Real-valued functions

1. | The real line

Definition 1. Let $(K, +, \cdot)$ be a field. We say that K, together with a total order relation $\leq$[1], is an *ordered field* if the following properties are satisfied:

1. If $x, y, z \in K$ are such that $x \leq y$, then $x+z \leq y+z$.
2. If $x, y \in K$ are such that $x \geq 0$ and $y \geq 0$, then $x \cdot y \geq 0$.

Definition 2. Let K be an ordered field and $A \subset K$. We say that A is *bounded from above* if $\exists M \in K$ (called *upper bound* of A) such that $x \leq M$ $\forall x \in A$. Analogously, we say that A is *bounded from below* if $\exists m \in K$ (called *lower bound* of A) such that $x \geq m$ $\forall x \in A$.

Definition 3. Let K be an ordered field and $A \subset K$ be a set bounded from above. We say that an upper bound α of A is the *supremum* of A, denoted by $\sup A$, if any other upper bound α' satisfies $\alpha' \geq \alpha$. Analogously if $B \subset K$ is a set bounded from below, we say that a lower bound β of B is the *infimum* of B, denoted by $\inf B$, if any other lower bound β' satisfies $\beta' \leq \beta$.

Proposition 4. Let K be an ordered field and $A \subset K$. If M is an upper bound of A, then $-M$ is a lower bound of $-A$. Similarly, if m is a lower bound of A, then $-m$ is a upper bound of $-A$

Proposition 5. Let K be an ordered field and $A, B \subset K$. If $\alpha = \sup A$ and $\beta = \inf B$, then:

$$-\alpha = \inf(-A) \qquad -\beta = \sup(-B)$$

Proposition 6. The supremum of a set, if exists, is unique.

Theorem 7 (Supremum axiom). There exists a unique field with the property that any bounded set from above has a supremum: the field of real numbers $\mathbb{R}$.

Proposition 8. Natural numbers are not bounded from above in $\mathbb{R}$.

Corollary 9 (Archimedean property). Let $\alpha \in \mathbb{R}$. Then, $\exists n \in \mathbb{N}$ such that $\alpha < n$.

Corollary 10. Let $\alpha \in \mathbb{R}_{>0}$. Then, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \alpha$.

Proposition 11. Let $x, y \in \mathbb{R}$ such that $x < y$. Then, there exist numbers $z \in \mathbb{R} \setminus \mathbb{Q}$ and $q \in \mathbb{Q}$ such that $x < z < y$ and $x < q < y$.

Definition 12. Given $x, y \in \mathbb{R}$ such that $x < y$ we define:

- *Open interval*: $(x, y) = \{z \in \mathbb{R} : x < z < y\}$.
- *Right-open interval*: $[x, y) = \{z \in \mathbb{R} : x \leq z < y\}$.
- *Left-open interval*: $(x, y] = \{z \in \mathbb{R} : x < z \leq y\}$.
- *Closed interval*: $[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$.

Lemma 13. Let K be an ordered field and $A \subset K$ be a set. If $\alpha = \sup A$, then $\forall \varepsilon > 0$ the interval $(\alpha - \varepsilon, \alpha]$ contains points of A.

Definition 14. Let $x \in \mathbb{R}$. We define the *absolute value* $|x|$ of x as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Lemma 15. Let $x, y \in \mathbb{R}$. Then:

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|xy| = |x||y|$
4. $|x + y| \leq |x| + |y|$ (*Triangular inequality*)

Definition 16. Let $x \in \mathbb{R}$. A *neighbourhood* of x is any open interval containing x.

Infinite and countable sets

Definition 17. A $X \neq \varnothing$ is *infinite* if there exist $\varnothing \neq A \subset X$ and $\phi : X \to A$ such that ϕ is a bijection. If no such A and ϕ exist, X is *finite*.

Proposition 18. Let X, Y be sets such that $X \subseteq Y$. If X is infinite, Y is infinite.

Proposition 19. Let $X \subset \mathbb{N}$. X is finite if and only if X is bounded.

Definition 20. Let A be a set. We say that A is *countable* if there exists a bijective function from A to $\mathbb{N}$. We say that A is *uncountable* if there is no such bijection.

Proposition 21. Any infinite subset of $\mathbb{N}$ is countable.

Corollary 22. Any subset of a countable set is either finite or countable.

Corollary 23. Let A be an infinite set. A is countable if and only if there exists an injective function from A to $\mathbb{N}$.

Proposition 24. If A and B are countable sets, then $A \times B$ is also countable.

Theorem 25. $\mathbb{Q}$ is countable.

Theorem 26. $\mathbb{R}$ is uncountable.

[1] See **??**

2. | Sequences

Limit notion

Definition 27. A *sequence of real numbers* is an enumerated collection of real numbers. More formally, a sequence is a function $a : \mathbb{N} \to \mathbb{R}$. The number $a(n)$ is usually denoted by a_n and the whole sequence by (a_n).

Definition 28. A sequence (a_n) is *bounded from above* if there is a real number M such that $a_n \leq M$ $\forall n \in \mathbb{N}$. Analogously, (a_n) is *bounded from below* if there is a real number m such that $a_n \geq m$ $\forall n \in \mathbb{N}$. Finally, we say that (a_n) is *bounded* if there exist $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$ $\forall n \in \mathbb{N}$.

Definition 29 (Limit). Let (a_n) be a sequence of real numbers and $\ell \in \mathbb{R}$. We say that $\lim\limits_{n\to\infty} a_n$ if $\forall \varepsilon > 0$ $\exists n_0$ such that $|a_n - \ell| < \varepsilon$ $\forall n > n_0$. We say that $\lim\limits_{n\to\infty} a_n = \pm\infty$ if $\forall M > 0$ $\exists n_0$ such that $\pm a_n > M$ $\forall n > n_0$.

Definition 30. We say a sequence is *convergent* if it has a limit, and *divergent* otherwise.

Lemma 31. The limit of a convergent sequence is unique.

Lemma 32. Let (a_n) be a convergent sequence. Then, (a_n) is bounded. Moreover, if $m \leq a_n \leq M$ $\forall n \in \mathbb{N}$, then $m \leq \lim\limits_{n\to\infty} a_n \leq M$.

