i_j}) \right) + \\ + R_k(f,a)$$

where

$$R_k(f,a) = \\ = \frac{1}{(k+1)!} \sum_{i_{k+1},\dots,i_1=1}^{n} \frac{\partial^{k+1} f}{\partial x_{i_{k+1}} \cdots \partial x_{i_1}}(\xi) \prod_{j=1}^{k+1} (x_{i_j} - a_{i_j}) = \\ = \mathrm{o}(\|x-a\|^k)$$

for some $\xi \in [a,x]$. In particular, for $k = 2$ we have:

$$f(x) = f(a) + Df(a)(x-a) + \frac{1}{2} Hf(a)(x-a,x-a) + \\ + R_2(f,a)$$

where $R_2(f,a) = \mathrm{o}(\|x-a\|^2)$.

Remark. In order to simplify the notation, we can make use of the *multi-index notation* and write:

$$f(x) = \sum_{|\boldsymbol{\alpha}| \leq m} \frac{\partial^{\boldsymbol{\alpha}} f(a)}{\boldsymbol{\alpha}!} (x-a)^{\boldsymbol{\alpha}} + R_m(f,a)$$

where

$$R_m(f,a) = \frac{1}{(m+1)!} \sum_{|\boldsymbol{\alpha}| = m+1} \frac{\partial^{\boldsymbol{\alpha}} f(a + c(x-a))}{\boldsymbol{\alpha}!} (x-a)^{\boldsymbol{\alpha}}$$

$c \in (0,1)$, and the multi-index $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ is a vector of non-negative integers and $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$. Moreover:

$$\boldsymbol{\alpha}! := \prod_{i=1}^{n} \alpha_i!$$
$$(x-a)^{\boldsymbol{\alpha}} := \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}$$
$$\partial^{\boldsymbol{\alpha}} f(a) := \frac{\partial^{\boldsymbol{\alpha}} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}(a)$$

Definition 82. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$. We say that f has a *local maximum* at $a \in U$ if $\exists B(a,r) \subset U$ such that $f(x) \leq f(a)$, $\forall x \in B(a,r)$. Analogously, we say that f has a *local minimum* at $a \in U$ if $\exists B(a,r) \subset U$ such that $f(x) \geq f(a)$, $\forall x \in B(a,r)$. A *local extremum* is either a local maximum or a local minimum. Moreover, if $f(x) \leq f(a)$ $\forall x \in U$, we say that f has a *global maximum* at $a \in U$. Similarly if $f(x) \geq f(a)$ $\forall x \in U$, we say that f has a *global minimum* at $a \in U$.

Proposition 83. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be a differentiable function at $a \in U$. If f has a local extremum at a, then $\boldsymbol{\nabla} f(a) = 0$.

Definition 84. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$. We say that $a \in U$ is a *critical point* of f if $\boldsymbol{\nabla} f(a) = 0$. We say that $a \in U$ is a *saddle point* if a is a critical point but not a local extremum.

Theorem 85. Let $\mathcal{Q}$ be a quadratic form. Then for all $x \neq 0$ we have:

$$\mathcal{Q} \text{ is defined positive} \iff \exists \lambda \in \mathbb{R}_{>0} : \mathcal{Q}(x) \geq \lambda \|x\|^2.$$
$$\mathcal{Q} \text{ is defined negative} \iff \exists \lambda \in \mathbb{R}_{<0} : \mathcal{Q}(x) \leq \lambda \|x\|^2.$$

Proposition 86 (Sylvester's criterion). Let $\mathbf{A} = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix. $\mathbf{A}$ is defined positive if and only if all its principal minors are positive, that is:

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$$

$\mathbf{A}$ is defined negative if and only if its principal minor of order k have sign $(-1)^k$, that is:

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots, (-1)^n \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$$

Theorem 87. Let $U \subseteq \mathbb{R}^2$ be an open set, $f : U \to \mathbb{R}$ a function of class $\mathcal{C}^2(U)$ and $a \in U : \boldsymbol{\nabla} f(a) = 0$. Let $\mathbf{H}f(a)$ be the Hessian matrix of f at a. Then:

[6] By default, if we omit to say the class of the diffeomorphism, we will refer to a diffeomorphism of class $\mathcal{C}^1$.