Lemma 33. Let (a_n) and (b_n) be convergent sequences with respective limits α and β. Then:

1. The sequences $(a_n + b_n)$ and $(a_n b_n)$ are convergents and
$$\lim_{n\to\infty} a_n + b_n = \alpha + \beta \qquad \lim_{n\to\infty} a_n \cdot b_n = \alpha \cdot \beta$$
2. If $\alpha \neq 0$, then $a_n \neq 0$ for n sufficiently large, the sequence $\left(\frac{b_n}{a_n}\right)$ is convergent and
$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{\beta}{\alpha}$$

Definition 34. Let (a_n) be a sequence. We say (a_n) is *monotonically increasing* if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$. Analogously, we say (a_n) is *monotonically decreasing* if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$[2]. Finally, we say (a_n) is *monotonic* if it is either monotonically increasing or monotonically decreasing.

Theorem 35. All monotonic and bounded sequences are convergent.

Lemma 36. Let (a_n) and (b_n) be two sequences verifying $a_n \leq b_n$ $\forall n \in \mathbb{N}$. Then, $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

Proposition 37 (Squeeze theorem). Let (a_n), (b_n) and (c_n) be three sequences verifying $a_n \leq b_n \leq c_n$ $\forall n \in \mathbb{N}$ and such that (a_n) and (c_n) are convergent. Suppose that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \ell$. Then, (b_n) is convergent and $\lim\limits_{n\to\infty} b_n = \ell$.

Lemma 38. Let $p \in \mathbb{R}_{>0}$ and $\alpha, x \in \mathbb{R}$. Then:

1. $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.
2. $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.
3. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
4. If $x > 1$, $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{x^n} = 0$.
5. If $x < 1$, $\lim\limits_{n\to\infty} x^n = 0$.

Theorem 39 (Root test). Let $(a_n) \geq 0$ be a sequence. Suppose that the limit $\ell = \lim\limits_{n\to\infty} \sqrt[n]{a_n}$ exists.

1. If $\ell < 1 \implies \lim\limits_{n\to\infty} a_n = 0$.
2. If $\ell > 1 \implies \lim\limits_{n\to\infty} a_n = +\infty$.

Theorem 40 (Ratio test). Let $(a_n) \geq 0$ be a sequence. Suppose that the limit $\ell = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists.

1. If $\ell < 1 \implies \lim\limits_{n\to\infty} a_n = 0$.
2. If $\ell > 1 \implies \lim\limits_{n\to\infty} a_n = +\infty$.

Theorem 41. Let $(a_n) \geq 0$ be a sequence. If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \ell$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \ell$.

The number e

Definition 42. We define the sequences (S_n) and (T_n) as:
$$S_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \qquad T_n = \left(1 + \frac{1}{n}\right)^n$$

Proposition 43. The sequences (S_n) and (T_n) are convergent and have the same limit. This limit is denoted by e and it's equal to $\mathrm{e} = 2.71828...$

Theorem 44. The number e is irrational.

Subsequences

Definition 45 (Subsequence). Let (a_n) be a sequence of real numbers and (k_n) be an increasing sequence of natural numbers. The sequence (a_{k_n}) is called a *subsequence* of (a_n).

Lemma 46. Let (a_n) be a sequence. If $\lim\limits_{n\to\infty} a_n = \ell$, then any subsequence of (a_n) has limit ℓ.

Definition 47. Let (a_n) be a sequence. We say p is an *accumulation point* of (a_n) if $\forall \varepsilon > 0$ and $\forall n_0 \in \mathbb{N}$ $\exists n > n_0$ such that $|a_n - p| < \varepsilon$.

Proposition 48. Let (a_n) be a sequence. p is an accumulation point of (a_n) if and only if there is a subsequence (a_{k_n}) of (a_n) with $\lim\limits_{n\to\infty} a_{k_n} = p$.

[2] If the inequalities are strict, we say that (a_n) is *strictly increasing* or *strictly decreasing*, respectively.

Corollary 49. A convergent sequence has its limit as the unique accumulation point.

Proposition 50. All sequences have a monotonic subsequence.

Theorem 51 (Bolzano-Weierstraß theorem). All bounded sequences have a convergent subsequence.

Proposition 52. Let (a_n) be a bounded sequence. Then, (a_n) is convergent if and only if it has a unique accumulation point.

Definition 53. Let (a_n) be a sequence. We define the *limit superior* of (a_n) as:

$$\limsup_{n\to\infty} a_n := \inf\{\sup\{x_m : m \geq n\} : n \geq 0\}$$

We define the *limit inferior* of (a_n) as:

$$\liminf_{n\to\infty} a_n := \sup\{\inf\{x_m : m \geq n\} : n \geq 0\}$$

Proposition 54. Let (a_n) be a sequence. Then $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ always exist and

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

If, moreover, (a_n) is bounded, then for all accumulation point $p \in \mathbb{R}$ of (a_n) we have:

$$\liminf_{n\to\infty} a_n \leq p \leq \limsup_{n\to\infty} a_n$$

Proposition 55. Let (a_n) be a bounded sequence. Then:

$$(a_n) \text{ is convergent } \iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

In this case we have:

$$\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$$

Cauchy condition

Definition 56 (Cauchy sequence). We say that a sequence (a_n) is a *Cauchy sequence* if $\forall\varepsilon > 0\ \exists n_0$ such that $|a_n - a_m| < \varepsilon\ \forall n, m > n_0$.

Theorem 57. A sequence is convergent if and only if it's a Cauchy sequence.

Theorem 58 (Stolz-Cesàro theorem). Let (a_n) be a strictly increasing sequence and (b_n) be any other sequence. Suppose that

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{a_n - a_{n-1}} = \ell \in \mathbb{R} \cup \{\pm\infty\}$$

Then:

1. If $\lim_{n\to\infty} a_n = \pm\infty$, $\lim_{n\to\infty} \frac{b_n}{a_n} = \ell$.
2. If $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} \frac{b_n}{a_n} = \ell$.

3. | Continuity

Limit of a function

Definition 59. Let $f : [a,b] \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. We say that ℓ is the *limit of the function f at the point x_0*, denoted by $\lim_{x\to x_0} f(x) = \ell$, if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $|x - x_0| < \delta$.

Lemma 60. Let $f : (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. Then, $\lim_{x\to x_0} f(x) = \ell$ if and only if for any sequence $(a_n) \subset (a,b) \setminus \{x_0\}$ with $\lim_{n\to\infty} a_n = x_0$ we have $\lim_{n\to\infty} f(a_n) = \ell$.

Lemma 61. The limit of a function at a point, if exists, is unique.

Proposition 62. Let $f, g : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$ and suppose that $\lim_{x\to x_0} f(x) = \ell_1$ and $\lim_{x\to x_0} g(x) = \ell_2$. Then, the following properties are satisfied:

1. $\lim_{x\to x_0} (f+g)(x) = \ell_1 + \ell_2$.
2. $\lim_{x\to x_0} (f \cdot g)(x) = \ell_1 \cdot \ell_2$.
3. If $\ell_1 > 0$, then $f(x) > 0$ on a neighbourhood of x_0. And if $\ell_1 < 0$, then $f(x) < 0$ on a neighbourhood of x_0. Moreover in both cases $\lim_{x\to x_0} \left(\frac{1}{f}\right)(x) = \frac{1}{\ell_1}$.

Definition 63. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. We say that f is *bounded* on I if there are $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M \quad \forall x \in I$$

Lemma 64. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$. If the limit of f at x_0 exists, then f is bounded on a neighbourhood of x_0.

Definition 65. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$. We say that the limit of f at x_0 is infinite, denoted by $\lim_{x\to x_0} f(x) = \pm\infty$, if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $\pm f(x) > \varepsilon$ whenever $|x - x_0| < \delta$.

Lemma 66. Let $f : (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. Then, $\lim_{x\to x_0} f(x) = \pm\infty$ if and only if for all sequence $(a_n) \subset (a,b) \setminus \{x_0\}$ with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = \pm\infty$.

Definition 67. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$. We say that ℓ is the *right-sided limit of f at x_0*, denoted by $\lim_{x\to x_0^+} f(x) = \ell$, if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x - x_0 < \delta$. Analogously, we say that ℓ is the *left-sided limit of f at x_0*, denoted by $\lim_{x\to x_0^-} f(x) = \ell$, if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x_0 - x < \delta$.

Lemma 68. Let $f : (a,b) \to \mathbb{R}$ and $x_0 \in (a,b)$. Then:

$$\lim_{x\to x_0} f(x) = \ell \iff \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \ell$$

Definition 69. Let $f : (a, \infty) \to \mathbb{R}$. We say that ℓ is the *limit of f at infinity*, denoted by $\lim_{x\to\infty} f(x) = \ell$, if $\forall\varepsilon > 0$ $\exists K > a$ such that $|f(x) - \ell| < \varepsilon$ for all $x > K$.

Definition 70. Let $f : (a, \infty) \to \mathbb{R}$. We say that the limit of f at infinity is infinity, denoted by $\lim_{x\to\infty} f(x) = \pm\infty$, if $\forall K > 0$ $\exists M > a$ such that $\pm f(x) > K$ for all $x > M$.

Continuity

Definition 71. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$. We say that f is *continuous at x_0* if the limit of f at x_0 exists and it's equal to $f(x_0)$[3]. We say that f is *continuous* on I if it's continuous at all points of I.

Lemma 72. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. f is continuous at $x_0 \in I$ if and only if for all sequence $(a_n) \subset I$ with $\lim_{n\to\infty} a_n = x_0$ we have that $\lim_{n\to\infty} f(a_n) = f(x_0)$.

Proposition 73. Let $I \subset \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$ be continuous functions at $x_0 \in I$. Then:

1. $f + g$ and $f \cdot g$ are continuous at x_0.
2. If $f(x_0) > 0$, then $f(x) > 0$ on a neighbourhood of x_0. And if $f(x_0) < 0$, then $f(x) < 0$ on a neighbourhood of x_0. Moreover, in both cases, $\frac{1}{f}$ is continuous at x_0.

Proposition 74. Let $I, J \subset \mathbb{R}$ be intervals, $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$. Let $x_0 \in I$ with $f(x_0) \in J$ and suppose that f is continuous at x_0 and g is continuous at $f(x_0)$. Then, $g \circ f$ is continuous at x_0.

Theorem 75 (Weierstraß theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, f is bounded on $[a, b]$. Moreover, $\exists m, M \in [a, b]$ such that:

$$f(m) \le f(x) \le f(M) \quad \forall x \in [a, b]$$

Theorem 76 (Bolzano's theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)\cdot f(b) < 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$.

Corollary 77 (Intermediate value theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $c \in \langle f(a), f(b)\rangle$[4]. Then, $\exists z \in (a, b)$ such that $f(z) = c$.

Corollary 78. All real numbers have a unique positive n-th root.

Continuity of inverse function

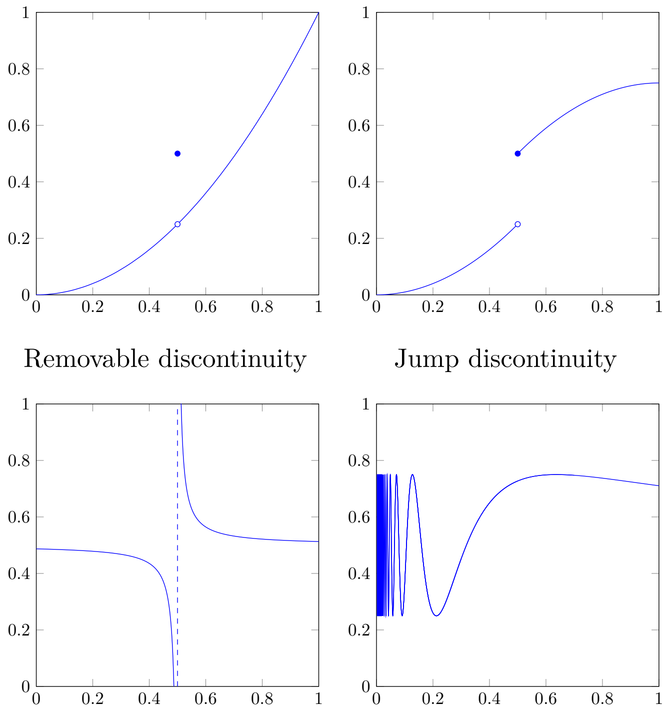
Definition 79. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. We say that f is *increasing* on I if $f(x) \le f(y)$ whenever $x \le y$. We say that f is *decreasing* on I if $f(x) \ge f(y)$ whenever $x \le y$[5]. We say that f is *monotonic* if it is either increasing or decreasing.

Theorem 80. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. If f is injective and continuous, then f is monotonic. Moreover, f^{-1} is also continuous on $f((a, b))$.

Classification of discontinuities

Definition 81. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Suppose f is not continuous at $x_0 \in I$. There are mainly four types of discontinuities:

1. *Removable discontinuity*: The limit $\lim_{x\to x_0} f(x)$ exists but
$$\lim_{x\to x_0} f(x) \ne f(x_0)$$
2. *Jump discontinuity*: The one-sided limits $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist but
$$\lim_{x\to x_0^+} f(x) \ne \lim_{x\to x_0^-} f(x)$$
3. *Discontinuity of the first kind*:
$$\text{Either } \lim_{x\to x_0^+} f(x) = \pm\infty \text{ or } \lim_{x\to x_0^-} f(x) = \pm\infty$$
4. *Discontinuity of the second kind*: One one-sided limit does not exist.