1. If $\mathbf{H}f(a)$ is defined positive, then f has a local minimum at a.
2. If $\mathbf{H}f(a)$ is defined negative, then f has a local maximum at a.
3. If $\mathbf{H}f(a)$ is undefined, then f has a saddle point at a.

Theorem 88 (Lagrange multipliers theorem). Let $f, g_i : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be functions of class $\mathcal{C}^1(U)$ for $i = 1, \ldots, k$ and $1 \leq k < n$. Let $S = \{x \in U : g_i(x) = 0,\ \forall i\}$ and $a \in S$ such that $f|_S(a)$ is a local extremum. If the vectors $\nabla g_1(a), \ldots, \nabla g_k(a)$ are linearly independent, then $\exists \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that:

$$\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a)$$

4. | Integral calculus

Integration over compact rectangles

Definition 89. A *rectangle* R of $\mathbb{R}^n$ is a product $R = I_1 \times \cdots \times I_n$ where $I_j \in \mathbb{R}$ are bounded and non-degenerate[7] intervals.

Definition 90. The *n-dimensional volume* (*length* if $n = 1$ and *surface* if $n = 2$) of a bounded rectangle $R = I_1 \times \cdots \times I_n$, $I_i = [a_i, b_i]$ is:

$$\mathrm{vol}(R) = \prod_{i=1}^{n} (b_i - a_i)$$

Definition 91. Given a rectangle $R = I_1 \times \cdots \times I_n$, a *partition* of R is the product $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ where $\mathcal{P}_j$ is a partition of the interval I_j. A partition $\mathcal{P}$ is *regular* if for all j, $\mathcal{P}_j$ is regular, that is, all subintervals in $\mathcal{P}_j$ have the same size. We denote by $\mathbf{P}(R)$ the set of all partitions of R.

Definition 92. Given two partitions $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ and $\mathcal{P}' = \mathcal{P}'_1 \times \cdots \times \mathcal{P}'_n$ of a rectangle R, we say that $\mathcal{P}'$ is *finer than* $\mathcal{P}$ if each $\mathcal{P}'_j$ is finer than $\mathcal{P}_j$.

Definition 93. Let $R \subset \mathbb{R}^n$ be a compact rectangle, $f : R \to \mathbb{R}$ be a bounded function and $\mathcal{P} \in \mathbf{P}(R)$. For each subrectangle R_j, $j = 1, \ldots, m$, determined by $\mathcal{P}$ let

$$m_j := \inf\{f(x) : x \in R_j\} \quad \text{and} \quad M_j := \sup\{f(x) : x \in R_j\}$$

We define the *lower sum* and the *upper sum* of f with respect to $\mathcal{P}$ as:

$$L(f, \mathcal{P}) = \sum_{j=1}^{m} m_j \mathrm{vol}(R_j) \qquad U(f, \mathcal{P}) = \sum_{j=1}^{m} M_j \mathrm{vol}(R_j)$$ [8]

Definition 94. Let $R \subset \mathbb{R}^n$ be a compact rectangle and $f : R \to \mathbb{R}$ be a bounded function. We define the *lower integral* and *upper integral* of f on R as

$$\underline{\int_R} f := \sup\{L(f, \mathcal{P}) : \mathcal{P} \in \mathbf{P}\}$$
$$\overline{\int_R} f := \inf\{U(f, \mathcal{P}) : \mathcal{P} \in \mathbf{P}\}$$

We say that f is *Riemann-integrable* on R if $\displaystyle\underline{\int_R} f = \overline{\int_R} f$.