Removable discontinuity | Jump discontinuity

Discontinuity of the first kind | Discontinuity of the second kind

Figure 1: Types of discontinuities

[3] If I contains one of its endpoints, the continuity in these points must be defined with the notion of one-sided limit.
[4] The interval $\langle a, b\rangle$ is defined as $\langle a, b\rangle := (\min(a, b), \max(a, b))$.
[5] If the inequalities are strict, we say that f is *strictly increasing* or *strictly decreasing*, respectively.

4. | Exponential and logarithmic functions

Lemma 82. Let $a \in \mathbb{R}_{>0}$ and $f : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = a^x$. The function f has the following properties:

1. $f(x+y) = f(x)f(y)$.
2. If $a > 1$, f is increasing. If $a < 1$, f is decreasing.
3. If $(a_n) \subset \mathbb{Q}$ is a sequence with $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} f(a_n) = 1$.

Lemma 83. Let $a, x \in \mathbb{R}$ be such that $a > 0$ and $(x_n) \subset \mathbb{Q}$ be a sequence with $\lim\limits_{n\to\infty} x_n = x$. Then, $\lim\limits_{n\to\infty} a^{x_n}$ exists and does not depend on the sequence (x_n). That is, if $(y_n) \subset \mathbb{Q}$ is another sequence with $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x$, then $\lim\limits_{n\to\infty} a^{x_n} = \lim\limits_{n\to\infty} a^{y_n}$.

Definition 84. Let $a \in \mathbb{R}_{>0}$. We define the *exponential function with base a* as the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(x) = \lim\limits_{n\to\infty} a^{x_n}$, where (x_n) is any sequence of rational numbers $\lim\limits_{n\to\infty} x_n = x$.

Proposition 85. The function g has the following properties:

1. If $x \in \mathbb{Q}$, $\tilde{f}(x) = a^x$.
2. $\tilde{f}(x+y) = \tilde{f}(x)\tilde{f}(y)$.
3. If $a > 1$, $\tilde{f}$ is increasing. If $a < 1$, $\tilde{f}$ is decreasing.
4. $\tilde{f}(x) > 0 \ \forall x \in \mathbb{R}$.
5. $\tilde{f}$ is continuous.
6. If $a > 1$, $\lim\limits_{x\to\infty} \tilde{f}(x) = \infty$ and $\lim\limits_{x\to-\infty} \tilde{f}(x) = 0$.

   If $a < 1$, $\lim\limits_{x\to\infty} \tilde{f}(x) = 0$ and $\lim\limits_{x\to-\infty} \tilde{f}(x) = \infty$[6].

Proposition 86. Let $a, x, y \in \mathbb{R}$ be such that $a > 0$. Then, $(a^x)^y = a^{xy}$.

Definition 87. Let $a \in \mathbb{R}_{>0}$. Since a^x is continuous and monotonic and its image is $(0, \infty)$, it has an associated inverse defined in $(0, \infty)$. This function is denoted by $\log_a(x)$ and it is called *logarithm with base a*[7].

Proposition 88. The logarithm with base $a \in \mathbb{R}_{>0}$ has the following properties:

1. $\log_a$ is continuous.
2. If $a > 1$, $\log_a$ is increasing. If $a < 1$, $\log_a$ is decreasing.
3. If $a > 1$, $\lim\limits_{x\to 0} \log_a(x) = -\infty$ and $\lim\limits_{x\to\infty} \log_a(x) = \infty$.

   If $a < 1$, $\lim\limits_{x\to 0} \log_a(x) = \infty$ and $\lim\limits_{x\to\infty} \log_a(x) = -\infty$.
4. $\log_a(xy) = \log_a(x) + \log_a(y)$.
5. $\log_a(x^y) = y \log_a(x)$.

Proposition 89. Let (a_n) be a sequence such that $\lim\limits_{n\to\infty} a_n = \infty$. Then:

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n}$$

Corollary 90. Let (a_n) be a sequence such that $\lim\limits_{n\to\infty} a_n = \infty$ and $x \in \mathbb{R}$. Then:

$$\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{a_n}\right)^{a_n}$$

Proposition 91. For all $x \in \mathbb{R}_{\geq 0}$ we have:

$$1 + x \leq \mathrm{e}^x \leq 1 + x\mathrm{e}^x$$

5. | Differentiation

Definition of derivative and elementary properties

Definition 92. Let $f : (a,b) \to \mathbb{R}$. We say that f is *differentiable at $x_0 \in (a,b)$* if the following limit exists:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

In this case, we denote this limit by $f'(x_0)$ and we refer to it as the *derivative of f at x_0*. We say f is *differentiable* on (a,b) if it is differentiable at each point of (a,b).

Proposition 93. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function at $x_0 \in I$. The *tangent line to the graph at the point $(x_0, f(x_0))$* is:

$$y(x) = f(x_0) + f'(x_0)(x - x_0)$$

That is, the derivative of f at x_0 is precisely the slope of the tangent line at the point x_0.

Lemma 94. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function at $x_0 \in I$. Then, f is continuous at x_0.

Differentiation rules

Proposition 95. Let f, g be two functions defined on a neighbourhood of a and differentiable at a. Then, $f + g$ and fg are differentiable at a and

1. $(f+a)'(a) = f'(a) + g'(a)$.
2. $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$.

If, moreover, $f(a) \neq 0$, then $\frac{1}{f}$ is defined on a neighbourhood of a, it is differentiable at a and

3. $\left(\dfrac{1}{f}\right)'(a) = -\dfrac{f'(a)}{f(a)^2}$.

Proposition 96 (Chain rule). Let $g : (a,b) \to (c,d)$ and $f : (c,d) \to \mathbb{R}$. Suppose that g is differentiable at $x \in (a,b)$ and f is differentiable at $g(x) \in (c,d)$. Then, $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

[6] From now on, we will denote $\tilde{f}(x)$ simply as $a^x \ \forall x \in \mathbb{R}$.

[7] If the base of the logarithm is the number e, it is common to denote $\log_{\mathrm{e}}(x)$ by $\ln(x)$.

Proposition 97 (Inverse function rule). Let $f : (a, b) \to \mathbb{R}$ be an injective and continuous function on (a, b) and differentiable at $c \in (a, b)$ with $f'(c) \neq 0$. Then, f^{-1} is differentiable at $f(c)$ and

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}$$

$f(x)$	$f'(x)$
x^α	$\alpha x^{\alpha-1}$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$1 + \tan^2(x) = \frac{1}{\cos^2(x)}$
$\cot(x)$	$-1 - \cot^2(x) = -\frac{1}{\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\operatorname{arccot}(x)$	$-\frac{1}{1+x^2}$

Table 1: Table of derivatives of elementary functions

Basic differentiation theorems

Definition 98. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $c \in I$. We say that c is a *local maximum* of f if exists an open interval $J \subset I$ with $c \in J$ such that $f(x) \leq f(c)$ $\forall x \in J$. We say that c is a *local minimum* of f if exists an open interval $J \subset I$ with $c \in J$ such that $f(x) \geq f(c)$ $\forall x \in J$. Finally, a *local extremum* is either a local maximum or a local minimum.

Proposition 99. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $c \in I$ be a local extremum of f. If f is differentiable at c, then $f'(c) = 0$.

Theorem 100 (Rolle's theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable function on (a, b). Suppose $f(a) = f(b)$. Then, there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 101 (Mean value theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then, there exists a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Corollary 102. Let f be a differentiable function on (a, b) verifying that $f'(x) = 0$ $\forall x \in (a, b)$. Then, f is constant in (a, b).

Corollary 103. Let f be a differentiable function on (a, b). If $f'(x) > 0$ $\forall x \in (a, b)$, then f is strictly increasing on (a, b). Similarly, if $f'(x) < 0$ $\forall x \in (a, b)$, then f is strictly decreasing on (a, b).

Corollary 104. Let f be a differentiable function on a neighbourhood of a and such that f' is continuous on this neighbourhood. Suppose that $f'(a) \neq 0$. Then, exists another neighbourhood of a on which f is invertible.

Theorem 105 (Cauchy's mean value theorem). Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ and differentiable on (a, b). Then, there exists a point $c \in (a, b)$ such that

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

Theorem 106 (L'Hôpital's rule). Let f, g be two functions defined on a neighbourhood of $a \in \mathbb{R} \cup \{\pm\infty\}$ and such that either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} g(x) = \infty$. Suppose, moreover, that the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Then, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists too and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Theorem 107 (Darboux's theorem). Let $f : (a, b) \to \mathbb{R}$ be a differentiable function and suppose that there exist $x, y \in (a, b)$, $x < y$, with $f'(x)f'(y) < 0$. Then, there exists $z \in (x, y)$ such that $f'(z) = 0$.

6. | Convexity and concavity

Definition 108. We say that $f : I \to \mathbb{R}$ is *convex* if given any two points $a, b \in I$, $a < b$, the segment between $(a, f(a))$ and $(b, f(b))$ lies above the graph on (a, b). That is:

$$f(bt + (1-t)a) \leq tf(b) + (1-t)f(a) \quad \forall t \in [0, 1]$$

We say that f is *concave* if given any two points $a, b \in I$, $a < b$, the segment between $(a, f(a))$ and $(b, f(b))$ lies below the graph on (a, b). That is:

$$f(bt + (1-t)a) \geq tf(b) + (1-t)f(a) \quad \forall t \in [0, 1]$$ [8]

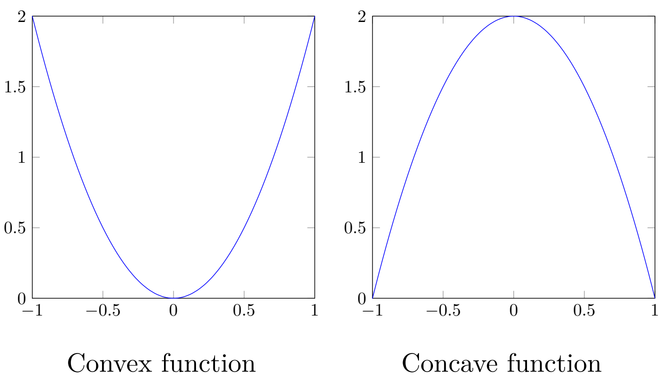


Figure 2

Lemma 109. A function f is convex on an interval I is and only if $-f$ if concave on I.

[8] If the inequalities are strict, we say that f is *strictly convex* or *strictly concave*, respectively.

Lemma 110. Let $f : I \to \mathbb{R}$. f is convex on I if and only if $\forall a, x, b \in I$ with $a < x < b$ we have:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

Or, equivalently:

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

Similarly, f is concave on I if and only if $\forall a, x, b \in I$ with $a < x < b$ we have:

$$\frac{f(x) - f(a)}{x - a} \geq \frac{f(b) - f(a)}{b - a}$$

Or, equivalently:

$$\frac{f(b) - f(a)}{b - a} \geq \frac{f(b) - f(x)}{b - x}$$

Proposition 111. Let f be a convex or concave function on an interval I. Then, f is continuous on I.

Lemma 112. Let f be a differentiable function and $a < b$ be such that $f(a) = f(b)$. Then:

- If f' is increasing, $f(x) \leq f(a)$ $\forall x \in (a, b)$.
- If f' is decreasing, $f(x) \geq f(a)$ $\forall x \in (a, b)$.

Theorem 113. Let f be a differentiable function on an interval I. Then:

- f is (strictly) convex if and only if f' is (strictly) increasing.
- f is (strictly) concave if and only if f' is (strictly) decreasing.

Theorem 114. Let f be a differentiable function on an interval I. Then, f is convex if and only if the graph lies above all its tangent lines. And similarly, f is concave if and only if the graph lies below all its tangent lines.

Definition 115. Let f be a differentiable function on an interval I. If the function $f' : I \to \mathbb{R}$ is differentiable at $a \in I$, we say that f is *two times differentiable at a*. If this happens in all points of I, we say that f is *two times differentiable* on I. In this case we denote the derivative of f' at the point a, $(f')'(a)$, by $f''(a)$ and we refer to it as *second derivative of f at a*.

Theorem 116. Let f be a function two times differentiable on I. Then:

1. f is convex on I if and only if $f''(x) \geq 0$ $\forall x \in I$.
2. f is concave on I if and only if $f''(x) \leq 0$ $\forall x \in I$.

Definition 117. Let $f : I \to \mathbb{R}$. We say that f is convex at $x \in I$ if exists a neighbourhood $J \subset I$ of x on which f is convex. Analogously, we say that f is concave at $x \in I$ if exists a neighbourhood $J \subset I$ of x on which f is concave.