Proposition 95. Let $R \subset \mathbb{R}^n$ be a compact rectangle and $f : R \to \mathbb{R}$ be a bounded function. f is Riemann-integrable if and only if $\forall \varepsilon\ \exists \mathcal{P} \in \mathbf{P}(R)$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.

Definition 96. Let $R \subset \mathbb{R}^n$ be a compact rectangle, $f : R \to \mathbb{R}$ be a bounded function, $\mathcal{P} \in \mathbf{P}(R)$ and ξ_j be an arbitrary point of the subrectangle R_j for $j = 1, \ldots, m$. Then, we define the *Riemann sum* of f associated to $\mathcal{P}$ as:

$$S(f, \mathcal{P}) = \sum_{j=1}^{m} f(\xi_j) \mathrm{vol}(R_j)$$

Theorem 97. Let $R \subset \mathbb{R}^n$ be a compact rectangle and $f : R \to \mathbb{R}$ be a bounded function. f is Riemann-integrable over R if and only if $\forall \varepsilon > 0\ \exists \mathcal{P}_\varepsilon \in \mathbf{P}(R)$ such that:

$$\left| S(f, \mathcal{P}) - \int_R f \right| = \left| \sum_{j=1}^{m} f(\xi_j) \mathrm{vol}(R_j) - \int_R f \right| < \varepsilon$$

for any $\mathcal{P} \in \mathbf{P}(R)$ finer than $\mathcal{P}_\varepsilon$ and for any $\xi_j \in R_j$.

Fubini's theorem

Theorem 98 (Fubini's theorem). Let $R_1 \subset \mathbb{R}^n$ and $R_2 \subset \mathbb{R}^m$ be closed rectangles and $f : R_1 \times R_2 \to \mathbb{R}$ be an integrable[9] function. Suppose for every $x_0 \in R_1$, $f(x_0, y)$ is integrable over R_2. Then, the function $g(x) = \displaystyle\int_{R_2} f(x, y)\,\mathrm{d}y$ is integrable over R_1 and

$$\int_{R_1 \times R_2} f(x, y) = \int_{R_1} \mathrm{d}x \int_{R_2} f(x, y)\,\mathrm{d}y$$

Similarly if for every $y_0 \in R_2$, $f(x, y_0)$ is integrable over R_1, then the function $h(y) = \displaystyle\int_{R_1} f(x, y)\,\mathrm{d}x$ is integrable over R_2 and

$$\int_{R_1 \times R_2} f(x, y) = \int_{R_2} \mathrm{d}y \int_{R_1} f(x, y)\,\mathrm{d}x$$

[7] That is, non-empty intervals with more than one point.

[8] We will omit the results related to these definitions because of they are a natural extension of results of single-variable functions course and can be deduced easily. That's why we only expose the most important ones here.

[9] As we have only defined Riemann-integration, it goes without saying that an *integrable function* means a *Riemann-integrable function*.

Corollary 99. Let $R_1 \subset \mathbb{R}^n$ and $R_2 \subset \mathbb{R}^m$ be closed rectangles and let $f : R_1 \times R_2 \to \mathbb{R}$ be a continuous function on $R_1 \times R_2$. Then:

$$\int_{R_1\times R_2} f = \int_{R_1} \mathrm{d}x \int_{R_2} f(x,y)\,\mathrm{d}y = \\ = \int_{R_2} \mathrm{d}y \int_{R_1} f(x,y)\,\mathrm{d}x$$

Corollary 100. Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ be a rectangle. If $f : R \to \mathbb{R}$ is a continuous function, then

$$\int_R f = \int_{a_n}^{b_n} \mathrm{d}x_n \int_{a_{n-1}}^{b_{n-1}} \mathrm{d}x_{n-1} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_n)\,\mathrm{d}x_1$$