Definition 118. Let f be a continuous function on I. We say $x \in I$ is an *inflection point* if exists $\delta > 0$ such that f is convex (or concave) on $(x - \delta, x]$ and concave (or convex) on $[x, x + \delta)$.

Proposition 119. Let f be a function two times differentiable on I. Then:

1. If a is an inflection point, $f''(a) = 0$.
2. Suppose that f'' is continuous at $a \in I$. Then:
   - If $f''(a) \geq 0$, f is convex at a.
   - If $f''(a) \leq 0$, f is concave at a.

7. | Polynomial approximation

Definition 120. Let f, g be two functions defined on a neighbourhood of $a \in \mathbb{R}$. We say that f and g have *contact of order $\geq n$ at a* if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x - a)^n} = 0$$

Definition 121. Let f be a function. Iterating the process in Theorem 115, one can define the notion of the *n-th derivative of f at the point $a \in \mathbb{R}$*, denoted by $f^{(n)}(a)$.

Definition 122. We say that a function f is of *class $\mathcal{C}^n$ at a point $a \in \mathbb{R}$*, $n \in \mathbb{N}$, if f is n times differentiable at a neighbourhood of a and $f^{(n)}$ is continuous in this neighbourhood. We say that f is of *class $\mathcal{C}^\infty$ at a* if f is of class $\mathcal{C}^n$ at a $\forall n \in \mathbb{N}$. Finally, if $p \in \mathbb{N} \cup \{\infty\}$, we say that f is of *class $\mathcal{C}^p$*, or $\mathcal{C}^p(I)$, on an interval I it it is of class $\mathcal{C}^p$ at all points of I.

Lemma 123. Let f, g be functions n times differentiable at $a \in \mathbb{R}$. Then:

1. If $f^{(i)}(a) = g^{(i)}(a)$, $i = 0, 1, \ldots, n$, and $f^{(n)}$ and $g^{(n)}$ are continuous at a, then f and g have contact of order $\geq n$.
2. If f and g have contact of order $\geq n$, then $f^{(i)}(a) = g^{(i)}(a)$, $i = 0, 1, \ldots, n$.

Theorem 124. Let f be a function n times differentiable at $a \in \mathbb{R}$. Then, the polynomial

$$P_{n,f,a}(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \\ + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

has contact with f of order $\geq n$ at a. This polynomial is called *Taylor polynomial of order n of f centered at a*.

Proposition 125. Let P and Q be polynomials of degree $\leq n$ with order of contact $\geq n$ at a point $a \in \mathbb{R}$. Then $P = Q$[9].

Theorem 126. Let f be a function n times differentiable at $a \in \mathbb{R}$. If $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$ then:

1. If n is odd, a isn't a local extremum of f.

[9] This means that the Taylor polynomial $P_{n,f,a}(x)$ is the unique polynomial which has contact with a function f of order $\geq n$ at a point a.

2. If n is even and $f^{(n)}(a) > 0$, a is a local minimum of f.
3. If n is even and $f^{(n)}(a) < 0$, a is a local maximum of f.

Theorem 127. Let f be a function $n+1$ times differentiable on a neighbourhood I of $a \in \mathbb{R}$. Let $P = P_{n,f,a}$, $R_n := f - P$ and $x \in I$. Then:

1. Cauchy's formula:
$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-a)$$
for some $\xi \in \langle a, x \rangle$.
2. Lagrange's formula:
$$R_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1}$$
for some $\eta \in \langle a, x \rangle$.
3. Integral formula: If $f^{(n+1)}$ is integrable[10] on $[a, x]$:
$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t$$

Definition 128. We say that f is *analytic at a* if it's of class $\mathcal{C}^\infty$ on a neighbourhood I of a and $\lim_{n\to\infty} R_n(x) = 0$ $\forall x \in I$.

$f(x)$	Taylor polynomials
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}$
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$
$(1+x)^\alpha$	$1 + \alpha x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}x^n$
$\arctan(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}$

Table 2: Taylor polynomials centered at 0 of some elementary functions

8. | Riemann integral

Construction of Riemann integral

Definition 129. Let $I = [a, b]$ be an interval. A *partition* $\mathcal{P}$ of I is a finite collection of points $a = t_0 < t_1 < \cdots < t_n = b$ of I. We denote by $\mathrm{P}(I)$ the set of all partitions of the interval I.

[10]See Theorem 134.

Definition 130. Let $f : I \to \mathbb{R}$ be a bounded function and $\mathcal{P} = \{t_i\}_{i=0}^n \in \mathrm{P}(I)$. We define the respective *lower sum* and *upper sum* of f associated with $\mathcal{P}$ as:
$$L(f, \mathcal{P}) = \sum_{i=1}^n m_i(t_i - t_{i-1}) \quad U(f, \mathcal{P}) = \sum_{i=1}^n M_i(t_i - t_{i-1})$$
where $m_i = \inf\{f(x_i) : x_i \in [t_{i-1}, t_i]\}$ and $M_i = \sup\{f(x_i) : x_i \in [t_{i-1}, t_i]\}$.

Definition 131. Let $\mathcal{P}, \mathcal{Q} \in \mathrm{P}(I)$ be two partitions. We say that $\mathcal{P}$ is *finer than* $\mathcal{Q}$, $\mathcal{Q} \prec \mathcal{P}$, if $\mathcal{Q} \subset \mathcal{P}$.

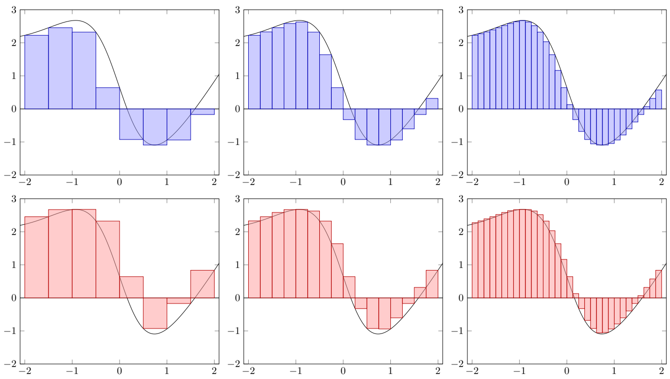


Figure 3: Lower (blue) and upper (red) sums of a function with three different partitions, each one finer than the previous one.

Proposition 132. Let $f : I \to \mathbb{R}$ be a bounded function and $\mathcal{P}, \mathcal{Q} \in \mathrm{P}(I)$ with $\mathcal{Q} \prec \mathcal{P}$. Then:
$$L(f, \mathcal{Q}) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{Q})$$

Definition 133. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. We define the *lower integral* of f on I as:
$$\underline{\int_a^b} f(x) \, \mathrm{d}x = \sup\{L(f, \mathcal{P}) : \mathcal{P} \in \mathrm{P}(I)\}$$
Analogously, we define the *upper integral* of f on I as:
$$\overline{\int_a^b} f(x) \, \mathrm{d}x = \inf\{U(f, \mathcal{P}) : \mathcal{P} \in \mathrm{P}(I)\}$$

Definition 134. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. We say that f is *integrable* on I if
$$\underline{\int_a^b} f(x) \, \mathrm{d}x = \overline{\int_a^b} f(x) \, \mathrm{d}x$$
In this case, we denote the integral of f on I by $\int_a^b f(x) \, \mathrm{d}x$.

Lemma 135. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then, f is integrable on I if and only if $\forall \varepsilon > 0$ $\exists \mathcal{P} \in \mathrm{P}(I)$ such that:
$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

Theorem 136. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a monotonic and bounded function. Then, f is integrable on I.

Definition 137. Let $f : I \to \mathbb{R}$ be a function. We say that f is *uniformly continuous* on I if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

Theorem 138. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function. Then, f is uniformly continuous at I.

Theorem 139. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function. Then, f is integrable on I.

Properties of the integral

Proposition 140. Let f, g be integrable functions on $[a, b]$ and $c \in \mathbb{R}$. Then, $f + g$ and cf are integrable on I and

$$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

Theorem 141. Let f be an integrable function on $[a, b]$ with $f([a, b]) \subseteq [c, d]$ and g be a continuous function on $[c, d]$. Then, $g \circ f$ is integrable on $[a, b]$.

Corollary 142. Let f be an integrable function on $[a, b]$. Then, f^2 is integrable on $[a, b]$. And if there exists $\delta > 0$ with $f(x) > \delta\ \forall x \in [a, b]$, then $\frac{1}{f}$ is integrable on $[a, b]$.

Corollary 143. Let f, g be integrable functions on $[a, b]$. Then, fg is integrable on $[a, b]$.

Inequalities involving integrals

Proposition 144. Let f, g be integrable functions on $[a, b]$ with $f(x) \leq g(x)\ \forall x \in [a, b]$. Then:

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

Corollary 145. Let f be an integrable function on $[a, b]$ with $m \leq f(x) \leq M\ \forall x \in [a, b]$. Then:

$$m(b - a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b - a)$$

If, moreover, f is continuous, there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b - a)$$

Proposition 146. Let f be an integrable function on $[a, b]$. Then, $|f|$ is integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$

Proposition 147. Let f be an integrable function on $[a, b]$ and g be a function defined on $[a, b]$ distinct to f on a finite number points. Then, g is integrable on $[a, b]$ and

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Fundamental theorem of calculus

Proposition 148. Let $f : [a, b] \to \mathbb{R}$ and $b \in (a, c)$. f is integrable on $[a, c]$ if and only if f is integrable on $[a, b]$ and on $[b, c]$. Moreover:

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

Theorem 149 (Fundamental theorem of calculus). Let f be an integrable function on $[a, b]$. Then,

$$F(t) = \int_a^t f(x)\,\mathrm{d}x$$

is a continuous function on $[a, b]$. If, moreover, f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$. Finally, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $F' = f$. In this last case, the function F is called *primitive function* of f.

Theorem 150. Let f be an integrable function on $[a, b]$ which has primitives. Then, these primitives are of the form:

$$F(t) = k + \int_a^t f(x)\,\mathrm{d}x$$

where $k \in \mathbb{R}$. Moreover they satisfy $F' = f$ and

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Corollary 151 (Integration by parts). Let f, g be integrable functions on $[a, b]$ with primitives F and G, respectively. Then:

$$\int_a^b F(x)g(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,\mathrm{d}x$$

Corollary 152 (Integration by substitution). Let $\varphi : [c, d] \to [a, b]$ be a function of class $\mathcal{C}^1$ such that $\varphi(c) = a$ and $\varphi(d) = b$ and f be a continuous function on $[a, b]$. Then:

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d (f \circ \varphi)(x)\varphi'(x)\,\mathrm{d}x$$

Riemann sums

Definition 153. Let $\mathcal{P} = \{t_i\}_{i=0}^n \in \mathrm{P}([a,b])$. A *Riemann sum* of f associated with $\mathcal{P}$, $S(f,\mathcal{P})$, is:

$$S(f,\mathcal{P}) = \sum_{i=1}^n f(x_i)(t_i - t_{i-1})$$

where $x_i \in [t_{i-1}, t_i]$.

Theorem 154. Let f be a continuous function on $[a,b]$. Then, $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $\mathcal{P} = \{t_i\}_{i=0}^n \in \mathrm{P}([a,b])$ with $t_i - t_{i-1} < \delta$, then:

$$\left| \int_a^b f(x)\,\mathrm{d}x - S(f,\mathcal{P}) \right| < \varepsilon$$

for all Riemann sums associated with $\mathcal{P}$.

Corollary 155. Let f be a continuous function on $[a,b]$ and let $\mathcal{P}_n = \{t_i\}_{i=0}^n \in \mathrm{P}([a,b])$ be a sequence of partitions of $[a,b]$ such that $t_i - t_{i-1} < 1/n$. Then, for all Riemann sums $S(f,\mathcal{P}_n)$ we have:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} S(f,\mathcal{P}_n)$$

Geometric applications

Definition 156. Let $f : [a,b] \to \mathbb{R}$ and $\mathcal{P} = \{t_i\}_{i=0}^n \in \mathrm{P}([a,b])$. We define the *length of the polygonal* approximating the arc length of f on $[a,b]$ as:

$$\ell(f,\mathcal{P}) = \sum_{i=1}^n \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

Lemma 157. Let $f : I \to \mathbb{R}$ and $\mathcal{P}, \mathcal{Q} \in \mathrm{P}(I)$ with $\mathcal{Q} \prec \mathcal{P}$. Then, $\ell(f,\mathcal{P}) \geq \ell(f,\mathcal{Q})$.