Definition 101. Let $D \subset \mathbb{R}^{n-1}$ be a compact set and $\varphi_1, \varphi_2 : D \to \mathbb{R}$ be continuous functions such that $\varphi_1(x) \leq \varphi_2(x)$ $\forall x \in D$. The set

$$S = \{(x, y) \subset \mathbb{R}^n : x \in D, \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

is called an *elementary region* in $\mathbb{R}^n$. In particular, if $n = 2$, we say S is *x-simple*. An elementary region in $V \subset \mathbb{R}^3$ is called *xy-simple* if it is of the form:

$$V = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U, \phi_1(x, y) \leq z \leq \phi_2(x, y)\}$$

where U is an elementary region in $\mathbb{R}^2$ and ϕ_1, ϕ_2 are continuous functions on U[10].

Theorem 102 (Fubini's theorem for elementary regions). Let $D \subset \mathbb{R}^{n-1}$ be a compact set, $\varphi_1, \varphi_2 : D \to \mathbb{R}$ be continuous functions such that $\varphi_1(x) \leq \varphi_2(x)$ $\forall x \in D$, $S = \{(x, y) \subset \mathbb{R}^n : x \in D, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ be an elementary region in $\mathbb{R}^n$ and $f : S \to \mathbb{R}$. If f is integrable over S and for all $x_0 \in D$ the function $f(x_0, y)$ is integrable over $[-M, M]$, $M \in \mathbb{R}$, then:

$$\int_S f = \int_D \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,\mathrm{d}y$$

Definition 103. Let $D \subset \mathbb{R}^{n-1}$ be a compact set, $\varphi_1, \varphi_2 : D \to \mathbb{R}$ be continuous functions such that $\varphi_1(x) \leq \varphi_2(x)$ $\forall x \in D$ and $S = \{(x, y) \subset \mathbb{R}^n : x \in D, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ an elementary region. We define the *n-dimensional volume* of S as

$$\mathrm{vol}(S) := \int_S \mathrm{d}x = \int_D \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_2(x)} \mathrm{d}y$$ [11]

Corollary 104 (Cavalieri's principle). Let $\Omega \subset R \times [a, b]$ be a set in $\mathbb{R}^n$ where $R \subset \mathbb{R}^{n-1}$ is a rectangle. For every $t \in [a, b]$ let

$$\Omega_t = \{(x, y) \in \Omega : y = t\} \subset \mathbb{R}^n$$

be the section of Ω corresponding to the hyperplane $y = t$. If $\nu(\Omega_t)$ is the $(n-1)$-dimensional volume (length if $n = 2$ and area if $n = 3$) of Ω_t, then:

$$\mathrm{vol}(\Omega) = \int_a^b \nu(\Omega_t)\,\mathrm{d}t$$

Definition 105 (Center of mass). The *center of mass* of an object with mass density $\rho(x, y, z)$ occupying a region $\Omega \subset \mathbb{R}^3$ is the point $(\overline{x}, \overline{y}, \overline{z}) \in \mathbb{R}^3$ whose coordinates are:

$$\overline{x} = \frac{1}{m}\int_\Omega x\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$
$$\overline{y} = \frac{1}{m}\int_\Omega y\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$
$$\overline{z} = \frac{1}{m}\int_\Omega z\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$

where $m = \displaystyle\int_\Omega \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is the total mass of the object.

Definition 106 (Moment of inertia). Given a body with mass density $\rho(x, y, z)$ occupying a region $\Omega \subset \mathbb{R}^3$ and a line $L \subset \mathbb{R}^3$, the *moment of inertia* of the body about the line L is:

$$I_L = \int_\Omega d(x, y, z)^2 \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where $d(x, y, z)$ denotes the distance from (x, y, z) to the line L. In particular, when L is the z-axis, then:

$$I_z = \int_\Omega (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and similarly for I_x and I_y. The moment of inertia of the body about the xy-plane is defined by:

$$I_{xy} = \int_\Omega z^2 \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and similarly for I_{yz} and I_{zx}.

Change of variable

Theorem 107 (Change of variable theorem). Let $U \subseteq \mathbb{R}^n$ be an open set and let $\boldsymbol{\varphi} : U \to \mathbb{R}^n$ be a diffeomorphism. If $f : \boldsymbol{\varphi}(U) \to \mathbb{R}$ is integrable on $\boldsymbol{\varphi}(U)$, then:

$$\int_{\boldsymbol{\varphi}(U)} f = \int_U (f \circ \boldsymbol{\varphi})|J\boldsymbol{\varphi}|$$

[10] Analogously we define *y-simple* regions in $\mathbb{R}^2$ and *yz-simple* or *xz-simple* regions in $\mathbb{R}^3$.

[11] In particular, we define the area of a region $S \subset \mathbb{R}^2$ as $\mathrm{area}(S) = \displaystyle\int_S \mathrm{d}x\,\mathrm{d}y$ and the volume of a region $\Omega \subset \mathbb{R}^3$ as $\mathrm{vol}(\Omega) = \displaystyle\int_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Corollary 108 (Integral in polar coordinates). Let $\varphi : U \subseteq [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ be such that:

$$\varphi(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

Then, we have $|J\varphi| = r$ and therefore:

$$\int_{\varphi(U)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_U f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

Corollary 109 (Integral in cylindrical coordinates). Let $\varphi : U \subseteq [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ be such that:

$$\varphi(r, \theta, z) \longmapsto (r\cos\theta, r\sin\theta, z)$$

Then, we have $|J\varphi| = r$ and therefore:

$$\int_{\varphi(U)} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_U f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

Corollary 110 (Integral in spherical coordinates). Let $\varphi : U \subseteq [0, \infty) \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$ be such that:

$$\varphi(\rho, \theta, \phi) \longmapsto (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

Then, we have $|J\varphi| = \rho^2 \sin\phi$ and therefore:

$$\int_{\varphi(U)} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_U f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

5. | Vector calculus

Arc-length and line integrals

Definition 111. Let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a parametrization of a curve and $\mathcal{P} = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Then, the *length of the polygonal* created from the vertices $\boldsymbol{\gamma}(t_i)$, $i = 1, \ldots, n$, is:

$$L(\boldsymbol{\gamma}, \mathcal{P}) = \sum_{i=1}^{n} \|\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1})\|$$

Definition 112. Let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a parametrization of a curve C. The *arc length* of C is

$$L(C) = \sup\{L(\boldsymbol{\gamma}, \mathcal{P}) : \mathcal{P} \in \mathbf{P}([a, b])\} \in [0, \infty]$$

Definition 113. We say that a curve C is *rectifiable* if it has a finite arc length, that is, if $L(C) < \infty$.

Proposition 114. Let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ be a parametrization of class $\mathcal{C}^1$ of a curve C. Then C is rectifiable and

$$L(C) = \int_a^b \|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t$$ [12]

Definition 115. Let $\mathbf{F} : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a vector field[13]. If all its component functions F_i are integrable, we define:

$$\int_U \mathbf{F} := \left(\int_U F_1, \ldots, \int_U F_n\right) \in \mathbb{R}^n$$

Definition 116. Let C be a curve in $\mathbb{R}^2$ parametrized by $\boldsymbol{\gamma} = (x(t), y(t))$. The unit tangent vector to the curve at time t is:

$$\mathbf{t} = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$$

The normal vector to the curve is $N(t) = (y'(t), -x'(t))$ and the unit normal vector to the curve is:

$$\mathbf{n} = \frac{N(t)}{\|N(t)\|}$$ [14]

Definition 117. Let C be a curve parametrized by $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ and $\varphi : [c, d] \to [a, b]$ be a diffeomorphism. The composition $\boldsymbol{\gamma} \circ \varphi : [c, d] \to \mathbb{R}^n$ is called a *reparametrization* of C.