Definition 158. Let $f : I \to \mathbb{R}$. If the set $\mathcal{L} := \{\ell(f,\mathcal{P}) : \mathcal{P} \in \mathrm{P}([a,b])\}$ is bounded from above, we say that the graph is *rectifiable* and we define its length $\ell(f,[a,b])$ as:

$$\ell(f,[a,b]) = \sup \mathcal{L}$$

Proposition 159. Let f be a function of class $\mathcal{C}^1([a,b])$. Then, f is rectifiable on $[a,b]$ and

$$\ell(f,[a,b]) = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

Definition 160. Let $\varphi : [a,b] \to \mathbb{R}^2$ with $\varphi(t) = (x(t), y(t))$ and $\mathcal{P} = \{t_i\}_{i=0}^n \in \mathrm{P}([a,b])$. We define the *length of the polygonal* approximating the arc length of φ on $[a,b]$ as:

$$\ell(\varphi,\mathcal{P}) = \sum_{i=1}^n \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

Proposition 161. Let $\varphi : [a,b] \to \mathbb{R}^2$ with $\varphi(t) = (x(t), y(t))$. Suppose that the functions $x(t)$, $y(t)$ are of class $\mathcal{C}^1([a,b])$. Then, the curve φ is rectifiable on $[a,b]$ and

$$\ell(\varphi,[a,b]) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}x$$

Lemma 162. Let f, g be continuous functions on $[a,b]$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $\mathcal{P} = \{t_i\}_{i=0}^n$ with $t_i - t_{i-1} < \delta$, then:

$$\left| \int_a^b \sqrt{f(x)^2 + g(x)^2}\,\mathrm{d}x - \right.$$
$$\left. - \sum_{i=1}^n (t_i - t_{i-1})\sqrt{f(c_i)^2 + g(d_i)^2} \right| < \varepsilon$$

for any $c_i, d_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$.

Lemma 163. Let f, g be continuous functions on $[a,b]$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $\mathcal{P} = \{t_i\}_{i=0}^n$ with $t_i - t_{i-1} < \delta$, then:

$$\left| \int_a^b f(x)g(x)\,\mathrm{d}x - \sum_{i=1}^n (t_i - t_{i-1}) f(c_i) g(d_i) \right| < \varepsilon$$

for any $c_i, d_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$.

Proposition 164 (Surface of revolution). Let $f : [a,b] \to \mathbb{R}_{>0}$ be a function of class $\mathcal{C}^1$. Then, the surface of the solid formed by rotating the area below the function $f(x)$ and between the lines $x = a$ and $x = b$ about the x-axis is given by:

$$S_x = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

Proposition 165 (Surface of revolution). Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. Then, the surface of the solid formed by rotating the area below the function $f(x)$ and between the lines $x = a$ and $x = b$ about the y-axis is given by:

$$S_y = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

Proposition 166 (Volume of revolution). Let $f, g : [a,b] \to \mathbb{R}_{>0}$ be bounded and integrable functions. Then, the volume of the solid formed by rotating the area between the curves of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ about the x-axis is given by:

$$V_x = \pi \int_a^b \left| f(x)^2 - g(x)^2 \right| \mathrm{d}x$$

Proposition 167 (Volume of revolution). Let $a > 0$ and $f, g : [a,b] \to \mathbb{R}$ be bounded and integrable functions. Then, the volume of the solid formed by rotating the area between the curves of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ about the y-axis is given by:

$$V_y = \pi \int_a^b x\,|f(x) - g(x)|\,\mathrm{d}x$$

Proposition 168 (Center of masses). The *center of masses* (x_0, y_0) of a thin plate with uniformly density ρ is:

$$x_0 = \frac{\int_a^b x\sqrt{1+f'(x)^2}dx}{\int_a^b \sqrt{1+f'(x)^2}dx} \quad y_0 = \frac{\int_a^b f(x)\sqrt{1+f'(x)^2}dx}{\int_a^b \sqrt{1+f'(x)^2}dx}$$

Calculation of primitives

Lemma 169. Let $P(x), Q(x) \in \mathbb{R}[x]$ be polynomials with $\deg P(x) < \deg Q(x)$. Suppose $Q(x)$ factorises as:

$$Q(x) = \prod_{i=1}^{n} (x - a_i)^{r_i} \prod_{i=1}^{m} (x^2 + b_i x + c_i)^{s_i}$$

with $b_i^2 - 4c_i < 0$ for $i = 1, \ldots, m$. Then, the function $\frac{P(x)}{Q(x)}$ can be expressed as:

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n}\sum_{j=1}^{r_i} \frac{A_i^j}{(x-a_i)^j} + \sum_{i=1}^{m}\sum_{j=1}^{s_i} \frac{M_i^j x + N_i^j}{(x^2 + b_i x + c_i)^j}$$

where $A_i^j, M_i^j, N_i^j \in \mathbb{R}\ \forall i, j$.

Proposition 170. Let $P(x), Q(x) \in \mathbb{R}[x]$ be polynomials. If $P(x) = C(x)Q(x) + R(x)$, then:

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \int C(x)\,\mathrm{d}x + \int \frac{R(x)}{Q(x)}\,\mathrm{d}x$$

where $\deg R(x) < \deg Q(x)$.

Lemma 171. Let $P(x), Q(x) \in \mathbb{R}[x]$ be polynomials with $\deg P(x) < \deg Q(x)$. Suppose $Q(x)$ factorises as:

$$Q(x) = \prod_{i=1}^{n} (x - a_i)^{r_i} \prod_{i=1}^{m} (x^2 + b_i x + c_i)^{s_i}$$

with $b_i^2 - 4c_i < 0$ for $i = 1, \ldots, m$. Then, the function $\frac{P(x)}{Q(x)}$ can be expressed as:

$$\frac{P(x)}{Q(x)} = \left(\frac{A_1(x)}{Q_1(x)}\right)' + \frac{A_2(x)}{Q_2(x)}$$

where $Q_2(x) = \prod_{i=1}^{n}(x - a_i) \prod_{i=1}^{m}(x^2 + b_i x + c_i)$, $Q_1(x) = \frac{Q(x)}{Q_2(x)}$ and $A_i \in \mathbb{R}[x]$ with $\deg A_i(x) < \deg Q_i(x)$, $i = 1, 2$.

Theorem 172 (Hermite reduction method). Let $P(x), Q(x) \in \mathbb{R}[x]$ be polynomials. Suppose

$$\frac{P(x)}{Q(x)} = \left(\frac{A_1(x)}{Q_1(x)}\right)' + \frac{A_2(x)}{Q_2(x)}$$

for some polynomials $Q_i(x), A_i(x) \in \mathbb{R}[x]$. Then:

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \frac{A_1(x)}{Q_1(x)} + \int \frac{A_2(x)}{Q_2(x)}\,\mathrm{d}x$$