Definition 118. Let C be a curve of class $\mathcal{C}^1$ parametrized by $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ an L be its arc length. We define the *arc length parameter* as:

$$s(t) = \int_a^t \|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t$$

We reparametrize C by $\rho(s) = \boldsymbol{\gamma}(t(s))$, $0 \leq s \leq L$. Then $\rho'(s)$ is a unit tangent vector to C and $\rho''(s)$ is perpendicular to C at the point $\rho(s)$.

Definition 119. Let C be a curve of class $\mathcal{C}^2$ and s be its arc length parameter. We define the *curvature* of C at the point $\rho(s)$ as

$$\kappa(\rho(s)) = \|\rho''(s)\|$$

Definition 120. Let $C = \{\boldsymbol{\gamma}(t) : t \in [a, b]\} \subset \mathbb{R}^n$ be a curve of class $\mathcal{C}^1$ and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous function. We define the *line integral* of f along C as:

$$\int_C f\,\mathrm{d}s = \int_a^b f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t$$ [15]

[12] It can be seen that the arc length of a curve does not depend on its parametrization.
[13] A *vector field* is nothing more than a vector-valued function.
[14] Observe that $-N(t)$ is also a normal vector to the curve but, by agreement, we take the one pointing to the right of the curve or, if the curve is closed, the one pointing outwards from the curve.
[15] It can be seen that this integral is independent of the parametrization of C.
[16] It can be seen that the latter integral is independent of the parametrization of C except for a factor of -1 that depends on the orientation of the parametrization.

Definition 121. Let $C = \{\boldsymbol{\gamma}(t) : t \in [a,b]\} \subset \mathbb{R}^n$ be a curve of class $\mathcal{C}^1$ and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. We define the *line integral* of $\mathbf{F}$ along C as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{t} \, \mathrm{d}s = \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \, \mathrm{d}t$$

where $\mathbf{t}$ is the unit tangent vector to C[16]. If C is closed, this integral is called the *circulation* of $\mathbf{F}$ around C.

Definition 122. A *Jordan arc* is the image of an injective continuous map $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$. A *Jordan closed curve* is the image of an injective continuous map $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ such that $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$.

Conservative vector fields

Definition 123. Let $U \subseteq \mathbb{R}^n$ be a domain and $f : U \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. We say that $\mathbf{F} : U \to \mathbb{R}^n$ is a *conservative* or a *gradient vector field* if

$$\mathbf{F}(x) = \boldsymbol{\nabla} f(x) \qquad \forall x \in U$$

The function f is called the *potential* of $\mathbf{F}$.

Theorem 124. Let $\mathbf{F} = \boldsymbol{\nabla} f$ be a conservative vector field on $U \subseteq \mathbb{R}^n$ and C be a closed curve that admits a parametrization $\boldsymbol{\gamma}(t) : [a,b] \to \mathbb{R}^n$ of class $\mathcal{C}^1(U)$. Then:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a))$$

Corollary 125. Let $\mathbf{F}$ be a conservative vector field on U and C be a closed curve that admits a parametrization of class $\mathcal{C}^1(U)$. Then $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0$.

Divergence, curl and Laplacian

Definition 126. Let $\mathbf{F} = (F_1, \ldots, F_n)$ be a vector field of class $\mathcal{C}^1(U)$, $U \subseteq \mathbb{R}^n$. The *divergence* of $\mathbf{F}$ is:

$$\mathbf{div}\,\mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i=1}^n \frac{\partial F_j}{\partial x_j}$$

Definition 127. Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field of class $\mathcal{C}^1(U)$, $U \subseteq \mathbb{R}^3$. The *curl* of $\mathbf{F}$ is:

$$\mathbf{rot}\,\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \\ = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Definition 128. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2(U)$, $U \subseteq \mathbb{R}^3$. The *Laplacian* of f is

$$\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_j^2}$$

Proposition 129. Let U be an open set of $\mathbb{R}^3$ and $f : U \to \mathbb{R}$, $\mathbf{g} : U \to \mathbb{R}^3$ be functions of class $\mathcal{C}^2(U)$. Then for all $x \in U$ we have:

$$\mathbf{rot}(\boldsymbol{\nabla} f) = 0 \qquad \mathbf{div}(\mathbf{rot}\,\mathbf{g}) = 0 \quad \text{and} \quad \mathbf{div}(\boldsymbol{\nabla} f) = \Delta f$$

Surface area and surface integrals

Proposition 130. Let S be the graph of a function $z = f(x,y)$ of class $\mathcal{C}^1(U)$, $U \subseteq \mathbb{R}^2$. Then

$$\operatorname{area}(S) = \int_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, \mathrm{d}x \, \mathrm{d}y$$

Definition 131. A *parametrized surface* $S \subset \mathbb{R}^3$ is the image of a map $\boldsymbol{\Phi} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ of class $\mathcal{C}^1(U)$ defined by $\boldsymbol{\Phi}(u,v) = (x(u,v), y(u,v), z(u,v))$.

Proposition 132. Let $S = \boldsymbol{\Phi}(U)$ be a surface in $\mathbb{R}^3$ parametrized by $\boldsymbol{\Phi} \in \mathcal{C}^1(U)$. Then the unit normal vector to S at the point $\boldsymbol{\Phi}(u,v)$ is

$$\mathbf{n}(u,v) = \frac{\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}}{\left\| \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right\|}$$

Proposition 133. Let $S = \boldsymbol{\Phi}(U)$ be a surface in $\mathbb{R}^3$ parametrized by $\boldsymbol{\Phi} \in \mathcal{C}^1(U)$. Then:

$$\operatorname{area}(S) = \int_U \left\| \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right\| \mathrm{d}u \, \mathrm{d}v$$

Definition 134. Let $S = \boldsymbol{\Phi}(U)$ be a surface in $\mathbb{R}^3$ parametrized by $\boldsymbol{\Phi} \in \mathcal{C}^1(U)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function whose domain contain S. We define the *surface integral* f over S as:

$$\int_S f \, \mathrm{d}S = \int_U f(\boldsymbol{\Phi}(u,v)) \left\| \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right\| \mathrm{d}u \, \mathrm{d}v$$ [17]

Definition 135. Let $S = \boldsymbol{\Phi}(U)$ be a surface in $\mathbb{R}^3$ parametrized by $\boldsymbol{\Phi} \in \mathcal{C}^1(U)$ and $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field whose domain contain S. We define the *surface integral* $\mathbf{f}$ over S or the *flux* of $\mathbf{f}$ across S as:

$$\int_S \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{f} \cdot \mathbf{n} \, \mathrm{d}S = \\ = \int_U \mathbf{f}(\boldsymbol{\Phi}(u,v)) \cdot \left(\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v$$

where $\mathbf{n}$ is the unit normal vector to S[18].

[17] It can be seen that this integral is independent of the parametrization of S.

[18] It can be seen that the latter integral is independent of the parametrization of S except for a factor of -1 that depends on the orientation of the normal vector $\mathbf{n}$.

Theorems of vector calculus on $\mathbb{R}^2$

Definition 136. Let $U \subseteq \mathbb{R}^3$ be an open set. A *differential 1-form* on U is an expression of the form

$$\omega = f_1 \, dx + f_2 \, dy + f_3 \, dz$$

where f_1, f_2, f_3 are scalar functions defined on U[19].

Theorem 137 (Green's theorem). Let $\mathbf{F} = (F_1, F_2)$ be a vector field of class $\mathcal{C}^1(U)$, $U \subseteq \mathbb{R}^2$, and $c = \partial U$ be the curve formed from the boundary of U[20]. Then:

$$\int_{\partial U} \mathbf{F} \cdot d\mathbf{s} = \int_U \mathbf{rot}\, \mathbf{F} \, dx \, dy$$ [21]

Corollary 138. Let U be a region in $\mathbb{R}^2$ and ∂U be its boundary. Then:

$$\text{area}(U) = \int_{\partial U} x \, dy = -\int_{\partial U} y \, dx = \frac{1}{2} \int_{\partial U} (x \, dy - y \, dx)$$

Theorem 139 (Divergence theorem on $\mathbb{R}^2$). Let $\mathbf{F} = (F_1, F_2)$ be a vector field of class $\mathcal{C}^1(U)$, $U \subseteq \mathbb{R}^2$ with boundary ∂U. Then:

$$\int_{\partial U} \mathbf{F} \cdot \mathbf{n} \, ds = \int_U \mathbf{div}\, \mathbf{F} \, dx \, dy$$ [22]

Theorems of vector calculus on $\mathbb{R}^3$

Theorem 140 (Stokes' theorem). Let S be a parametrized surface of class $\mathcal{C}^1$ and ∂S be its boundary. Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field of class $\mathcal{C}^1$ in a domain containing $S \cup \partial S$. Then:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_S \mathbf{rot}\, \mathbf{F} \cdot \mathbf{n} \, dS$$

Corollary 141. Let $a \in \mathbb{R}^3$ and $\mathbf{n}$ be a unit vector. Suppose $D_r = D(a, r)$ is a disk of radius r centered at a and perpendicular to $\mathbf{n}$. Let $\mathbf{F}$ be a vector field of class $\mathcal{C}^1(D_r)$. Then:

$$\mathbf{rot}\, \mathbf{F}(a) \cdot \mathbf{n} = \lim_{r \to 0} \frac{1}{\text{area}(D_r)} \int_{\partial D_r} \mathbf{F} \cdot d\mathbf{s}$$

Therefore, the $\mathbf{n}$-th component of $\mathbf{rot}\, \mathbf{F}(a)$ is the circulation of $\mathbf{F}$ in a small circular surface perpendicular to $\mathbf{n}$, per unit of area.

Definition 142. A region of $\mathbb{R}^3$ is *symmetric* if is xy-simple, yz-simple and xz-simple.

Theorem 143 (Divergence theorem on $\mathbb{R}^3$). Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field of class $\mathcal{C}^1$ on a symmetric region $V \subset \mathbb{R}^3$ with boundary ∂V. Then:

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = \int_V \mathbf{div}\, \mathbf{F} \, dx \, dy \, dz$$

Corollary 144. Let $B_r = B(a, r)$ be a ball of radius r centered at $a \in \mathbb{R}^3$ and $\mathbf{F}$ be a vector field of class $\mathcal{C}^1(B_r)$. Then:

$$\mathbf{div}\, \mathbf{F}(a) = \lim_{r \to 0} \frac{1}{\text{vol}(B_r)} \int_{\partial B_r} \mathbf{F} \cdot \mathbf{n} \, dS$$

Therefore, $\mathbf{div}\, \mathbf{F}(a)$ is the flux of $\mathbf{F}$ outward form a, in the normal direction across the surface of a small ball centered on a, per unit of volume.

[19] Extending this notion, we can define *2-forms* and *3-forms* as:

$$\begin{aligned} \omega &= f_1 \, dx \, dy + f_2 \, dy \, dz + f_3 \, dz \, dy && \text{2-form} \\ \omega &= f \, dx \, dy \, dz && \text{3-form} \end{aligned}$$

[20] It goes without saying that the orientation is chosen positive, that is counterclockwise.

[21] Alternatively, using differential forms, we get

$$\int_{\partial U} (F_1 \, dx + F_2 \, dy) = \int_U \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \, dy$$

[22] The first integral represents the flux of $\mathbf{F}$ across the curve ∂U.